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On Composition Operators of Fibonacci Matrix and Applications of Hausdorff Measure of Noncompactness *

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ABSTRACT: The aim of the paper is introduced the composition of the two infinite matrices $\Lambda = (\lambda_{nk})$ and $\widehat{F} = (f_{nk})$. Further, we determine the α-, β-, γ-duals of new spaces and also construct the basis for the space $\ell_p^{\lambda}(\widehat{F})$. Additionally, we characterize some matrix classes on the spaces $\ell_\infty^{\lambda}(\widehat{F})$ and $\ell_p^{\lambda}(\widehat{F})$. We also investigate some geometric properties concerning Banach-Saks type p. Finally we characterize the subclasses $\mathcal{K}(X : Y)$ of compact operators by applying the Hausdorff measure of noncompactness, where $X \in \{\ell_\infty^{\lambda}(\widehat{F}), \ell_p^{\lambda}(\widehat{F})\}$ and $Y \in \{c_0, c, \ell_\infty, \ell_1, bv\}$, and $1 \leq p < \infty$.

Key Words: Fibonacci numbers, Matrix Transformations, Hausdorff measure of noncompactness, Compact operator, Banach-Saks type p.

Contents

1. Introduction

Define the sequence $(f_n)_{n\in\mathbb{N}_0}$ of Fibonacci numbers given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Fibonacci numbers have many interesting properties and applications. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, some basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \quad \text{(golden ratio)},$$
$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}_0),$$
$$\sum_k \frac{1}{f_k} \text{ converges},$$
$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula)}.$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$ (see [24]).
Let ω be the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. By ℓ_∞, c, c_0 and ℓ_p, we denote the sets of all bounded, convergent, null and p-absolutely summable sequences,

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respectively. Here and after, we suppose unless stated otherwise that $1 \leq p < \infty$ and $q = p/(p-1)$. Also, we use the conventions that $e = (1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}_0$.

Let X and Y be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}_0$. Then we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k)_{k\in\mathbb{N}_0} \in X$, the sequence $Ax = \{A_n(x)\}_{n\in\mathbb{N}_0}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_k a_{nk}x_k \quad \text{for each} \quad n \in \mathbb{N}_0. \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X : Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}_0$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$.

A sequence space X is called an FK-space if it is complete linear metric space with continuous coordinates $p_n : X \to \mathbb{R}$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_n) \in X$ and every $n \in \mathbb{N}_0$. A BK-space is a normed FK-space, that is, a BK-space is a Banach space with continuous coordinates. ℓ_p is a BK-space with the norm

$$\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$$

and c_0, c and ℓ_∞ are BK-spaces with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}_0} |x_k|$.

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if every $x \in X$, there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\left\| x - \sum_{n=0}^{m} \alpha_n b_n \right\| \to 0$, as $m \to \infty$.

The matrix domain plays an important role to construct a new sequence space. In studies on the sequence spaces, generally there are some approaches. Most important of them are determination of topologies, matrix mappings and inclusion relations. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{1.2}$$

It is easy to see that X_A is a sequence space whenever X is a sequence space. In the past, several authors studied matrix transformations on sequence spaces that are the matrix domain of the difference operator, or of the matrices of some classical methods of summability in different sequence spaces, for instance we refer to [7,8,9,19,20,21,22,25,28,36,37,38] and references therein. The Hausdorff measure of non-compactness of linear operators given by infinite matrices in some special classes of sequence spaces were studied in [1,6,27,29,32].

The α, β and γ-duals X^α, X^β and X^γ of a sequence space X are respectively defined by

$$\begin{aligned} X^\alpha &:= \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \quad \text{for all} \quad x = (x_k) \in X\}, \\ X^\beta &:= \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \quad \text{for all} \quad x = (x_k) \in X\}, \\ X^\gamma &:= \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \quad \text{for all} \quad x = (x_k) \in X\}, \end{aligned}$$

where cs and bs are the spaces of all convergent and bounded series, respectively (see [2,31]).

If $X \supset \phi$ is a BK-space and $a = (a_k) \in \omega$, then we write

$$\|a\|_X^* = \sup\left\{\left|\sum_k a_k x_k\right| : \|x\| = 1\right\}.$$

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of compact operators by $\mathcal{K}(X : Y)$.

Let us recall some definitions and well-known results.

Definition 1.1. *Let (X,d) be a metric space, Q be a bounded subset of X nd $B(x,r)=\{y\in X: d(x,y)<r\}$. Then the Hausdorff measure of non-compactness $\chi(Q)$ of Q is defined by*

$$\chi(Q):=\inf\left\{\epsilon>0: Q\subset\bigcup_{i=1}^{n}B(x_i,r_i), x_i\in X, r_i<\epsilon \quad (i=1,2,\ldots,n), n\in\mathbb{N}\right\}.$$

The following results can be found in [3,26].
If Q, Q_1 and Q_2 are bounded subsets of the metric space (X,d), then we have

$$\begin{aligned}
&\chi(Q)=0 \text{ if and only if } Q \text{ is a totally bounded set},\\
&\chi(Q)=\chi(\bar{Q}),\\
&Q_1\subset Q_2 \text{ implies } \chi(Q_1)\leq\chi(Q_2),\\
&\chi(Q_1\cup Q_2)=\max\{\chi(Q_1),\chi(Q_2)\},\\
&\chi(Q_1\cap Q_2)\leq\min\{\chi(Q_1),\chi(Q_2)\}.
\end{aligned}$$

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have

$$\begin{aligned}
&\chi(Q_1+Q_2)\leq\chi(Q_1)+\chi(Q_2),\\
&\chi(Q+x)=\chi(Q) \text{ for all } x\in X,\\
&\chi(\lambda Q)=|\lambda|\,\chi(Q) \text{ for all } \lambda\in\mathbb{C}.
\end{aligned}$$

Definition 1.2. *Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y. Then, the operator $L:X\to Y$ is called $(\chi_1:\chi_2)$-bounded if $L(Q)$ is bounded subset of Y for every subset Q of X and there exists a positive constant K such that $\chi_2(L(Q))\leq K\chi_1(Q)$ for every bounded subset Q of X. If an operator L is $(\chi_1:\chi_2)$-bounded, then the number*

$$\|L\|_{(\chi_1:\chi_2)}=\inf\{K>0:\chi_2(L(Q))\leq K\chi_1(Q) \quad \textit{for all bounded} \quad Q\subset X\}$$

is called $(\chi_1:\chi_2)$- measure of non-compactness of L. In particular, if $\chi_1=\chi_2=\chi$, then we write $\|L\|_\chi$ instead of $\|L\|_{(\chi:\chi)}$.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of non-compactness, and it can be given as follows:
Let X and Y be Banach spaces and $L\in B(X:Y)$. Then the Hausdorff measure of non-compactness $\|L\|_\chi$ of L can be given by $\|L\|_\chi=\chi(L(S_X))$, where $S_X=\{x\in X:\|x\|=1\}$ and we have L is compact if and only if $\|L\|_\chi=0$. We also have $\|L\|=\sup\limits_{x\in S_X}\|Lx\|_Y$.

