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A Maximization Algorithm of Pseudo-convex Quadratic Functions on Closed Convex Sets in Euclidean Spaces

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ABSTRACT: We give an algorithm to find maxima of pseudo-convex quadratic functions on closed convex sets and show its convergence. Some computational results are given at the end.

Key Words: Pseudo-convexity, quadratic function, maxima, algorithm, computation.

Contents

1. Introduction

Our aim in this paper is the search for the maxima of vectorial pseudo-convex quadratic functions. The motivation behind the choice of these functions is mainly *computation*. We shall give an algorithm to find such maxima and give some computational results at the end. Many papers were devoted to the numerical search of minima and maxima of convex functions, we cite for example [1,8]. In this paper, we show that some results of Enkhbat and Ibaraki [1,8] given in a context of convex functions can be carried on to pseudo-convex functions.

We will give some necessary and sufficient conditions of optimality in the third section. We will also derive an algorithm to apply this program. We deal with the convergence of the algorithm in the fourth section. Some numerical results from problems that were treated in [1,2,7,8,9,10] are given at the end for illustration.

Consider a quadratic and pseudo-convex $f\colon \mathbb{R}^n \to \mathbb{R}$. The problem we are interested in is:

$$(P) \qquad \begin{cases} \text{maximize } f(x), \\ \text{for } x \in C, \text{ a closed convex set.} \end{cases}$$

Recall that a differentiable function f is pseudo-convex (cf. [6]) if:

$$\langle \nabla f(x), y-x\rangle \geq 0 \Rightarrow f(y) \geq f(x), \quad \forall x, y \in C.$$

Since f is quadratic, there is a square real matrix Q of order n and $x, p \in \mathbb{R}^n$ such that:

$$f(x) = \frac{1}{2}\langle Qx, x\rangle + \langle p, x\rangle.$$

The derivative of f at x is $\nabla f(x) = Qx + p$.

In this work, we do not suppose that Q is symmetric, nevertheless we can always transform our problem via a change of basis and a change of variable to the maximization of a function of the form $f(x) = f(x_0) + \frac{1}{2}\langle Dy, y\rangle$ where D is a diagonal matrix and y that will be clarified below.

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2. Properties of the function under study

We begin by the following result.

Proposition 2.1. *Let $f(x) = \frac{1}{2}\langle Qx, x\rangle + \langle p, x\rangle$, where Q is not necessarily a symmetric matrix. We can make a change of variable to obtain*

$$f(x) = f(x_0) + \frac{1}{2}\langle Dy, y\rangle,$$

where D is a diagonal matrix and x_0 is a vector such that $\nabla f\,(x_0) = 0$.

Proof. **Claim:** Consider two matrices defined as follows:

$$A = [a_{\text{ij}}]_{1\leq i,j\leq n} \quad \text{and} \quad Q = [q_{\text{ij}}]_{1\leq i,j\leq n},$$

with $a_{\text{ii}} = q_{\text{ii}}$ and $\forall i \neq j,\ a_{\text{ij}} = \frac{1}{2}(q_{\text{ij}} + q_{\text{ji}})$. Then, A is symmetric and $\langle Qx, x\rangle = \langle Ax, x\rangle$.

Indeed, by construction of the matrix A, $a_{\text{ij}} = a_{\text{ji}}$. So A is symmetric. On the other hand,

$$\begin{aligned}
\langle Qx, x\rangle &= \sum_{j=1}^{n}\sum_{i=1}^{n} q_{ji}\, x_i x_j \\
&= \sum_{i=1}^{n} q_{ii} x_{ii}^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{i=1}^{n} (q_{ji} + q_{ij}) x_i x_j \\
&= \sum_{i=1}^{n} a_{\text{ii}} x_{\text{ii}}^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{i=1}^{n} (a_{\text{ji}} + a_{\text{ij}}) x_i x_j \\
&= \langle A.x, x\rangle
\end{aligned}$$

For the second part, we refer to Best [4] who shows that if $f(x) = \frac{1}{2}\langle Ax, x\rangle + \langle p, x\rangle$, where A is a symmetric matrix. We can make a change of variable and obtain

$$f(x) = f(x_0) + \frac{1}{2}\langle Dy, y\rangle,$$

where D is a diagonal matrix and x_0 is a vector such that $\nabla f\,(x_0) = 0$.

