Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–15.
ISSN-0037-8712 in press
doi:10.5269/bspm.44841

Existence Results for Perturbed Fourth-order Kirchhoff Type Elliptic Problems with Singular Term

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ABSTRACT: Under appropriate growth conditions on the nonlinearity, the existence of multiple solutions for a perturbed nonlocal fourth-order Kirchhoff-type problem involving the Hardy term:

$$\Delta_p^2 u - \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x,u),$$

is established. Our main tools are based on variational methods and some critical points theorems. We give some examples to illustrate the obtained results.

Key Words: p-biharmonic, Kirchhoff-type problem, Navier condition, Hardy-Rellich inequality, Variational methods.

Contents

1. Introduction

The purpose of this paper is to establish the existence of multiple solutions for the following perturbed nonlocal fourth-order problem of Kirchhoff-type under Navier boundary condition

$$\begin{cases} \Delta_p^2 u - \left[M(\int_\Omega |\nabla u|^p dx)\right]^{p-1}\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x,u) & x\in\Omega, \\ u = \Delta u = 0 & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ and, $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ denote the p-biharmonic operator and the p-Laplacian operator, respectively, $1 < p < \frac{N}{2}, \Omega \subseteq \mathbb{R}^N$ is an open bounded domain containing the origin in $\mathbb{R}^N$, the boundary $\partial\Omega$ is smooth, and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \le M(t) \le m_1$ for all $t \ge 0$, and $\lambda > 0$, $\mu \ge 0$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory function.

Biharmonic equations can describe the static form change of a beam or the sport of a rigid body. For example, this type of equation furnishes a model for studying traveling wave in suspension bridges (see [21]). Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree theory, and variational methods and critical point theory, and we refer the reader to [1,5,7,8,9,10,13,16,17,18,19,20,23,24,25,26,27,30,34,36] and references therein.

On the other hand, singular elliptic problems have been intensively studied in recent years, see for example, [3,14,15,22,28,29,31,35,37] and the references. Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena and applied economical models. For instance, nonlinear singular boundary value problems arise in the context

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted October 03, 2018. Published March 26, 2019

of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids.

Recently, motivated by this large interest, the problem,

$$\begin{cases} \Delta_p^2 u = \frac{|u|^{p-2}u}{|x|^{2p}} + g(\lambda, x, u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $g :]0, +\infty[\times \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable function, has been extensively investigated. For instance, Xie and Wang, in [35] proved that the problem (1.2) has infinitely many solutions with positive energy levels. Later, Xu and Bai [37] studied the infinitely many solutions for perturbed Kirchhoff type elliptic problems with Hardy potential

$$\begin{cases} M\big(\int_\Omega |\Delta u|^p dx\big)\Delta_p^2 u - a\frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x, u) + \mu g(x, u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Li in [22] considered the fourth order elliptic problem with Navier boundary conditions

$$\begin{cases} \Delta_p^2 u + \frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x, u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

and proved that, the problem (1.4) admits at least two distinct solutions.

Our goal of this work is to show the existence three solutions and two solutions for the following p-harmonic equation:

$$\begin{cases} \Delta_p^2 u - \big[M(\int_\Omega |\nabla u|^p dx)\big]^{p-1}\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x, u) & x \in \Omega, \\ u = \Delta u = 0 & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is bounded domaine in $\mathbb{R}^N (N \geq 5)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < N/2$, and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a carathéodory function such that

(f$_1$) $|f(x, t)| \leq a_1 + a_2|t|^{q-1}, \quad \forall (x, t) \in \Omega \times \mathbb{R}$,

for some non-negative constants a_1, a_2 and $q \in]1, p^*[$, where

$$p^* := \frac{pN}{N - 2p}$$

and

(M$_1$) $M : [0, +\infty[\longrightarrow \mathbb{R}$ be a continuous function such that there are two positive constants m_0 and m_1 with

$$m_0 \leq M(t) \leq m_1 \qquad \forall t \geq 0.$$

Recall that a function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is said carathéodory function, if

(C1) the function $x \longrightarrow f(x, y)$ is measurable for every $y \in \mathbb{R}$;

(C2) the function $y \longrightarrow f(x, y)$ is continuous for a.e. $x \in \Omega$.

The plan of the paper is as follows: Section 2 contains some preliminary lemmas. In Section 3, using of three critical points theorems obtained in [6] which we recall in the next section (Theorems 2.4) we ensure the existence of at least three weak solutions for the problem (1.1). Finally Section 4 contains our main results and their proofs to obtain the existence of at least two weak solutions for the problem (1.1).