2. The sequence spaces $\ell_p^\lambda(\widehat{F})$, $(1\leq p\leq\infty)$ of non-absolute type

Das and Hazarika [10] introduced the spaces $c_0^\lambda(\widehat{F})$ and $c^\lambda(\widehat{F})$ derived by the composition of the two infinite matrices Λ and $\widehat{F}$, and obtain some interesting results in terms of the domain of the product of two infinite matrices.

In this section we introduce the spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ derived by the composition of the two infinite matrices Λ and $\widehat{F}$, and show that these spaces are the BK-spaces of non-absolute type which are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively.
We assume throughout this paper that $\lambda=(\lambda_k)_{k\in\mathbb{N}_0}$ is strictly increasing sequence of positive reals tending to ∞, that is, $0<\lambda_0<\lambda_1<\cdots$ and $\lambda_k\to\infty$, as $k\to\infty$.
The sequence spaces ℓ_p^λ and ℓ_∞^λ of non absolute type have been introduced by Mursaleen and Noman (see [30]) as follows:

$$\begin{aligned}
\ell_p^\lambda &:= \left\{x=(x_k)\in\omega:\sum_{n=0}^{\infty}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k\right|^p<\infty\right\},\\
\ell_\infty^\lambda &:= \left\{x=(x_k)\in\omega:\sup_{n\in\mathbb{N}}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k\right|<\infty\right\}.
\end{aligned}$$

Define the matrix $\Lambda = (\lambda_{nk})$ by

$$\lambda_{nk} := \begin{cases} \frac{1}{\lambda_n}(\lambda_k - \lambda_{k-1}) & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}_0$ (see [30,33]). Then, with the notation of (1.2), one can redefine the spaces ℓ_p^λ and ℓ_∞^λ as $\ell_p^\lambda = (\ell_p)_\Lambda$ and $\ell_\infty^\lambda = (\ell_\infty)_\Lambda$.

Let $n \in \mathbb{N}_0$ and f_n be the *nth* Fibonacci number. The infinite matrix $\widehat{F} = (f_{nk})$ was defined by Kara [18] as follows:

$$f_{nk} := \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n \end{cases}$$

for all $k, n \in \mathbb{N}_0$. Define the sequence $y = (y_n)$, which will be frequently used, by the $\widehat{F}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \widehat{F}_n(x) := \begin{cases} x_0 & , \quad n = 0, \\ \frac{f_n}{f_{n+1}}x_n - \frac{f_{n+1}}{f_n}x_{n-1} & , \quad n \geq 1. \end{cases}$$

We employ the technique for obtaining a new sequence space by means of matrix domain. We thus introduce the sequence spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ defined as follows:

$$\ell_p^\lambda(\widehat{F}) = \left\{ x = (x_k) \in \omega : \sum_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1} \right) \right|^p < \infty \right\},$$

$$\ell_\infty^\lambda(\widehat{F}) = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}_0} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1} \right) \right| < \infty \right\}.$$

We use the convention that any term with negative subscript is equal to zero, e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$. With the notation of (1.2), we can redefine the spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ as follows:

$$\ell_p^\lambda(\widehat{F}) = \left(\ell_p^\lambda\right)_{\widehat{F}} \quad \text{and} \quad \ell_\infty^\lambda(\widehat{F}) = \left(\ell_\infty^\lambda\right)_{\widehat{F}}. \tag{2.1}$$

It is immediate by (2.1) that the sets $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ are linear spaces with coordinatewise addition and scalar multiplication. On the other hand, we define the matrix $E = (e_{nk})$ for all $n, k \in \mathbb{N}_0$ by

$$e_{nk} := \begin{cases} \frac{1}{\lambda_n}\left[(\lambda_k - \lambda_{k-1})\frac{f_k}{f_{k+1}} - (\lambda_{k+1} - \lambda_k)\frac{f_{k+2}}{f_{k+1}} \right] & , \quad k < n, \\ \frac{1}{\lambda_n}(\lambda_n - \lambda_{n-1})\frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad k > n. \end{cases} \tag{2.2}$$

Then it can be easily seen that

$$E_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1} \right)$$

holds for all $n \in \mathbb{N}_0$ and $x = (x_k) \in \omega$ which leads us to the fact that

$$\ell_p^\lambda(\widehat{F}) = (\ell_p)_E \quad \text{and} \quad \ell_\infty^\lambda(\widehat{F}) = (\ell_\infty)_E. \tag{2.3}$$

Since E is a triangle, it has a unique inverse $E^{-1} = (g_{nk})$ for all $n, k \in \mathbb{N}_0$ given by

$$g_{nk} := \begin{cases} \lambda_k f_{n+1}^2 \left[\frac{1}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} - \frac{1}{(\lambda_{k+1} - \lambda_k) f_{k+1} f_{k+2}} \right] & , \quad 0 \leq k < n, \\ \frac{\lambda_n f_{n+1}^2}{(\lambda_n - \lambda_{n-1}) f_n f_{n+1}} & , \quad k = n, \\ 0 & , \quad k > n. \end{cases}$$

Further for any sequence $x=(x_k)$ we define the sequence $y=(y_k)$ such that $y=Ex$, that is,

$$\begin{aligned} y_k=E_k(x) &= \sum_{j=0}^{k-1}\frac{1}{\lambda_k}\left[(\lambda_j-\lambda_{j-1})\frac{f_j}{f_{j+1}}-(\lambda_{j+1}-\lambda_j)\frac{f_{j+2}}{f_{j+1}}\right]x_j \\ &\quad +\frac{1}{\lambda_k}(\lambda_k-\lambda_{k-1})\frac{f_k}{f_{k+1}}x_k \end{aligned} \tag{2.4}$$

for all $k\in\mathbb{N}_0$.
Now we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The sequence spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ are BK-spaces with the norms*

$$\begin{aligned} \|x\|_{\ell_p^\lambda(\widehat{F})} &= \|Ex\|_p=\left(\sum_n|E_n(x)|^p\right)^{1/p}, \\ \|x\|_{\ell_\infty^\lambda(\widehat{F})} &= \|Ex\|_\infty=\sup_{n\in\mathbb{N}}|E_n(x)|. \end{aligned}$$

Proof. Since (2.3) holds and ℓ_p and ℓ_∞ are BK-spaces with respect to their natural norms and the matrix E is a triangle, Theorem 4.3.12 of Wilansky [39] gives the fact that $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ are BK-spaces with the given norms.