Indeed, A is real symmetric, so there is an orthogonal matrix S and a diagonal matrix D such that $A = S.D.S^T$ and $S^\top = S^{-1}$. Also, $\mathrm{A}x_0 + p = 0$ implies that x_0 is a solution of a linear system. Consider the vector y such that $x = S.y + x_0$.

□

Therefore, whatever the real matrix Q that defines the function $f(x)$, we can always find a symmetric matrix D such that

$$f(x) = \frac{1}{2}\langle Q.x, x\rangle + \langle p, x\rangle = f(x_0) + \frac{1}{2}\langle D.y, y\rangle$$

That way we have an equivalent formulation of f with terms of the form y_i^2 only and no $y_i.y_j$.

Characterization of pseudoconvex quadratic functions

Denote by $\nu(D)$ the number of negative eigenvalues in D and $\pi(D)$ the number of positive eigenvalues.

Suppose in the sequel that $D = (\lambda_i)_i$ is a diagonal matrix and $\nu\,(D) = 1$, $\pi\,(D) = k - 1$ and $k \leq n$. We sort the eigenvalues of the matrix D so that $\lambda_1 \leq 0$. Denote the sets:

$$T_k^+ = \left\{x \in \mathbb{R}^k;\ \sum_{i=1}^{k} \lambda_i.x_i^2 < 0;\ x_1 > 0\right\} \quad \text{and} \quad T_k^- = \left\{x \in \mathbb{R}^k;\ \sum_{i=1}^{k} \lambda_i.x_i^2 < 0;\ x_1 < 0\right\}$$

$$T^+ = \{x \in \mathbb{R}^n;\ f(x) < 0;\ x_1 > 0\} \quad \text{and} \quad T^- = \{x \in \mathbb{R}^n;\ f(x) < 0;\ x_1 > 0\}$$

According to Greub [3], the following sets represent the *solid cones*:

$$\overline{T^+} = \{x \in \mathbb{R}^n;\ f(x) \leq 0;\ x_1 \geq 0\} \text{ and } \overline{T^-} = \{x \in \mathbb{R}^n;\ f(x) \leq 0;\ x_1 \geq 0\}$$

And according to Ferland, [2], if the real diagonal matrix D satisfies $\nu(D) = 1$, then the quadratic form $f(x) = \langle Dx, x\rangle$ is pseudoconvex on either of the sets $\overline{T^+} \setminus N$ and $\overline{T^-} \setminus N$ defined by:

$$\overline{T^+} \setminus N = \left\{x \in \mathbb{R}^n;\ x \in \overline{T^+} \text{ and } Dx \neq\ 0\right\} \text{ and } \overline{T^-} \setminus N = \left\{x \in \mathbb{R}^n;\ x \in \overline{T^-} \text{ and } Dx \neq 0\right\}$$

Proposition 2.2. *If a function f is quadratic pseudoconvex, then we can easily check that:*

(i) $\forall x, y \in C$, *if* $\langle Ax, y - x\rangle \geq 0$, *then* $f(y) \geq f(x)$

(ii) $\forall x, y \in C$,*if* $f(y) < f(x)$, *then* $\langle Ax, y - x\rangle < 0$.