2. Preliminaries

Here and in the sequel, X will denote the space $W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega)$. By the Hardy-Rellich inequality (see [11]), we know that

$$\int_\Omega \frac{|u|^p}{|x|^{2p}}dx \leq \frac{1}{H}\int_\Omega |\Delta u|^p dx, \tag{2.1}$$

where the best constant is

$$H = \Big(\frac{(p-1)N(N-2p)}{p^2}\Big)^p. \tag{2.2}$$

Obviously, for any $\mu \in [0, H)$,

$$\begin{aligned}(1-\frac{\mu}{H})\Big(\int_\Omega(|\Delta u|^p + |\nabla u|^p)dx\Big) &\leq \int_\Omega(|\Delta u|^p + |\nabla u|^p - \mu\frac{|u|^p}{|x|^{2p}})dx\\ &\leq \int_\Omega(|\Delta u|^p + |\nabla u|^p)dx.\end{aligned}$$

In $W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega)$, for $\mu \in [0, H]$, we define

$$\|u\| = \Big(\int_\Omega(|\Delta u|^p + |\nabla u|^p - \mu\frac{|u|^p}{|x|^{2p}})dx\Big)^{\frac{1}{p}},$$

this norm is equivalent to $\big(\int_\Omega(|\Delta u|^p + |\nabla u|^p)dx\big)^{\frac{1}{p}}$.

From now let us assume that $\mu \in [0, H[$. Moreover, set $p^* := \frac{pN}{N-2p}$. By the Sobolev embedding theorem there exists a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c\|u\| \ \ (\forall u \in X) \tag{2.3}$$

see, for instance, [33]. Fixing $q \in [1, p^*[$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q\|u\| \ \ (\forall u \in X) \tag{2.4}$$

and, in the particular, the embedding $X \hookrightarrow L^q(\Omega)$ is compact.

Let us define the functionals Φ, $\Psi : X \longrightarrow \mathbb{R}$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\Big(\int_\Omega |\Delta u|^p dx + \tilde{M}(\int_\Omega|\nabla u|^p dx) - \mu\int_\Omega\frac{|u|^p}{|x|^{2p}}dx\Big),\\ \Psi(u) &= \int_\Omega F(x,u)dx,\end{aligned} \tag{2.5}$$

where

$$\tilde{M}(t) = \int_0^t [M(s)]^{p-1}ds \qquad t \geq 0,$$

and

$$F(x,u) = \int_0^u f(x,t)dt, \qquad (x,t) \in \Omega\times\mathbb{R}.$$

In this article, we assume that the following condition holds,

(M1) $M : [0, +\infty[\longrightarrow \mathbb{R}$ is continuous function. Add there are two positive constants m_0, m_1 such that

$$m_0 \leq M(t) \leq m_1, \qquad \forall t \geq 0. \tag{2.6}$$

Then

$$m_0^{p-1}t \leq \tilde{M(t)} \leq m_1^{p-1}t. \tag{2.7}$$

Throughout the paper, denote

$$M^- = \min\{1, m_0^{p-1}\}$$

and

$$M^+ = \max\{1, m_1^{p-1}\},$$

then

$$\begin{aligned}\int_\Omega |\Delta u|^p dx + &\tilde{M}(\int_\Omega |\nabla u|^p dx) - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx \\ &\leq \int_\Omega |\Delta u|^p dx + m_1^{p-1} \int_\Omega |\nabla u|^p dx - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx \\ &\leq M^+ \|u\|^p, \end{aligned} \tag{2.8}$$

and Since, by (2.1), for any $\mu \in [0, H]$

$$\int_\Omega \big(|\Delta u|^p - \mu \frac{|u|^p}{|x|^{2p}}\big) dx \geq 0,$$

we have

$$\begin{aligned}\int_\Omega |\Delta u|^p dx + &\tilde{M}(\int_\Omega |\nabla u|^p dx) - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx \\ &\geq \int_\Omega |\Delta u|^p dx + m_0^{p-1} \int_\Omega |\nabla u|^p dx - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx \\ &\geq M^- \|u\|^p. \end{aligned} \tag{2.9}$$

We need the following propositions in the proofs of Theorems. The proof of this propositions are similar to the proof in [2, Proposition 3.3 and Theorem 3.4].

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function satisfies condition* (f$_1$). *Then we have the following result:*

(1) $\Psi \in C^1(X, \mathbb{R})$ *and for u, v in X, we have*

$$\Psi^{'}(u)(v) = \int_\Omega f(x, u(x))v(x)dx.$$

(2) *The operator $\Psi^{'} : X \to X^*$ is compact.*

Proof. (1) By condition (f$_1$), we have

$$|F(x, u)| \leq a_1|u| + \frac{a_2}{q}|u|^q.$$

Then the Nemytskii operator properties implies that Ψ is a C^1 operator in $L^q(\Omega)$. Since there is a continuous embedding of X into $L^q(\Omega)$, the function Ψ is also C^1 in X and

$$\Psi^{'}(u)(v) = \int_\Omega f(x, u(x))v(x)dx.$$

(2) It is enough to show that $\Psi^{'}$ is strongly continuous in X. Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u$. Since, the embedding of X into $L^q(\Omega)$ is compact, there exists a subsequence, noted also $\{u_n\}$, such that $u_n \to u$ in $L^q(\Omega)$. According to the Krasnoselski's theorem, the Nemytskii operator