Remark 2.2. *One can easily check that the absolute property is not satisfied by $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$, that is, $\|x\|_{\ell_p^\lambda(\widehat{F})}\neq\||x|\|_{\ell_p^\lambda(\widehat{F})}$ and $\|x\|_{\ell_\infty^\lambda(\widehat{F})}\neq\||x|\|_{\ell_\infty^\lambda(\widehat{F})}$. This shows that $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ are sequence spaces of non-absolute type, where $|x|=(|x_k|)$.*

Theorem 2.3. *The sequence spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ of non-absolute type are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively, that is, $\ell_p^\lambda(\widehat{F})\cong\ell_p$ and $\ell_\infty^\lambda(\widehat{F})\cong\ell_\infty$.*

Proof. To prove the fact $\ell_p^\lambda(\widehat{F})\cong\ell_p$, we should show the existence of a linear bijection between the spaces $\ell_p^\lambda(\widehat{F})$ and ℓ_p. Consider the transformation T defined with the notation of (2.4), from $\ell_p^\lambda(\widehat{F})$ to ℓ_p by $Tx=y=Ex\in\ell_p$ for every $x\in\ell_p^\lambda(\widehat{F})$. Since T has a matrix representation, the linearity of T is clear. Further it is trivial that $x=\theta$ whenever $Tx=\theta$. Hence T is injective.
Further let $y\in(y_k)\in\ell_p$. Now we define the sequence $x=(x_k)$ by

$$x_k=\sum_{j=0}^{k}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{f_{k+1}^2\lambda_i y_i}{(\lambda_j-\lambda_{j-1})f_jf_{j+1}}$$

for all $k\in\mathbb{N}_0$. It is immediate by the fact that $Ex=y\in\ell_p$ that $x\in\ell_p^\lambda(\widehat{F})$. Hence T is surjective.
Moreover for every $x\in\ell_p^\lambda(\widehat{F})$ we have $\|Tx\|_p=\|y\|_p=\|Ex\|_p=\|x\|_{\ell_p^\lambda(\widehat{F})}$ which means that T is norm preserving. Consequently T is a linear bijection which shows that $\ell_p^\lambda(\widehat{F})$ and ℓ_p are linearly isomorphic. Similarly one can show that $\ell_\infty^\lambda(\widehat{F})\cong\ell_\infty$. So, we omit the details. This concludes the proof.

Theorem 2.4. *Except the case $p=2$, the space $\ell_p^\lambda(\widehat{F})$ is not an inner product space and hence is not a Hilbert space.*

Proof. We have to prove that the space $\ell_2^\lambda(\widehat{F})$ is the only Hilbert space among the $\ell_p^\lambda(\widehat{F})$ spaces. Since the space $\ell_2^\lambda(\widehat{F})$ is the BK-space with the norm $\|x\|_{\ell_2^\lambda(\widehat{F})}=\|Ex\|_2$ by Theorem 2.1 and its norm can be obtained from an inner product, i.e., the equality

$$\|x\|_{\ell_2^\lambda(\widehat{F})}=\langle x,x\rangle^{1/2}=\langle Ex,Ex\rangle_2^{1/2}$$

holds for all $x \in \ell_2^\lambda(\widehat{F})$, the space $\ell_2^\lambda(\widehat{F})$ is a Hilbert space, where $\langle \cdot, \cdot \rangle_2$ denotes the inner product on ℓ_2. Let us consider the sequences $u = (u_k)$ and $v = (v_k)$ defined by

$$u_k := \begin{cases} 1 & , \quad k = 0, \\ f_2^2 + f_2 & , \quad k = 1 \\ f_3^2 \left(1 + \frac{1}{f_2}\right) - \frac{\lambda_1 f_3}{(\lambda_2 - \lambda_1) f_2} & , \quad k \geq 2, \end{cases}$$

$$v_k := \begin{cases} 1 & , \quad k = 0, \\ f_2^2 - \left(\frac{\lambda_1 + \lambda_0}{\lambda_1 - \lambda_0}\right) f_2 & , \quad k = 1, \\ f_3^2 \left[1 - \frac{\lambda_1 + \lambda_0}{(\lambda_1 - \lambda_0) f_2}\right] + \frac{\lambda_1 f_3}{(\lambda_2 - \lambda_1) f_2} & , \quad k \geq 2. \end{cases}$$

Thus we have $Eu = (1, 1, 0, 0, \ldots)$ and $Ev = (1, -1, 0, 0, \ldots)$. Therefore it can be easily seen with $p \neq 2$ that

$$\|u + v\|_{\ell_p^\lambda(\widehat{F})}^2 + \|u - v\|_{\ell_p^\lambda(\widehat{F})}^2 = 8 \neq 4\left(2^{2/p}\right) = 2\left(\|u\|_{\ell_p^\lambda(\widehat{F})}^2 + \|v\|_{\ell_p^\lambda(\widehat{F})}^2\right),$$

i.e., the norm $\|\cdot\|_{\ell_p^\lambda(\widehat{F})}$ with $p \neq 2$ doesn't satisfy the parallelogram identity. This means that the norm $\|\cdot\|_{\ell_p^\lambda(\widehat{F})}$ can't be obtained from an inner product. Hence $\ell_p^\lambda(\widehat{F})$ with $p \neq 2$ is not a Hilbert space.

Remark 2.5. *$\ell_\infty^\lambda(\widehat{F})$ is not an Hilbert space.*

Theorem 2.6. *If $1 \leq p < q < \infty$, then the inclusion $\ell_p^\lambda(\widehat{F}) \subset \ell_q^\lambda(\widehat{F})$ strictly holds.*

Proof. Let $1 \leq p < q < \infty$. Since $\ell_p \subset \ell_q$, we have $\ell_p^\lambda(\widehat{F}) \subset \ell_q^\lambda(\widehat{F})$. Further since the inclusion $\ell_p \subset \ell_q$ is strict, there exists a sequence $x = (x_k) \in \ell_q$ but not in ℓ_p. Let us now define the sequence $y = (y_k)$ in terms of the sequence $x = (x_i)$ as follows:

$$y_k = \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i f_{k+1}^2}{(\lambda_j - \lambda_{j-1}) f_j f_{j+1}} x_i$$

for all $k \in \mathbb{N}_0$. Then we have for all $n \in \mathbb{N}_0$ that

$$E_n(y) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1}\right) = x_n$$

which shows that $Ey = x \in \ell_q \backslash \ell_p$. Hence $y \in \ell_q^\lambda(\widehat{F})$ but is not in $\ell_p^\lambda(\widehat{F})$. That is to say that the inclusion $\ell_p^\lambda(\widehat{F}) \subset \ell_q^\lambda(\widehat{F})$ is strict. This concludes the proof.