(iii) *When f is pseudoconvex, it is quasiconvex.*

(iv) $\forall x, y \in C$, *if* $f(y) = f(x)$, *then* $\langle Ax, y - x\rangle \leq 0$

3. Optimality Conditions and Algorithm

We define a level set as follows:

$$C_x = \{y \in \mathbb{R}^n;\ f(y) = f(x)\}$$

Theorem 3.1 (Hassouni and Jaddar [5])**.** *A vector x^* is a solution of (P), if and only if*

$$\forall y \in C_{x^*} \text{ and } \forall x \in C,\ \langle Ay, x - y\rangle \leq 0.$$

For the construction of the algorithm to solve our problem, we define the following functions:

$$Y(y) = \max_{y \in C_x} \langle Ay, x - y\rangle \quad \text{ and } \quad X(x) = \max_{y \in C} Y(y).$$

Theorem 3.2. *If $X(x^*) \leq 0$, then x^* is a solution of (P).*

Proof. By definition of $X(x)$ and $Y(y)$, $\forall y \in C_x$:

$$X(x) = \max_{y \in C} Y(y) \geq Y(y) = \max_{y \in C_x} \langle A.y, x - y\rangle \geq \langle Ay, x - y\rangle$$

Hence, $\forall y \in C_x$, $X(x) \geq \langle Ay, x - y\rangle$. So, for x^*, we have

$$\forall y \in C_{x^*},\ 0 \geq X(x^*) \geq \langle Ay, x^* - y\rangle.$$

By Theorem 3, we have that x^* is a solution of (P). □

Algorithm

See Figure 1 below for our algorithm to find the maxima of f.

Lemma 3.3. *The sequence $(f(x^{(k)})_k$ is strictly increasing.*

Proof. For the general case, we suppose that $x^{(k)}$ is not a solution of the problem.

First, let's prove that the sequence is increasing. Since $x^{(k)}$ is not a solution, $X\left(x^{(k)}\right) > 0$. Hence

$$\langle A.y^{(k)}, x^{(k+1)} - y^{(k)}\rangle > 0 \Longrightarrow \langle Ay^{(k)}, x^{(k+1)} - y^{(k)}\rangle \geq 0$$

By (i), $f\left(x^{(k+1)}\right) \geq f\left(y^{(k)}\right) = f\left(x^{(k)}\right)$. So the sequence $(f(x^{(k)})_k$ is increasing.

Let's prove now that the sequence is strictly increasing.

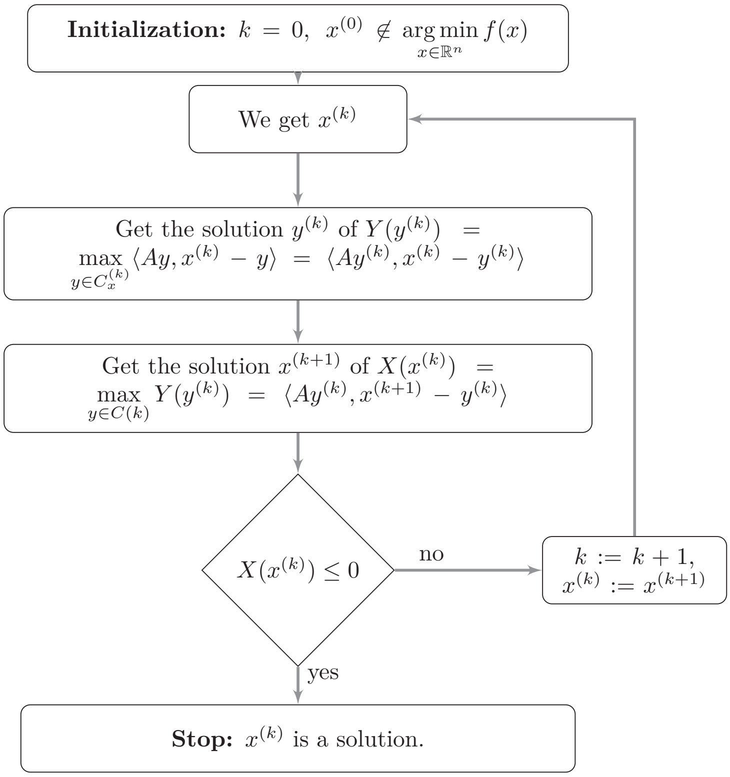

Figure 1: Algorithm

By absurdum, suppose that $\exists k$ such that $f\left(x^{(k+1)}\right) = f\left(x^{(k)}\right)$. Then, $f\left(y^{(k)}\right) = f\left(x^{(k+1)}\right)$, and by (iv), $\langle Ay^{(k)}, x^{(k+1)}\rangle \leq \langle Ay^{(k)}, y^{(k)}\rangle$, so