$$\begin{aligned} N_f : &L^q \to L^{\frac{q}{q-1}} \\ &u \mapsto f(., u) \end{aligned}$$

is continuous. Hence, $N_f(u_n) \to N_f(u)$ in $L^{\frac{q}{q-1}}(\Omega)$. Then by Holder's inequality and embedding of X into $L^q(\Omega)$, we have

$$\begin{aligned} |\big(\Psi^{'}(u_n) - \Psi^{'}(u)\big)v| &= |\int_\Omega \big(f(x.u_n) - f(x,u)\big)v(x)dx| \\ &\leq c_q \|N_f(u_n) - N_f(u)\|_{\frac{q}{q-1}} \|v\|. \end{aligned}$$

Thus, $\Psi^{'}(u_n) \to \Psi^{'}(u)$ in X^*. This completes the proof. $\square$

Definition 2.2. Let X be a reflexive real Banach space. The operator $T : X \to X^*$ is said to satisfy the (S_+) condition if the assumptions $\limsup_{n\to+\infty} \big\langle T(u_n) - T(u_0), u_n - u_0\big\rangle \leq 0$ and $u_n \rightharpoonup u_0$ in X imply $u_n \to u_0$ in X.

Proposition 2.3. *Let $\mu \in [0, H]$ and $T : X \longrightarrow X^*$ be the operator defined by*

$$\begin{aligned} T(u)v := \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx + \Big[M\big(\int_\Omega |\nabla u|dx\big)\Big]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \\ - \mu \int_\Omega \frac{|u|^{p-2}}{|x|^{2p}} uv dx, \end{aligned}$$

for every $u, v \in X$. Then T admits a continuous inverse on X^*.

Proof. since

$$\begin{aligned} T(u)u &= \int_\Omega |\Delta u|^p dx + \Big[M\big(\int_\Omega |\nabla u|dx\big)\Big]^{p-1} \int_\Omega |\nabla u|^p dx - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx, \\ &\geq \int_\Omega |\Delta u|^p dx + m_0^{p-1} \int_\Omega |\nabla u|^p dx - \mu \int_\Omega \frac{|u|^p}{|x|^{2p}} dx, \\ &\geq M^- \|u\|^p, \end{aligned}$$

then T is coercive. Consequently, thanks to a Minty-Browder theorem [38], the operator T is surjection. For any $x, y \in \mathbb{R}^N$, we have the following elementary inequalities from which we can get the strictly monotonicity of T:

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq \begin{cases} C_p|x-y|^p, & if\, p \geq 2 \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & if\, 1 < p < 2, \end{cases} \tag{2.10}$$

where $\langle ., . \rangle$ denotes the usual inner product in $\mathbb{R}^N$, for every x, $y \in \mathbb{R}^N$. Indeed, for $1 < p < 2$, it is easy to see that

$$\begin{aligned} \big\langle T(u) - T(v),\, u - v\big\rangle &\geq \int_\Omega (|\Delta u|^{p-2}\Delta u - |\Delta v|^{p-2}\Delta v)(\Delta u - \Delta v)dx \\ &+ m_0^{p-1}\Big[\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)dx\Big] \\ &- \mu \int_\Omega (\frac{|u|^{p-2}}{|x|^{2p}}u - \frac{|v|^{p-2}}{|x|^{2p}}v)(u-v)dx \\ &\geq C_p\Big[\int_\Omega \big(\frac{|\Delta u - \Delta v|^2}{(|\Delta u| + |\Delta v|)^{2-p}} + m_0^{p-1}\frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\ &- \mu \frac{|u-v|^2}{|x|^{2p}(|u| + |v|)^{2-p}}\big)dx\Big] > 0, \end{aligned}$$

and for, $p \geq 2$, we also observe that

$$\begin{aligned} \big\langle T(u) - T(v),\, u - v\big\rangle &\geq C_p \int_\Omega \big(|\Delta u - \Delta v|^p + m_0^{p-1}|\nabla u - \nabla v|^p - \mu \frac{|u-v|^p}{|x|^{2p}}\big)dx \\ &\geq C_p M^- \|u - v\|^p > 0, \end{aligned}$$

which means that T is strictly monotone. Thus T is injective and admits an inverse mapping. T^{-1} is continuous. Indeed, let $\{f_n\}$ be a sequence of X^* such that $f_n \longrightarrow f$ in X^*. Let u_n and u in X such that

$$T^{-1}(f_n) = u_n \text{ and } T^{-1}(f) = u.$$

By the coercivity of T, the sequence $\{u_n\}$ is bounded in the reflexive space X. This means that there exist a subsequence that we call again $\{u_n\}$, such that $u_n \rightharpoonup \hat{u}$ in X which implies

$$\lim_{n\to+\infty} \langle T(u_n) - T(u), u_n - \hat{u} \rangle = \lim_{n\to+\infty} \langle f_n - f, u_n - \hat{u} \rangle = 0.$$