Theorem 2.7. *The inclusions $\ell_p^\lambda(\widehat{F}) \subset c_0^\lambda(\widehat{F}) \subset c^\lambda(\widehat{F}) \subset \ell_\infty^\lambda(\widehat{F})$ strictly hold.*

Proof. It is trivial that the inclusion $c_0^\lambda(\widehat{F}) \subset c^\lambda(\widehat{F})$ strictly holds. Let $x = (x_k) \in \ell_p^\lambda(\widehat{F})$. This means that $Ex \in \ell_p$. Since $\ell_p \subset c_0$, $Ex \in c_0$ which gives $x \in c_0^\lambda(\widehat{F})$. Hence $\ell_p^\lambda(\widehat{F}) \subset c_0^\lambda(\widehat{F})$ holds. Now we have to show that the inclusion is strict.
Let us define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i f_{k+1}^2}{(\lambda_j - \lambda_{j-1}) (i+1)^{1/p} f_j f_{j+1}}$$

for all $k \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}_0$ we have $E_n(x) = (n+1)^{-1/p}$ which shows that Ex is not in ℓ_p but is in c_0. Thus the sequence x belongs to the set $c_0^\lambda(\widehat{F}) \setminus \ell_p^\lambda(\widehat{F})$. Hence the inclusion $\ell_p^\lambda(\widehat{F}) \subset c_0^\lambda(\widehat{F})$ is strict. Since $c \subset \ell_\infty$ holds, we have $c^\lambda(\widehat{F}) \subset \ell_\infty^\lambda(\widehat{F})$. Let us consider the sequence $y = (y_k)$ defined by

$$y_k = \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j} \frac{\lambda_i f_{k+1}^2}{(\lambda_j - \lambda_{j-1}) f_j f_{j+1}}$$

for all $k \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}_0$ we have $E_n(y) = (-1)^n$ which shows that $Ey \in \ell_\infty \backslash c$. Thus $y \in \ell_\infty^\lambda(\widehat{F}) \backslash c^\lambda(\widehat{F})$. Therefore the inclusion $c^\lambda(\widehat{F}) \subset \ell_\infty^\lambda(\widehat{F})$ is strict. This completes the proof.

Theorem 2.8. *The inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{F})$ strictly holds.*

Proof. Let $x = (x_k) \in \ell_\infty$. Then we have

$$\|x\|_{\ell_\infty^\lambda(\widehat{F})} = \sup_{n\in\mathbb{N}_0} |E_n(x)| = \sup_{n\in\mathbb{N}_0} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right|$$

Since $\frac{f_k}{f_{k+1}} \leq 1$ and $\frac{f_{k+1}}{f_k} \leq 2$ for all $k \in \mathbb{N}_0$, one can see that

$$\|x\|_{\ell_\infty^\lambda(\widehat{F})} \leq 4\|x\|_\infty . \sup_{n\in\mathbb{N}_0} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \right| \leq 4\|x\|_\infty .$$

Hence the inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{F})$ holds.
Now let us consider $t = (t_k)$ defined by

$$t_k := \begin{cases} 1 & , \quad k = 0, \\ f_{k+1}^2 \left(\sum_{j=1}^{k} \frac{1}{f_j f_{j+1}} + 1 \right) & , \quad k \geq 1. \end{cases}$$

Then we obtain $E_n(t) = 1$ for all $n \in \mathbb{N}_0$ which leads to the fact that $t \in \ell_\infty^\lambda(\widehat{F}) \backslash \ell_\infty$. Hence the inclusion $\ell_\infty \subset \ell_\infty^\lambda(\widehat{F})$ is strict. This completes the proof.

Theorem 2.9. *If the inclusion $\ell_p \subset \ell_p^\lambda(\widehat{F})$ holds, then $(1/\lambda_n) \in \ell_p$.*

Proof. Let us assume that the inclusion $\ell_p \subset \ell_p^\lambda(\widehat{F})$ holds and consider the sequence $e^{(0)} = \{1, 0, 0, 0, \ldots\} \in \ell_p$. Then we have $e^{(0)} \in \ell_p^\lambda(\widehat{F})$ by our assumption and hence $Ee^{(0)} \in \ell_p$. We have $E_n\left(e^{(0)}\right) = \left(3\lambda_0 - 2\lambda_1\right)/\lambda_n$ and therefore we obtain

$$\sum_n \left| E_n\left(e^{(0)}\right) \right|^p = |3\lambda_0 - 2\lambda_1|^p \sum_n \left(\frac{1}{\lambda_n} \right)^p < \infty$$

which implies that $(1/\lambda_n) \in \ell_p$. This completes the proof.

Lemma 2.10. *[30] If $(1/\lambda_n) \in \ell_1$, then $M = \sup_{n\in\mathbb{N}_0} \sum_{n=k}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} < \infty$.*

Theorem 2.11. *If $(1/\lambda_n) \in \ell_1$, then the inclusion $\ell_p \subset \ell_p^\lambda(\widehat{F})$ strictly holds.*

Proof. Let $x = (x_k) \in \ell_p$ with $p > 1$. Then by applying Hölder's inequality we have

$$\begin{aligned} |E_n(x)| &= \left| \sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) \left(\frac{f_k}{f_{k-1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \right| \\ &\leq \left[\sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) \left| \frac{f_k}{f_{k-1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right|^p \right]^{1/p} \left[\sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) \right]^{1-1/p} \end{aligned}$$

which gives

$$|E_n(x)|^p \leq \sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) \left| \frac{f_k}{f_{k-1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right|^p .$$

By Lemma 2.10 we have

$$|E_n(x)|^p \leq \sum_{k=0}^{n} M \left| \frac{f_k}{f_{k-1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right|^p$$

and hence

$$\|x\|^p_{\ell_p^\lambda(\widehat{F})} \leq M2^{2p-1}\left(\sum_k |x_k|^p + \sum_k |x_{k-1}|^p\right) \leq M2^{2p}\|x\|^p_{\ell_p^\lambda}.$$

Therefore $\|x\|_{\ell_p^\lambda(\widehat{F})} \leq 4M^{1/p}\|x\|_p < \infty$. This shows that $x \in \ell_p^\lambda(\widehat{F})$. Hence the inclusion $\ell_p \subset \ell_p^\lambda(\widehat{F})$ holds for $p > 1$.
Now let us consider the sequence $v = (v_k)$ defined by

$$v_k := \begin{cases} 1 & , \quad k = 0, \\ f_2^2 - \frac{\lambda_0 f_2}{(\lambda_1 - \lambda_0) f_1} & , \quad k \geq 1. \end{cases}$$

Then we have $Ev = e^{(0)} \in \ell_p$. Therefore $v \in \ell_p^\lambda(\widehat{F}) \setminus \ell_p$. This means that the inclusion $\ell_p \subset \ell_p^\lambda(\widehat{F})$ is strict.
Similarly one can show that the inclusion $\ell_1 \subset \ell_1^\lambda(\widehat{F})$ also strictly holds.

It is known from Theorem 2.3 of Jarrah and Malkowsky [16] that if T is a triangle then the domain λ_T of T in a normed sequence space λ has a basis if and only if λ has a basis. As a direct consequence of this fact, since the transformation T defined from $\ell_p^\lambda(\widehat{F})$ to ℓ_p, is an isomorphism, the inverse image of the basis $\left\{e^{(k)}\right\}_{k\in\mathbb{N}_0}$ of the space ℓ_p is the basis for the new space $\ell_p^\lambda(\widehat{F})$ with $1 \leq p < \infty$, we have

Corollary 2.12. *Define the sequence $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}_0}$ for every fixed $k \in \mathbb{N}_0$ by*

$$b_n^{(k)} := \begin{cases} 0 & , \quad n < k, \\ \frac{\lambda_k f_{n+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} & , \quad n = k, \\ \frac{\lambda_k f_{n+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} - \frac{\lambda_k f_{n+1}^2}{(\lambda_{k+1} - \lambda_k) f_{k+1} f_{k+2}} & , \quad n > k. \end{cases}$$

Then the following statements hold:

(i) The space $\ell_\infty^\lambda(\widehat{F})$ has no Schauder basis.