$$\langle A.y^{(k)}, x^{(k+1)} - y^{(k)}\rangle \leq 0$$

But, since $x^{(k)}$ is not a solution of the problem and satisfies $\langle Ay^{(k)}, x^{(k+1)} - y^{(k)}\rangle > 0$.

A contradiction.

□

Lemma 3.4.

$$\exists L \in \mathbb{R} \textit{ such that } \lim_{k\to\infty} f(x^{(k)}) = L$$

Proof. The sequence $(f(x^{(k)})_k$ is strictly increasing and bounded from above by $f(x^*)$, where x^* is a solution of (P). So for all $k = 1, 2, \ldots$, we have $f\left(x^{(0)}\right) < f(x^{(k)}) \leq f(x^*)$, and the function $f(x)$ is bounded. Since it's also continuous, we deduce that:

$$\text{There exists } L \in \mathbb{R} \text{ such that } f(x^{(k)}) = L.$$

□

Lemma 3.5.

$$\lim_{k\to\infty} X(x^{(k)}) = 0$$

Proof. We have

$$\begin{aligned} X(x^{(k)}) &= \langle Ay^{(k)}, x^{(k+1)} - y^{(k)} \rangle \\ &= \langle Ay^{(k)}, x^{(k+1)} \rangle - 2f(y^{(k)}) \\ &= \langle A(y^{(k)} - x^{(k+1)} + x^{(k+1)}), x^{(k+1)} \rangle - 2f(x^{(k)}) \\ &= \langle A(y^{(k)} - x^{(k+1)}), x^{(k+1)} \rangle + \langle Ax^{(k+1)}, x^{(k+1)} \rangle - 2f(x^{(k)}) \\ &= 2\left(f\left(x^{(k+1)}\right) - f\left(x^{(k)}\right)\right) + \ \langle A(y^{(k)} - x^{(k+1)}), x^{(k+1)} \rangle \end{aligned}$$

By Lemma 3.3, $f\left(x^{(k+1)}\right) > f\left(x^{(k)}\right) = f\left(y^{(k)}\right)$. By (ii), $\langle Ax^{(k+1)}, y^{(k)} - x^{(k+1)} \rangle < 0$. Hence, $X(x^{(k)}) < 2\left(f\left(x^{(k+1)}\right) - f\left(x^{(k)}\right)\right)$. Since $X(x^{(k)}) = \max_{x \in C} Y(y^{(k)}) \geq \max_{y \in C_x(k)} \langle A.y, x^{(k)} - y \rangle \geq 0$. We get $0 \leq X(x^{(k)}) < 2\left(f\left(x^{(k+1)}\right) - f\left(x^{(k)}\right)\right)$. which gives us $\lim_{k \to \infty} X(x^{(k)}) = 0$. □