Now we prove that T is a mapping of type (S_+), it follows that

$$u_n \to \hat{u} \text{ in X.} \tag{2.11}$$

Indeed let $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle T(u_n) - T(u), u_n - u \rangle \leq 0$. Since T is strictly monotone, then

$$\limsup_{n\to+\infty} \langle K'(u_n) - K'(u), u_n - u \rangle \leq 0,$$

where $K' : X \to X^*$ defined as

$$K(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx \qquad \forall u \in X,$$

and

$$\langle K'(u), v \rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$

for every $v \in X$. Then $u_n \to u$ in X (see Theorem 3.1 of [12]). So, T is a mapping of (S_+) type. On the other hand since T is the Fréchet derivative, it follows that T is continuous, thus from (2.11) we have,

$$T(u_n) \to T(\hat{u}) = T(u) \text{ in X}^*.$$

Hence, taking into account that T is an injection, we have $u = \hat{u}$. This completes the proof. □

To prove our main result in section 3, we use a three critical point theorem of [6]. We recall it in a convenient form.

Theorem 2.4 ([6, Theorem 2.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $X^*, \Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_{X\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(ii) *for each $\lambda \in \Lambda_r := \left]\frac{\Psi(\bar{x})}{\Phi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Other toll is the following abstract result.

Theorem 2.5 ([4, Theorem 3.2])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\Phi(u)<r}\Psi(u) < +\infty$ and assume that, for each $\lambda \in]0, \frac{r}{\sup_{\Phi(u)<r}\Psi(u)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in]0, \frac{r}{\sup_{\Phi(u)<r}\Psi(u)}[$, the functional I_λ admits two distinct critical points.*

We say that a function $u \in X$ is a (weak) solution of the problem (1.1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx + \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx$$
$$-\mu\int_\Omega \frac{|u|^{p-2}}{|x|^{2p}}uvdx - \lambda\int_\Omega f(x,u)vdx = 0$$

for every $v \in X$.

3. Existence of three weak solutions

In this section, we formulate our main results on the existence of at least three weak solutions for the problem (1.1).

Fix $x^0 \in \Omega$ and pick $s > 0$ such that $B(x^0, s) \subset \Omega$ where $B(x^0, s)$ denotes the ball with center at x^0 and radius of s. Put

$$\theta_1 := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\Big|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\Big|^p r^{N-1}dr,$$

$$\theta_2 := \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})}\left[\Sigma_{i=1}^N\left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9(x_i - x_i^0)}{s\ell}\right)^2\right]^{\frac{p}{2}} dx,$$

where Γ denotes the Gamma function, and

$$L := \theta_1 + \theta_2. \tag{3.1}$$

We present our first existence result as follows. We recall that c_q is the constant of the embedding $W_0^{1,p} \cap W^{2,p} \hookrightarrow L^q(\Omega)$ for each $q \in]p, p^*[$, and c_1 stands for c_q with q=1.

Theorem 3.1. *Suppose* (M_1) *and* $\mu \in [0, H[$ *hold(with* H *is as in* (2.2)*). Also let* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Carathéodory function, satisfying condition* (f_1)*. Moreover, assume that*

(f_2) *there exist* $r > 0$ *and* $d > 0$ *with* $r < \frac{M^-}{p}d^pL(1 - \frac{\mu}{H})$ *such that*

$$\varpi_r := \frac{1}{r}\Big\{a_1c_1(\frac{pr}{M^-})^{\frac{1}{p}} + \frac{a_2}{q}c_q^q(\frac{pr}{M^-})^{\frac{q}{p}}\Big\} < \frac{p\int_{\Omega\setminus B(x^0,\frac{s}{2})}F(x,d)dx}{d^pM^+L};$$

(f_3) $\int_{\Omega\setminus B(x^0,\frac{s}{2})}F(x,\zeta)dx \geq 0$ *for each* $\zeta \in [0, d]$*;*

(f_4) *there exist* $a \in [0, +\infty[$ *and* $\gamma \in (1, p)$ *such that*

$$F(x,t) \leq a(1 + |t|^\gamma).$$

Then, for every $\lambda \in \Lambda := \Big]\frac{d^pM^+L}{p\int_{\Omega\setminus B(x^0,\frac{s}{2})}F(x,d)dx}, \frac{1}{\varpi_r}\Big[$ *the problem* (1.1) *possesses at least three weak solutions.*