(ii) The sequence $\left\{b^{(k)}\right\}_{k\in\mathbb{N}_0}$ is a basis for the space $\ell_p^\lambda(\widehat{F})$ and every $x \in \ell_p^\lambda(\widehat{F})$ has a unique representation of the form $x = \sum_k \alpha_k b^{(k)}$, where $\alpha_k = E_k(x)$ for all $k \in \mathbb{N}_0$.

Corollary 2.13. *While the space $\ell_p^\lambda(\widehat{F})$ is separable but $\ell_\infty^\lambda(\widehat{F})$ is not separable.*

3. The α-, β- and γ-duals of the spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$

In this section we determine the α, β and γ-duals of the sequence spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$ of non-absolute type.
We assume throughout that the sequences $x = (x_k)$ and $y = (y_k)$ are connected by the relation (2.4). Let $A = (a_{nk})$ be an infinite matrix. Now we may begin with quoting the following lemmas which are required for proving the next theorems.

Lemma 3.1. *[35] $A = (a_{nk}) \in (\ell_p : \ell_1)$ if and only if*

(i) For $1 < p \leq \infty$,

$$\sup_{K\in\mathcal{F}} \sum_k \left|\sum_{n\in K} a_{nk}\right|^q < \infty.$$

(ii) For $p = 1$,

$$\sup_{k\in\mathbb{N}_0} \sum_n |a_{nk}| < \infty.$$

Lemma 3.2. *[35] The following statements hold:*

(i) Let $1 < p < \infty$. Then, $A = (a_{nk}) \in (\ell_p : c)$ if and only if

$$\lim_{n\to\infty} a_{nk} \text{ exists for each fixed } k \in \mathbb{N}_0, \tag{3.1}$$

$$\sup_{n\in\mathbb{N}_0} \sum_k |a_{nk}|^q < \infty. \tag{3.2}$$

(ii) $A = (a_{nk}) \in (\ell_1 : c)$ if and only if (3.1) holds and

$$\sup_{n,k\in\mathbb{N}_0} |a_{nk}| < \infty. \tag{3.3}$$

(iii) $A = (a_{nk}) \in (\ell_\infty : c)$ if and only if

$$\sup_{n\in\mathbb{N}_0} \sum_k |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} \sum_k \left| a_{nk} - \lim_{n\to\infty} a_{nk} \right| = 0.$$

Lemma 3.3. *[35] Let $1 < p \leq \infty$. Then the following statements hold:*

(i) $A = (a_{nk}) \in (\ell_p : \ell_\infty)$ if and only if (3.2) holds.

(ii) $A = (a_{nk}) \in (\ell_1 : \ell_\infty)$ if and only if (3.3) holds.

Theorem 3.4. *Define the sets d_1 and d_2 by*

$$d_1 := \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} b_{nk} \right|^q < \infty \right\},$$

$$d_2 := \left\{ a = (a_k) \in \omega : \sup_{k\in\mathbb{N}_0} \sum_n |b_{nk}| < \infty \right\},$$

where the matrix $B = (b_{nk})$ is defined via the sequence $a = (a_n) \in \omega$ by

$$b_{nk} := \begin{cases} \left[\frac{\lambda_k f_{n+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} - \frac{\lambda_k f_{n+1}^2}{(\lambda_{k+1} - \lambda_k) f_{k+1} f_{k+2}} \right] a_n & , \quad k < n, \\ \frac{\lambda_k f_{n+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} & , \quad k = n, \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}_0$. Then $\left[\ell_1^\lambda(\widehat{F})\right]^\alpha = d_2$ and $\left[\ell_p^\lambda(\widehat{F})\right]^\alpha = d_1$.

Proof. Let $a = (a_n) \in \omega$. Then we immediately derive by (2.4) that

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j f_{n+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} a_n y_j = B_n(y) \tag{3.4}$$

for all $n \in \mathbb{N}_0$. Thus we observe by (3.4) that $ax = (a_n x_n) \in \ell_1$ when $x = (x_k) \in \ell_p^\lambda(\widehat{F})$ if and only if $By \in \ell_1$ when $y = (y_k) \in \ell_p$, i.e., $a = (a_n)$ is in the α-dual of the space $\ell_p^\lambda(\widehat{F})$ if and only if $B \in (\ell_p : \ell_1)$. Therefore we see by Lemma 3.1 that $a \in \left[\ell_p^\lambda(\widehat{F})\right]^\alpha$ iff

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} b_{nk} \right|^q < \infty$$

which gives that $\left[\ell_p^{\lambda}(\widehat{F})\right]^{\alpha} = d_1$.
Similarly we get from (3.4) that $a \in \left[\ell_1^{\lambda}(\widehat{F})\right]^{\alpha}$ if and only if $B \in (\ell_1 : \ell_1)$ which is equivalent to

$$\sup_{k\in\mathbb{N}_0} \sum_n |b_{nk}| < \infty.$$

This leads us to the desired result that $\left[\ell_1^{\lambda}(\widehat{F})\right]^{\alpha} = d_2$.

Theorem 3.5. *Define the sets d_3, d_4, d_5, d_6, d_7 and d_8 by*

$$\begin{aligned}
d_3 &:= \left\{ a=(a_k)\in\omega : \sum_{j=k+1}^{\infty} a_j f_{j+1}^2 \text{ exists for each } k\in\mathbb{N}_0 \right\},\\
d_4 &:= \left\{ a=(a_k)\in\omega : \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{n-1} |\bar{a}_k(n)|^q < \infty \right\},\\
d_5 &:= \left\{ a=(a_k)\in\omega : \sup_{n\in\mathbb{N}_0} \left| \frac{\lambda_n f_{n+1}^2}{(\lambda_n-\lambda_{n-1})f_n f_{n+1}} a_n \right| < \infty \right\},\\
d_6 &:= \left\{ a=(a_k)\in\omega : \sup_{n,k\in\mathbb{N}_0} |\bar{a}_k(n)| < \infty \right\},\\
d_7 &:= \left\{ a=(a_k)\in\omega : \lim_{n\to\infty} \sum_k |\bar{a}_k(n) - \bar{a}_k| = 0 \right\},\\
d_8 &:= \left\{ a=(a_k)\in\omega : \sup_{n\in\mathbb{N}_0} \sum_k |\bar{a}_k(n)| < \infty \right\},
\end{aligned}$$

where

$$\begin{aligned}
\bar{a}_k(n) &= \lambda_k \left\{ \frac{a_k f_{k+1}^2}{(\lambda_k-\lambda_{k-1})f_k f_{k+1}} \right.\\
&\quad \left. + \left\{ \left[\frac{1}{(\lambda_k-\lambda_{k-1})f_k f_{k+1}} - \frac{1}{(\lambda_{k+1}-\lambda_k)f_{k+1}f_{k+2}} \right] \sum_{j=k+1}^{n} f_{j+1}^2 a_j \right\},
\end{aligned}$$

with $k<n$ and $\bar{a}_k = \lim_{n\to\infty} \bar{a}_k(n)$. Then $\left[\ell_p^{\lambda}(\widehat{F})\right]^{\beta} = d_3 \cap d_4 \cap d_5$, $\left[\ell_1^{\lambda}(\widehat{F})\right]^{\beta} = d_3 \cap d_5 \cap d_6$ and $\left[\ell_\infty^{\lambda}(\widehat{F})\right]^{\beta} = d_4 \cap d_7 \cap d_8$, where $1<p<\infty$.