4. Numerical Simulations

Now, that we have an algorithm to find solutions of (P), we will use it on the following set of problems:

$$P_1 \quad \begin{array}{c} f(x) = x_1^2 + x_2^2 + x_3^2 + (x_3 - x_4)^2 \to \max \\ -2.3 \leq x_i \leq 2.7, \qquad i = 1,2,3,4 \end{array} \tag{4.1}$$

$$P_2 \quad \begin{array}{rl} \multicolumn{2}{c}{f(x) = 4(x_1-1)^2 + 25(x_2-2)^2 \to \max} \\ 8.3x_1 + 20.5x_2 & \leq 170.15 \\ -7.5x_1 + 18x_2 & \leq 135 \\ -10.5x_1 + 7.7x_2 & \leq 80.85 \\ -3.7x_1 - 10.2x_2 & \leq 37.74 \\ -2.7x_1 - 13x_2 & \leq 35.1 \\ 4.5x_1 - 7x_2 & \leq 31.5 \\ -20 \leq x_1 \leq 20, & -20 \leq x_2 \leq 20 \end{array} \tag{4.2}$$

$$P_3 \quad \begin{array}{c} \|x\|^2 \to \max \\ -(n-i+1) \leq x_i \leq n + 0.5i \qquad i = 1,2,\ldots,n \end{array} \tag{4.3}$$

$$P_4 \quad \begin{array}{rl} \multicolumn{2}{c}{f(x) = -x_1^2 + x_2^2} \\ -x_1 - x_2 & \leq -6 \\ 0.4x_1 - x_2 & \leq 1 \\ -x_1 + x_2 & \leq -2 \\ x_1 + x_2 & \leq 13 \\ 0.5x_1 + x_2 & \leq 8.5 \end{array} \tag{4.4}$$

$$P_5 \quad \begin{array}{c} f(x) = -0.5x_1^2 - 2x_1x_2 - 7x_1x_3 - 5x_1 \\ i \leq x_i \leq n + 3i, \ i = 1,2,3 \end{array} \tag{4.5}$$

$$P_6 \quad \begin{array}{c} f(x) = -x_1^2 + \sum_{i>1} \|x\|^2 \\ i + 0.5 \leq x_i \leq n + 0.5i, \ i = 1,2,\ldots,n \end{array} \tag{4.6}$$

The results of the numerical simulations are as follows (n is the number of variables):

Problem	n	$x^{(0)}$	$f(x^{(0)})$	$x^{(*)}$	$f(x^{(*)})$
P_1	4	$x_i^{(0)} = -2.29$	15.7323	$(2.7; 2.7; 2.7; -2.3)^T$	46.87
P_2	2	(-7.21907; 0.65376)	315.5215	$(0.97; 7.91)^T$	871.946
P_3	10	$x_i^{(0)} = 0.01$	0.001	$x_1^* = 10.5$, $x_{i>1}^* = x_{i-1}^* + 0.5$	1646.25

P_4	30	$x_i^{(0)} = 0.01$	0.003	$x_1^* = 30.5$, $x_{i>1}^* = x_{i-1}^* + 0.5$	43313.75
P_3	70	$x_i^{(0)} = 0.01$	0.007	$x_1^* = 70.5$, $x_{i>1}^* = x_{i-1}^* + 0.5$	546148.75
P_3	100	$x_i^{(0)} = 0.01$	0.01	$x_1^* = 100.5$, $x_{i>1}^* = x_{i-1}^* + 0.5$	1589587.5
P_4	2	$x_1 = 4.01$ $x_2 = 1.01$	-15.6	$x_1 = 4$, $x_2 = 2$	-12.00
P_5	3	$x_1^{(0)} = 6.01$, $x_{i\neq 1}^{(0)} = x_{i-1}^{(0)} + 3$	-661.671	$x^* = (1, 2, 3)^T$	-30.50
P_6	5	$x_1^{(0)} = 1.51$, $x_{i>1}^{(0)} = x_{i-1}^{(0)} + 1$	67.0403	$x_1^* = 1.5$, $x_2^* = 6.0$, $x_{i>2}^* = x_{i-2}^* + 0.5$	181.250
P_6	20	$x_1^{(0)} = 1.51$, $x_{i>1}^{(0)} = x_{i-1}^{(0)} + 1$	3084.84	$x_1^* = 1.5$, $x_2^* = 21.0$, $x_{i>2}^* = x_{i-2}^* + 0.5$	12495.0

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