Proof. In order to apply Theorem 2.4 to our problem, We introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{p}\Big(\int_\Omega |\Delta u|^p dx + \tilde{M}(\int_\Omega |\nabla u|^p dx) - \mu\int_\Omega \frac{|u|^p}{|x|^{2p}}dx\Big),$$

and

$$\Psi(u) = \int_\Omega F(x,u)dx,$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Now we show that the functionals Φ and Ψ satisfy the required conditions. We easily observe that $\Phi(0) = \Psi(0) = 0$. By proposition 2.1 we know that Ψ is a differentiable functional whose differential at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx$$

for every $v \in X$, and $\Psi^{'} : X \to X^*$ is a compact operator. Moreover it is well known thatΨ is sequentially weakly upper semicontinuous, and Φ is continuously differentiable whose differential at the point $u \in X$ is $\Phi'(u) \in X^*$, given by

$$\begin{aligned}\Phi^{'}(u)(v) = \int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx + \big[M\big(\int_\Omega |\nabla u|^p dx\big)\big]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx \\ - a\int_\Omega \frac{|u|^{p-2}}{|x|^{2p}}uvdx, \end{aligned}\tag{3.2}$$

for every $v \in X$, while Proposition 2.3 gives that its Gâteaux derivative admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous.

Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Now, let $\bar{v} \in X$defined by

$$\bar{v}(x) = \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x^0, s) \\ d\Big(\frac{4}{s^3}l^3 - \frac{12}{s^2}l^2 + \frac{9}{s}l - 1\Big) & x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}) \\ d & x \in B(x^0, \frac{s}{2}). \end{cases}\tag{3.3}$$

A direct calculation shows

$$\frac{\partial \bar{v}(x)}{\partial x_i} = \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ d\Big(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9(x_i - x_i^0)}{sl}\Big) & x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}), \end{cases}$$

and

$$\frac{\partial^2 \bar{v}(x)}{\partial x_i^2} = \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ d\Big(\frac{12[(x_i - x_i^0)^2 + l^2]}{ls^3} - \frac{24}{s^2} + \frac{9[l^2 - (x_i - x_i^0)^2]}{l^3 s}\Big) & x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}), \end{cases}$$

and so that

$$\sum_{i=1}^N \frac{\partial^2 \bar{v}(x)}{\partial x_i^2} = \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ d\Big(\frac{12l(N+1)}{s^3} - \frac{24}{s^2} + \frac{9(N-1)}{ls}\Big) & x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}). \end{cases}$$

It is easy to see that

$$\int_\Omega |\Delta \bar{v}(x)|^p dx = \theta_1 d^p \qquad \int_\Omega |\nabla \bar{v}(x)|^p dx = \theta_2 d^p \tag{3.4}$$

Therefore

$$\begin{aligned}\Phi(\bar{v}) &= \frac{1}{p}\Big(\int_\Omega |\Delta \bar{v}|^p dx + \tilde{M}(\int_\Omega |\nabla \bar{v}|^p dx) - \mu\int_\Omega \frac{|\bar{v}|^p}{|x|^{2p}}dx\Big) \\ &\le \frac{1}{p}\int_\Omega |\Delta \bar{v}|^p dx + \frac{1}{p}\tilde{M}(\int_\Omega |\nabla \bar{v}|^p dx) \\ &= \frac{1}{p}\theta_1 d^p + \frac{1}{p}\tilde{M}(\theta_2 d^p) \\ &\le \frac{1}{p}(\theta_1 + m_1^{p-1}\theta_2)d^p \le \frac{M^+}{p}Ld^p, \end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\Phi(\bar{v}) &\geq \frac{1}{p}\Big(\int_\Omega |\Delta\bar{v}|^p dx + m_0^{p-1}\int_\Omega |\nabla\bar{v}|^p dx - \mu\int_\Omega \frac{|\bar{v}|^p}{|x|^{2p}}dx\Big)\\ &\geq \frac{M^-}{p}\Big(\int_\Omega |\Delta\bar{v}|^p dx + \int_\Omega |\nabla\bar{v}|^p dx - \mu\int_\Omega \frac{|\bar{v}|^p}{|x|^{2p}}dx\Big)\\ &\geq \frac{M^-}{p}\Big((1-\frac{\mu}{H})\theta_1 d^p + \theta_2 d^p\Big) \geq \frac{M^-}{p} d^p L(1-\frac{\mu}{H}).\end{aligned} \tag{3.6}$$

Due to (f_3), one has that

$$\Psi(\bar{v}) = \int_\Omega F(x,\bar{v})dx \geq \int_{B(x^0,\frac{s}{2})} F(x,\bar{v})dx = \int_{B(x^0,\frac{s}{2})} F(x,d)dx,$$

so, thanks to (3.5) we get

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} \geq \frac{p\int_{B(x^0,\frac{s}{2})} F(x,d)dx}{M^+ L d^P}. \tag{3.7}$$

From $r < \frac{M^-}{p} d^p L(1-\frac{\mu}{H})$, one has $r < \Phi(\bar{v})$.

On the other hand, due to (2.9), we get

$$\|u\| < (\frac{pr}{M^-})^{\frac{1}{p}}, \tag{3.8}$$

for every $u \in X$ and $\Phi(u) < r$.