Proof. Let $a=(a_k)\in\omega$ and consider the equality

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \left[\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i f_{k+1}^2}{(\lambda_j-\lambda_{j-1})f_j f_{j+1}} y_i \right] \right\} a_k\\
&= \sum_{k=0}^{n-1} \lambda_k \left\{ \frac{a_k f_{k+1}^2}{(\lambda_k-\lambda_{k-1})f_k f_{k+1}} \right.\\
&\quad \left. + \left[\frac{1}{(\lambda_k-\lambda_{k-1})f_k f_{k+1}} - \frac{1}{(\lambda_{k+1}-\lambda_k)f_{k+1}f_{k+2}} \right] \sum_{j=k+1}^{n} f_{j+1}^2 a_j \right\} y_k\\
&\quad + \frac{\lambda_n f_{n+1}^2}{(\lambda_n-\lambda_{n-1})f_n f_{n+1}} a_n y_n\\
&= \sum_{k=0}^{n-1} \bar{a}_k(n) y_k + \frac{\lambda_n f_{n+1}^2}{(\lambda_n-\lambda_{n-1})f_n f_{n+1}} a_n y_n\\
&= T_n(y) \text{ for all } n\in\mathbb{N}_0,
\end{aligned}$$

where $T=(t_{nk})$ is defined by

$$t_{nk}:=\begin{cases}\bar{a}_k(n) & , \quad k<n,\\ \frac{\lambda_n f_{n+1}^2}{(\lambda_n-\lambda_{n-1})f_n f_{n+1}} & , \quad k=n,\\ 0 & , \quad k>n\end{cases}$$

for all $n,k\in\mathbb{N}_0$. Then we have $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in\ell_p^\lambda(\widehat{F})$ if and only if $Ty\in c$ whenever $y=(y_k)\in\ell_p$. Therefore $a=(a_k)\in\left[\ell_p^\lambda(\widehat{F})\right]^\beta$ if and only if $T\in(\ell_p:c)$ with $1\le p\le\infty$. Then, we derive by using Lemma 3.2 for $1<p<\infty$ that

$$\sum_{j=k+1}^{\infty}a_jf_{j+1}^2 \text{ exists for each } k\in\mathbb{N}_0,$$
$$\sup_{n\in\mathbb{N}_0}\sum_{k=0}^{n-1}|\bar{a}_k(n)|^q<\infty,$$
$$\sup_{n\in\mathbb{N}_0}\left|\frac{\lambda_n f_{n+1}^2}{(\lambda_n-\lambda_{n-1})f_nf_{n+1}}a_n\right|<\infty.$$

Therefore we conclude that $\left[\ell_p^\lambda(\widehat{F})\right]^\beta=d_3\cap d_4\cap d_5$ for $1<p<\infty$.
Similarly, for $p=1$ and $p=\infty$ we can see by using Parts (ii) and (iii) of Lemma 3.2 that $\left[\ell_1^\lambda(\widehat{F})\right]^\beta=d_3\cap d_5\cap d_6$ and $\left[\ell_\infty^\lambda(\widehat{F})\right]^\beta=d_4\cap d_7\cap d_8$. This completes the proof.

Theorem 3.6. *Let $1<p\le\infty$. Then we have $\left[\ell_1^\lambda(\widehat{F})\right]^\gamma=d_5\cap d_6$ and $\left[\ell_p^\lambda(\widehat{F})\right]^\gamma=d_5\cap d_8$.*

Proof. This is obtained in the similar way used in the proof of Theorem 3.5 with Lemma 3.3 instead of Lemma 3.2.

4. Some matrix transformations on the sequence spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$

In this section we characterize the classes

$$\left(\ell_p^\lambda(\widehat{F}):\ell_\infty\right),\ \left(\ell_p^\lambda(\widehat{F}):c_0\right),\ \left(\ell_p^\lambda(\widehat{F}):c\right),$$
$$\left(\ell_p^\lambda(\widehat{F}):\ell_1\right),\ \left(\ell_1^\lambda(\widehat{F}):\ell_p\right),\text{ and }\left(\ell_\infty^\lambda(\widehat{F}):\ell_p\right),$$

of matrix transformations where $1\le p\le\infty$.
We assume that the sequences x and y are connected by $y=Ex$. We write for simplicity in notation that

$$e_{nk}(m)=\lambda_k\left\{\frac{f_{k+1}^2a_{nk}}{(\lambda_k-\lambda_{k-1})f_kf_{k+1}}+\left[\frac{1}{(\lambda_k-\lambda_{k-1})f_kf_{k+1}}-\frac{1}{(\lambda_{k+1}-\lambda_k)f_{k+1}f_{k+2}}\right]\sum_{j=k+1}^{m}f_{j+1}^2a_{nj}\right\},$$

where $k<m$ and

$$e_{nk}=\lambda_k\left\{\frac{f_{k+1}^2a_{nk}}{(\lambda_k-\lambda_{k-1})f_kf_{k+1}}+\left[\frac{1}{(\lambda_k-\lambda_{k-1})f_kf_{k+1}}-\frac{1}{(\lambda_{k+1}-\lambda_k)f_{k+1}f_{k+2}}\right]\sum_{j=k+1}^{\infty}f_{j+1}^2a_{nj}\right\}\tag{4.1}$$

for all $k,m,n\in\mathbb{N}_0$ provided the convergence of the series.
Now we quote the following lemmas which are needed in proving our theorems:

Lemma 4.1. *[35] $A = (a_{nk}) \in (\ell_p : c_0)$ if and only if*

(i) For $p = 1$,

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \in \mathbb{N}_0, \qquad \sup_{n,k\in\mathbb{N}_0} |a_{nk}| < \infty. \tag{4.2}$$

(ii) For $1 < p < \infty$, (4.2) holds and

$$\sup_{n\in\mathbb{N}_0} \sum_k |a_{nk}|^q < \infty.$$

(iii) For $p = \infty$,

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0.$$

Lemma 4.2. *[35] $A = (a_{nk}) \in (\ell_1 : \ell_p)$ if and only if*

$$\sup_{k\in\mathbb{N}_0} \sum_n |a_{nk}|^p < \infty$$

Lemma 4.3. *[35] Let $1 < p < \infty$. Then, $A = (a_{nk}) \in (\ell_\infty : \ell_p)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right|^p < \infty.$$

