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Willmore Function on Curvatures of The Curve-Surface Pair Under Mobius Transformation*

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ABSTRACT: We find a geometric invariant of the curve-surface pairs on Willmore functions with the mean and Gauss curvatures. Similar to the work in [5,19], in this work, we define Willmore functions on curve–surface pair and give new characterizations about Willmore functions with necessary and sufficient condition with strip theory in Euclidean 3-space for the first time. In this paper Willmore function on curvatures of the curve-surface pair under Möbiüs transformation is provided invariant.

Key Words: Curve-surface pair, mobius transformation, curvature, Willmore function.

Contents

1. Introduction

Möbiüs differential geometry is a classical subject that was extensively developed in the nineteenth and early twntieth centuries, culminating with the publication of *Blascke's Vorlesungen über Differentialgeometrie III:Differentialgeometrie der Kreise und Kugeln* [3] in 1929.

In 3-dimensional Euclidean Space, a regular curve is described by its curvatures k_1 and k_2 and also a curve-surface pair is described by its curvatures k_n, k_g and t_r. The relations between the curvatures of a curve-surface pair and the curvatures of the curve can be seen in many differential books and papers. Möbius transformations are the automorphisms of the extended complex plane $\mathbb{C}_\infty : \mathbb{C} \cup \{\infty\}$, that is the metamorphic bijections [24]. $\mathbb{M} : \mathbb{C}_\infty \to \mathbb{C}_\infty$. A möbius transformation $\mathbb{M}$ has the form

$$\mathbb{M}(z) = \frac{az+b}{cz+d}; a,b,c,d \in \mathbb{C} \text{ and } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. \tag{1}$$

The set of all Möbius transformations is a group under composition. The Möbius transformation with $c = 0$ form the subgroup of similarities. such transformations have the form

$$S(Z) = AZ + B; A, B \in \mathbb{C}, A \neq 0. \tag{2}$$

The transformation $J(Z) = \frac{1}{Z}$ is called an inversion. Every Möbius transformation $\mathbb{M}$ of the form (2) is a composition of finitely many similarities and inversions [5,9].

Several authors includinf Fubini [21], Thomsen[22] and White [23] have proven that the two form $H^2 - KdA$ is Möbiüs invariant. It is called Willmore functional [5,19].

In this paper we provide that Willmore function on curve-surface of the curve-surface pair under Möbiüs transformation is invariant.

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2. The Curve-Surface pairs

In this section, we give some basic definitions from differential geometry and curve-surface pairs

Definition 2.1. *Let M and α be a surface in E^3 and a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a curve-surface pair and is shown as (α, M).*

Definition 2.2. *Let $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ and $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ be the curve and curve-surface pair's vector fields . The curve-surface pair's tangent vector field, normal vector field and binormal vector field is given by $\overrightarrow{t} = \overrightarrow{\xi}$, $\overrightarrow{\zeta} = \overrightarrow{N}$ and $\overrightarrow{\eta} = \overrightarrow{\zeta} \Lambda \overrightarrow{\xi}$* $[7, 10-18]$.

2.0.1. Curvatures of the curve-surface pair and Curvatures of the Curve. Let $k_n = -b$, $k_g = c$, $t_r = a$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip $[7, 10-18]$.

Let $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ be the curve-surface pair's vector fields on α. Then we have

$$\begin{array}{rcl} \acute{\xi} & = & c\eta - b\zeta \\ \acute{\eta} & = & -c\xi + a\zeta \\ \acute{\zeta} & = & b\xi - a\eta \end{array} . \tag{3}$$

We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r. Let k_n, k_g and t_r be the -b, c, a $[4, 6]$. From (3) we have $\acute{\xi} = c\eta - b\zeta$. If we substitude $\overrightarrow{\xi} = \overrightarrow{t}$ in last equation, we obtain

$$\acute{\xi} = \kappa n$$

and

$$\begin{array}{rcl} b & = & -\kappa \sin\varphi \\ c & = & \kappa \cos\varphi \end{array} \tag{4}$$

$[7, 8, 10-18]$. From last two equations we obtain,

$$\kappa^2 = b^2 + c^2.$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a curve-surface pair $[4, 5, 7, 10-18]$.

By using similar operations, we obtain a new equation as follows

$$\tau = a + \frac{\acute{b}c - b\acute{c}}{b^2 + c^2}$$

($[4, 5, 7, 10-18]$). This equation is a relation between τ (torsion or second curvature of α) and a, b, c curvatures of a curve-surface pair that belongs to the curve α.

And also we can write

$$a = \acute{\varphi} + \tau.$$

The special case: ıf φ =constant, then $\acute{\varphi} = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

Definition 2.3. *Let α be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α, M) is called a curvature curve-surface pair* $[4, 5, 7, 10-18]$.

2.1. Willmore Function on Curvatures of the Curve-Surface Pair Under Möbiüs

The most outstanding problem in Möbiüs differential geometry is the Willmore Conjecture [5,19]. This conjecture is most naturally formulated in terms of surfaces in R^3 rather than S^3. Let $f : M^2 \to R^3$ be a compact surface immersed in R^3[5,19]. Let κ and τ denote principal curvatures of f, $H = (\kappa + \tau)/2$ and $K = \kappa\tau$ denote the mean and Gauss curvatures of f, respectively [5,19]. In 1965 Willmore [5,19] proposed the study of the functional. So it can be written $\tau(f, M^2)$ on the curve surface pair

$$\tau(f, M^2) = \int\limits_{M^2} \frac{\left[\sqrt{b^2+c^2} + \left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right)\right]^2}{2} dA$$

where dA is the area form on (f, M^2) induced by the immersion f. Several authors includinf Fubini [21], Thomsen[22] and White [23] have proven that the two form $H^2 - KdA$ is Möbiüs invariant. It so-called Willmore functional. Now it is:

$$W(f, M^2) = \int\limits_{M^2} \left\{ \frac{\left[\sqrt{b^2+c^2} + \left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right)\right]^2}{2} - \sqrt{b^2+c^2}\left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right) \right\} dA$$

is Möbiüs invariant on curve-surface pair. Thus the Gauss-Bonnet Theorem states that

$$\int\limits_{M^2} \sqrt{b^2+c^2}\left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right) dA = 2\pi\chi(f, M^2)$$

, where $\chi(f, M^2)$ is the Euler characteristic of (f, M^2),we have

$$W(f, M^2) = \int\limits_{M^2} \left\{ \frac{\left[\sqrt{b^2+c^2} + \left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right)\right]^2}{2} - \sqrt{b^2+c^2}\left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right) \right\} dA = \tau(f, M^2) - 2\pi\chi(f, M^2)$$

and then $\tau(f, M^2) = W(f, M^2) + 2\pi\chi(f, M^2)$ is also Möbiüs invariant. Note that

$$\frac{\left[\sqrt{b^2+c^2} + \left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right)\right]^2}{2} - \sqrt{b^2+c^2}\left(a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right) = \frac{1}{4}\left[\sqrt{b^2+c^2} - a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}\right]^2$$

so the Willmore funtional on curve-surface pair has the property that its integrand is non-negative, it vanishes at umbilic point where $\sqrt{b^2+c^2} = a + \frac{\acute{b}c - b\acute{c}}{b^2+c^2}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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