Now, from (2.4) and by using (3.8), one has

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,u(x))dx &\leq a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_L(\Omega)\\ &< a_1 c_1(\frac{pr}{M^-})^{\frac{1}{p}} + \frac{a_2}{q}c_q^q(\frac{pr}{M^-})^{\frac{q}{p}},\end{aligned}$$

for every $u \in X$ such that $\Phi(u) < r$. Hence

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \frac{1}{r}\Big(a_1 c_1(\frac{pr}{M^-})^{\frac{1}{p}} + \frac{a_2}{q}c_q^q(\frac{pr}{M^-})^{\frac{q}{p}}\Big),$$

and so condition (i) of Theorem 2.4 is verified.

Now we prove that I_λ is coercive. From (2.4) one has

$$\int_\Omega |u(x)|^\gamma dx \leq c_\gamma^\gamma \|u\|^\gamma.$$

and so, for each $u \in X$ with $\|u\| \geq \max\{1, \frac{1}{c_\gamma}\}$, from ($f_4$) and (2.9) we have

$$\begin{aligned}I_\lambda(u) &:= \Phi(u) - \lambda\Psi(u)\\ &\geq \frac{M^-}{p}\|u\|^p - \lambda c\{\mathrm{meas}(\Omega) + c_\gamma^\gamma\|u\|\},\end{aligned}$$

where meas(Ω) denotes the Lebesgue measure of the open set Ω. Since $\gamma < p$, coercivity of I_λ is obtained. Then, taking into account the fact that the weak solutions of the problem (1.1) are exactly critical points of the functional I_λ, and

$$\Lambda \subseteq \Big]\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\Big[$$

we have the desired conclusion. □

Remark 3.2. We observe that, if $f(x,0) \neq 0$, then by Theorem 3.1, we obtain the existence of at least three non-zero weak solutions.

Example 3.3. *The following function verifies the assumptions requested in Theorem 3.1. Let $r > 1$ be a real number and $1 < \gamma < p < q < p^*$. We consider the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x,t) = \begin{cases} 1 + \alpha(x)|t|^{q-1}, & x \in \Omega,\, t \leq r, \\ 1 + \alpha(x) r^{q-\gamma} t^{\gamma-1}, & x \in \Omega,\, t > r, \end{cases}$$

where $\alpha : \Omega \to \mathbb{R}$ be a Borel, bounded and positive function. condition (f_1) *is easily verified. Taking into account that*

$$F(x,t) = \begin{cases} \leq 0 & x \in \Omega,\, t \leq 0 \\ t + \alpha(x)\frac{|t|^q}{q}, & x \in \Omega,\, 0 < t \leq r, \\ t + \alpha(x)\left(\frac{r^q}{q} + \frac{r^{q-\gamma}}{\gamma} t^\gamma - \frac{r^q}{\gamma}\right), & x \in \Omega,\, t > r, \end{cases}$$

one has $F(x,t) \geq 0$ for each $(x,t) \in \Omega \times [0, +\infty[$ and (f_3) *is verified. Finally, we observe that*

$$F(x,t) = \begin{cases} \leq 0 & x \in \Omega,\, t \leq 0 \\ \leq r + \alpha(x)\frac{|r|^q}{\gamma}, & x \in \Omega,\, 0 < t \leq r, \\ \leq \left(r + \alpha(x)\frac{r^q}{\gamma}\right) t^\gamma, & x \in \Omega,\, t > r, \end{cases}$$

and since $F(x,t) \leq \left(r + \alpha(x)\frac{r^q}{\gamma}\right)\left(1 + |t|^\gamma\right)$ for each $(x,t) \in \Omega \times \mathbb{R}$, (f_4) *is verified.*

4. Existence of two weak solutions

In this section, our goal is to obtain the existence of two distinct weak solutions for the problem (1.1).

Theorem 4.1. *Suppose* (M_1) *and $\mu \in [0, H[$ hold(with H is as in (2.2)). Also let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that* (f_1) *holds. Moreover, assume that*

- (f_5) *there exist $\theta > p$ and $t_0 > 0$ such that*

$$0 < \theta F(x,t) \leq t f(x,t),$$

for each $x \in \Omega$ and $|t| \geq t_0$.

- (f_6) *For every $t > 0$*

$$\tilde{M}(t) \geq t[M(t)]^{p-1}.$$

Then, for each $\lambda \in]0, \lambda^*[$, the problem (1.1) admits at least two distinct weak solutions, where

$$\lambda^* = \frac{1}{a_1 c_1 \left(\frac{p}{M^-}\right)^{1/p} + a_2 \frac{c_q^q}{q}\left(\frac{p}{M^-}\right)^{q/p}}.$$