Theorem 4.4. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(i) Let $1 < p < \infty$. Then $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_\infty\right)$ if and only if

$$\sum_{j=k+1}^{\infty} a_{nj} f_{j+1}^2 \quad \text{exists for each} \quad k \in \mathbb{N}_0, \tag{4.3}$$

$$\left\{ \frac{\lambda_k f_{k+1}^2}{(\lambda_k - \lambda_{k-1}) f_k f_{k+1}} a_{nk} \right\} \in \ell_\infty \quad \text{for each} \quad n \in \mathbb{N}_0, \tag{4.4}$$

$$\sup_{k\in\mathbb{N}_0} \sum_n |e_{nk}|^q < \infty, \tag{4.5}$$

$$(a_{nk})_{k\in\mathbb{N}_0} \in d_3 \cap d_4 \cap d_5. \tag{4.6}$$

(ii) $A \in \left(\ell_1^\lambda(\widehat{F}) : \ell_\infty\right)$ if and only if (4.3) and (4.4) hold, and

$$\sup_{n,k\in\mathbb{N}_0} |e_{nk}| < \infty. \tag{4.7}$$

(iii) $A \in \left(\ell_\infty^\lambda(\widehat{F}) : \ell_\infty\right)$ if and only if (4.3) and (4.4) hold, and

$$\sup_{k\in\mathbb{N}_0} \sum_n |e_{nk}| < \infty, \qquad \lim_{m\to\infty} \sum_n |e_{nk}(m) - e_{nk}| = 0. \tag{4.8}$$

Proof. (i) Suppose that the conditions (4.3)-(4.6) hold and let $x = (x_k) \in \ell_p^\lambda(\widehat{F})$ for $1 < p < \infty$. Then we have by Theorem 3.5 that $(a_{nk})_{k\in\mathbb{N}_0} \in \left[\ell_p^\lambda(\widehat{F})\right]^\beta$ for all $n \in \mathbb{N}_0$ and this implies that Ax exists. Also it is clear that the associated sequence $y = (y_k)$ such that $Tx = y$ is in the space $\ell_p \subset c_0$.
Let us now consider the following equality derived from the mth partial sum of the series $\sum_k a_{nk}x_k$ by using the relation $y = Ex$:

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} e_{nk}(m)y_k + \frac{f_{m+1}^2\lambda_m}{f_m f_{m+1}(\lambda_m - \lambda_{m-1})}a_{nm}y_m \tag{4.9}$$

for all $m, n \in \mathbb{N}_0$. By using the conditions (4.3)-(4.5), we obtain from (4.9), as $m \to \infty$, that

$$\sum_k a_{nk}x_k = \sum_k e_{nk}y_k \text{ for all } n \in \mathbb{N}_0. \tag{4.10}$$

Furthermore $E = (e_{nk}) \in (\ell_p : \ell_\infty)$ by Lemma 3.3, we have $Ey \in \ell_\infty$. Therefore one can see by applying Hölder's inequality that

$$\|Ax\|_\infty = \sup_{n\in\mathbb{N}_0}\left|\sum_k a_{nk}x_k\right| \le \sup_{n\in\mathbb{N}_0}\left(\sum_k |e_{nk}|^q\right)^{1/q}\left(\sum_k |y_k|^p\right)^{1/p} < \infty$$

which shows that $Ax \in \ell_\infty$. Hence, $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_\infty\right)$.
Conversely suppose that $A = (a_{nk}) \in \left(\ell_p^\lambda(\widehat{F}) : \ell_\infty\right)$, where $1 < p < \infty$. Then $(a_{nk})_{k\in\mathbb{N}_0} \in \left[\ell_p^\lambda(\widehat{F})\right]^\beta$ for all $n \in \mathbb{N}_0$ which implies the necessity of the condition (4.6) with Theorem 3.5. Since $(a_{nk})_{k\in\mathbb{N}_0} \in \left[\ell_p^\lambda(\widehat{F})\right]^\beta$ for all $n \in \mathbb{N}_0$, (4.10) holds for all $x \in \ell_p^\lambda(\widehat{F})$ and $y \in \ell_p$.
Let us now consider the linear functional f_n on $\ell_p^\lambda(\widehat{F})$ by

$$f_n(x) = \sum_k a_{nk}x_k \text{ for all } n \in \mathbb{N}_0.$$

Then, since $\ell_p^\lambda(\widehat{F})$ and ℓ_p are norm isomorphic, it should follow with (4.10) that

$$\|f_n\| = \|E_n\|_q = \left(\sum_k |e_{nk}|^q\right)^{1/q} \text{ for all } n \in \mathbb{N}_0,$$

where $E_n = (e_{nk})_{k\in\mathbb{N}_0} \in \ell_q$.
This just show that the functional defined by the rows of A on $\ell_p^\lambda(\widehat{F})$ are pointwise bounded. Then, we deduce by Banach-Steinhaus Theorem that these functionals are uniformly bounded. Hence there exists a constant $M > 0$ such that $\|f_n\| \le M$ for all $n \in \mathbb{N}_0$ which gives us $\sup_{k\in\mathbb{N}_0}\sum_n |e_{nk}|^q < \infty$.
This completes the proof of Part (i).
Similarly, Parts (ii) and (iii) can be proved by Parts (i) and (ii) of Lemma 3.3.

Theorem 4.5. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(i) $A \in \left(\ell_1^\lambda(\widehat{F}) : c\right)$ *if and only if (4.3) and (4.4) hold, and*

$$\lim_{n\to\infty} e_{nk} = \alpha_k \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{4.11}$$

(ii) Let $1 < p < \infty$. Then $A \in \left(\ell_p^\lambda(\widehat{F}) : c\right)$ *if and only if (4.3)-(4.6) and (4.11) hold.*

(iii) $A \in \left(\ell_\infty^\lambda(\widehat{F}) : c\right)$ *if and only if (4.3), (4.4) and (4.8) hold, and*

$$\lim_{n\to\infty}\sum_k |e_{nk} - \alpha_k| = 0.$$

Proof. Assume that A satisfies the conditions (4.3)-(4.6) and (4.11), and $x \in \ell_p^\lambda(\widehat{F})$, where $1 < p < \infty$. Then Ax exists and by using (4.11), we have for every $k \in \mathbb{N}_0$ that $|e_{nk}|^q \to |\alpha_k|^q$, as $n \to \infty$, which leads us with (4.5) to the following inequality

$$\sum_{j=0}^{k} |\alpha_j|^q \leq \sup_{n\in\mathbb{N}_0} \sum_{j=0}^{k} |e_{nj}|^q = M < \infty$$

which holds for every $k \in \mathbb{N}_0$. This shows that $(\alpha_k) \in \ell_q$. Since $x \in \ell_p^\lambda(\widehat{F})$, we have $y \in \ell_p$. Therefore by Hölder's inequality we derive that $(\alpha_k y_k) \in \ell_1$ for each $y \in \ell_p$.