Proof. Our aim is to apply Theorem 2.5 to problem (1.1) in the case $r = 1$ to the space $X = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and to the functional $\Phi, \Psi : X \to \mathbb{R}$ defined in the proof of Theorem 3.1. First we prove that $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$I_\lambda(u_n) \to c, \qquad \text{and} \qquad I'_\lambda(u_n) \to 0, \tag{4.1}$$

contains a convergent subsequence. Due to (4.1), we can actually assume there is a constant C such that

$$|I_\lambda(u_n)| \leq C \qquad \text{and} \qquad |I'_\lambda(u_n)| \leq C\|u_n\|, \qquad \text{in} X \tag{4.2}$$

for every n. By (4.2) we can write

$$
\begin{aligned}
C+\frac{1}{\theta}.o(1)\|u_n\| &\geq I_\lambda(u_n)-\frac{1}{\theta}I_\lambda^{'}(u_n)u_n\\
&=\frac{1}{p}\Big(\int_\Omega|\Delta u_n|^p dx+\tilde{M}\big(\int_\Omega|\nabla u_n|^p dx\big)-\mu\int_\Omega\frac{|u_n|^p}{|x|^{2p}}dx\Big)\\
&-\frac{1}{\theta}\Big(\int_\Omega|\Delta u_n|^p dx+\big[M\big(\int_\Omega|\nabla u_n|^p dx\big)\big]^{p-1}\int_\Omega|\nabla u_n|^p dx\\
&-\mu\int_\Omega\frac{|u_n|^p}{|x|^{2p}}dx\Big)-\lambda\int_\Omega F(x,u_n)dx+\frac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n dx\\
&\geq(\frac{1}{p}-\frac{1}{\theta})\Big(\int_\Omega|\Delta u_n|^p dx+\big[M\big(\int_\Omega|\nabla u_n|^p dx\big)\big]^{p-1}\int_\Omega|\nabla u_n|^p dx\\
&-\mu\int_\Omega\frac{|u_n|^p}{|x|^{2p}}dx\Big)\\
&+\lambda\int_{\{x\in\Omega:|u_n|\leq t_0\}}\big(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\big)dx\\
&+\lambda\int_{\{x\in\Omega:|u_n|>t_0\}}\big(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\big)dx\\
&\geq(\frac{1}{p}-\frac{1}{\theta})\|u_n\|^p M^- - C_1.
\end{aligned}
$$

Which of course implies that $\{u_n\}$ is bounded in X. By the Eberlian-Smulyan theorem, passing to a subsequence if necessary, we can assume that $u_n\rightharpoonup u$ in X and $u_n\to u$ in $L^q(\Omega)$, so $\big\langle I_\lambda^{'}(u_n),u_n-u\big\rangle\to 0$. By (f$_1$), we have

$$
\int_\Omega|f(x,u_n)(u_n-u)|dx\leq\Big(a_1\mathrm{meas}(\Omega)+a_2\|u_n^{q-1}\|_{L^{\frac{q}{q-1}}(\Omega)}\Big)\|u_n-u\|_{L^q(\Omega)},
$$

since $u_n\to u$ in $L^q(\Omega)$, we have

$$
\lim_{n\to+\infty}\int_\Omega|f(x,u_n)||u_n-u|dx=0.
$$

But

$$
\Big\langle\Phi^{'}(u_n),u_n-u\Big\rangle=\Big\langle I_\lambda^{'}(u_n),u_n-u\Big\rangle+\lambda\int_\Omega f(x,u_n)(u_n-u)dx,
$$

hence,

$$
\limsup_{n\to+\infty}\Big\langle\Phi^{'}(u_n),u_n-u\Big\rangle\leq 0.
$$

Since $\Phi^{'}$verifies (S_+) condition, we have $u_n\to u$ in X and so I_λ satisfies (PS)-condition. From (f$_5$), by standard computations, there is a positive constant C such that

$$
F(x,t)\geq C|t|^\theta \tag{4.3}
$$

for all $x\in\Omega$ and $|t|>t_0$. In fact, setting $a(x):=\min_{|\zeta|=t_0}F(x,\zeta)$ and

$$
\varphi_t(s):=F(x,st),\qquad\forall s>0, \tag{4.4}
$$

by (f$_5$), for every $x\in\Omega$ and $|t|>t_0$ one has

$$
0<\theta\varphi_t(s)=\theta F(x,st)\leq stf(x,st)=s\varphi_t'(s),\qquad\forall s>\frac{t_0}{|t|}.
$$

Therefore,

$$
\int_{t_0/|t|}^1\frac{\varphi_t'(s)}{\varphi_t(s)}ds\geq\int_{t_0/|t|}^1\frac{\theta}{s}ds.
$$

Then

$$\varphi_t(1) \geq \varphi_t\Big(\frac{t_0}{|t|}\Big)\frac{|t|^\theta}{t_0^\theta}.$$

Taking into account of (4.4), we obtain

$$F(x,t) \geq F\Big(x, \frac{t_0}{|t|}t\Big)\frac{|t|^\theta}{t_0^\theta} \geq a(x)\frac{|t|^\theta}{t_0^\theta} \geq C|t|^\theta,$$

where $C > 0$ is a constant. Thus (4.3) is proved. Fixed $u_0 \in X \setminus \{0\}$, for each $t > 1$ one has

$$\begin{aligned} I_\lambda(tu_0) &= \Phi(tu_0) - \lambda\Psi(tu_0) \\ &\leq \frac{M^+}{p}t^p\|u_0\|^p - \lambda C t^\theta \int_\Omega |u_0|^\theta dx. \end{aligned}$$

Since $\theta > p$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0, \lambda^*[$, from (2.9) it follows that

$$\|u\| < \Big(\frac{p}{M^-}\Big)^{1/p} \tag{4.5}$$

for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty, 1[)$. Moreover, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1), (4.5) and the compact embedding $X \hookrightarrow L^q(\Omega)$ imply that, for each $u \in \Phi^{-1}(]-\infty, 1[)$ we have

$$\begin{aligned} \Psi(u) &\leq a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)} \\ &\leq a_1c_1\|u\| + a_2\frac{c_q^q}{q}\|u\|^q \\ &< a_1c_1\Big(\frac{p}{M^-}\Big)^{1/p} + a_2\frac{c_q^q}{q}\Big(\frac{p}{M^-}\Big)^{q/p}, \end{aligned}$$

and so,

$$\sup_{\Phi(u)<1}\Psi(u) \leq a_1c_1\Big(\frac{p}{M^-}\Big)^{1/p} + a_2\frac{c_q^q}{q}\Big(\frac{p}{M^-}\Big)^{q/p} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{4.6}$$

From (4.6) one has

$$\lambda \in]0, \lambda^*[\subset \Big]o, \frac{1}{\sup_{\Phi(u)<1}\Psi(u)}\Big[.$$

So all hypotheses of theorem 4.1 are verified. Therefore, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). $\square$

Example 4.2. *We consider the function f defined by*

$$f(x,u) = \begin{cases} u^{q-1}, & x \in \Omega, u \geq 0, \\ -(-u)^{q-1}, & x \in \Omega, u < 0, \end{cases} \tag{4.7}$$

for each $(x,u) \in \Omega \times \mathbb{R}$, where $1 < p < q < p^*$. We prove that f verifies the assumption requested in Theorem 4.1. Condition (f_1) is easily verified. We observe that

$$f(x,u)u - \theta F(x,u) = |u|^q - \frac{\theta}{q}|u|^q = (1 - \frac{\theta}{q})|u|^q,$$

for each $(x,u) \in \Omega \times \mathbb{R}$. Thus for every θ such that $p < \theta < q$ Condition (f_5) is verified too.

Now we consider the following special case of problem (1.1):

$$\begin{cases} \Delta(\Delta u) - \big[a + be^{-(\int_\Omega |\nabla u|^2 dx)}\big]\Delta u - \mu\frac{u}{|x|^4} = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{4.8}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N > 4$, $0 < \mu < (\frac{N(N-4)}{4})^2$, a and b are positive constants.
Set
$$M(t) = a + be^{-t} \qquad t \geq 0,$$
then
$$a \leq M(t) \leq a + b,$$
and
$$\tilde{M}(t) - tM(t) = b(1 - e^{-t} - te^{-t}).$$
Let $g(t) = -(e^{-t} + te^{-t})$, then $g(0) = -1$ and $g'(t) = te^{-t} \geq 0$ for all $t \geq 0$. Thus for all $t \geq 0$ we have
$$\tilde{M}(t) \geq tM(t).$$

Hence the condition (f_6) is satisfied. In view of Theorem 4.1, we have the following corollary.

Corollary 4.3. *Assume $f(x, u)$ satisfies* (f_1) *and* (f_5), *Then, for each $\lambda \in]0, \lambda^*[$, the problem* (4.8) *admits at least two distinct weak solutions, where*
$$\lambda^* = \frac{1}{a_1 c_1 \sqrt{\frac{2}{\min\{1,a\}}} + a_2 \frac{c_q^q}{q} \sqrt{\left(\frac{2}{\min\{1,a\}}\right)^q}}.$$

Example 4.4. *Let Ω is a bounded smooth domain of $\mathbb{R}^N$, $4 < N < 8$, $0 < \mu < (\frac{N(N-4)}{4})^2$, a and b are positive constants. Then the problem*
$$\begin{cases} \Delta(\Delta u) - \left[a + be^{-(\int_\Omega |\nabla u|^2 dx)}\right]\Delta u - \mu \frac{u}{|x|^4} = \lambda(1 + u^3) & x \in \Omega, \\ u = \Delta u = 0 & x \in \partial\Omega, \end{cases} \tag{4.9}$$
Then, for each $\lambda \in]0, \lambda^*[$, the problem (4.9) *admits at least two distinct weak solutions, where λ^* introduced in the statement of Corollary 4.3.*

In fact, if $N < 8$, then $p^* = \frac{2N}{N-4} > 4$. Hence (f_1) is satisfied, and for (f_5) we have
$$F(u) = \int_0^u (1 + t^3)dt = u + \frac{1}{4}u^4.$$
If we put $\theta = 3$ and $t_0 = 2$ then (f_5) is satisfied too.

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