Now for any given $\epsilon > 0$, choose a fixed $k_0 \in \mathbb{N}_0$ such that

$$\left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{1/p} \leq \frac{\epsilon}{4M^{1/q}}.$$

Then it follows from (4.11) that there is $m_0 \in \mathbb{N}_0$ such that

$$\left|\sum_{k=0}^{k_0} (e_{nk} - \alpha_k)\, y_k\right| < \frac{\epsilon}{2} \quad \text{for all} \quad n \geq n_0.$$

Therefore by using (4.10), we get for all $n \geq n_0$ that

$$\begin{aligned}
\left|\sum_k a_{nk}x_k - \sum_k \alpha_k y_k\right| &= \left|\sum_k (e_{nk} - \alpha_k)\, y_k\right| \\
&\leq \left|\sum_{k=0}^{k_0} (e_{nk} - \alpha_k)\, y_k\right| + \left|\sum_{k=k_0+1}^{\infty} (e_{nk} - \alpha_k)\, y_k\right| \\
&< \frac{\epsilon}{2} + \left[\sum_{k=k_0+1}^{\infty} (|e_{nk}| + |\alpha_k|)^q\right]^{1/q} \left[\sum_{k=k_0+1}^{\infty} |y_k|^p\right]^{1/p} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4M^{1/q}} \left[\left(\sum_{k=k_0+1}^{\infty} |e_{nk}|^q\right)^{1/q} + \left(\sum_{k=k_0+1}^{\infty} |\alpha_k|^q\right)^{1/q}\right] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4M^{1/q}} 2M^{1/q} = \epsilon.
\end{aligned}$$

Hence $A_n(x) \to \sum_k \alpha_k y_k$, as $n \to \infty$, which means that $Ax \in c$ i.e. $A \in \left(\ell_p^\lambda(\widehat{F}) : c\right)$.

Conversely let $A \in \left(\ell_p^\lambda(\widehat{F}) : c\right)$ with $1 < p < \infty$. Since $c \subset \ell_\infty$, we have $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_\infty\right)$. Thus, the necessity of the conditions (4.3)-(4.6) is immediately obtained by Theorem 4.4, which together imply that (4.10) holds for all $x \in \ell_p^\lambda(\widehat{F})$. Since $Ax \in c$ by the hypothesis, we get by (4.10) that $Ey \in c$ which means that $E = (e_{nk}) \in (\ell_p : c)$. The necessity of (4.11) is immediate by Lemma 3.2. This completes the proof of Part (i).

Since Parts (i) and (iii) can be proved similarly, we omit their proof.

Theorem 4.6. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(i) $A \in \left(\ell_1^\lambda(\widehat{F}) : c_0\right)$ if and only if (4.3) and (4.4) hold, and

$$\lim_{n\to\infty} e_{nk} = 0 \text{ for all } k \in \mathbb{N}_0. \tag{4.12}$$

(ii) Let $1 < p < \infty$. Then $A \in \left(\ell_p^\lambda(\widehat{F}) : c_0\right)$ if and only if (4.3)-(4.6) and (4.12) hold.

(iii) $A \in \left(\ell_\infty^\lambda(\widehat{F}) : c_0\right)$ *if and only if (4.3), (4.4) and (4.8) hold, and*

$$\lim_{n\to\infty} \sum_k |e_{nk}| = 0.$$

Proof. It is natural that Theorem 4.6 can be proved in the same method used in the proof of Theorem 4.5 with Lemma 3.2 and so we omit the detail.

Theorem 4.7. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(i) $A \in \left(\ell_1^\lambda(\widehat{F}) : \ell_1\right)$ *if and only if (4.3), (4.4) and (4.7) hold, and*

$$\sup_{n\in\mathbb{N}_0} \sum_n |e_{nk}| < \infty.$$

(ii) Let $1 < p < \infty$. Then $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_1\right)$ *if and only if (4.3)-(4.6) hold, and*

$$\sup_{F\in\mathcal{F}} \sum_k \left|\sum_{n\in F} e_{nk}\right|^q < \infty. \tag{4.13}$$

(iii) $A \in \left(\ell_\infty^\lambda(\widehat{F}) : \ell_1\right)$ *if and only if (4.3), (4.4) and (4.8) hold and*

$$\sup_{F\in\mathcal{F}} \sum_k \left|\sum_{n\in F} e_{nk}\right| < \infty.$$

Proof. (ii) Suppose that A satisfies the conditions (4.3)-(4.6) and (4.13) and take any $x \in \ell_p^\lambda(\widehat{F})$ with $1 < p < \infty$. We have by Theorem 3.5 that $(a_{nk})_{k\in\mathbb{N}_0} \in \left[\ell_p^\lambda(\widehat{F})\right]^\beta$ for all $n \in \mathbb{N}_0$ and this implies that Ax exists. Besides it follows by combining (4.13) with Lemma 3.1 that $E \in (\ell_p : \ell_1)$ and so we have $Ey \in \ell_1$. Also we derive from (4.3)-(4.6) that the relation (4.10) holds which yields that $Ay \in \ell_1$ and so $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_1\right)$.
Conversely assume that $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_1\right)$ with $1 < p < \infty$. Since $\ell_1 \subset \ell_\infty$, we get $A \in \left(\ell_p^\lambda(\widehat{F}) : \ell_\infty\right)$. Thus Theorem 4.4 implies the necessity of the conditions (4.3)-(4.6) which leads to the relation (4.10). Since $Ax \in \ell_1$, we deduce by (4.10) that $Ey \in \ell_1$ which means $E \in (\ell_p : \ell_1)$. Now the necessity of (4.13) is immediate by Lemma 3.1. This completes the proof of Part (ii).
Parts (i) and (iii) can be proved in the similar way, so we omit the details.

Theorem 4.8. $A = (a_{nk}) \in \left(\ell_1^\lambda(\widehat{F}) : \ell_p\right)$ *if and only if (4.3), (4.4) hold, and*

$$\sup_{k\in\mathbb{N}_0} \sum_n |e_{nk}|^p < \infty. \tag{4.14}$$

Proof. Suppose that the conditions (4.3), (4.4) and (4.14) hold and take $x \in \ell_1^\lambda(\widehat{F})$. Then we have by Theorem 3.5 that $(a_{nk})_{k\in\mathbb{N}_0} \in \left[\ell_1^\lambda(\widehat{F})\right]^\beta$ for all $n \in \mathbb{N}_0$ which implies that Ax exists. From (4.14), we have

$$\sup_{k\in\mathbb{N}_0} |e_{nk}| \le \sup_{k\in\mathbb{N}_0} \left(\sum_n |e_{nk}|^p\right)^{1/p} < \infty \quad \text{for each} \quad n \in \mathbb{N}_0.$$

Hence $\sum_n |e_{nk}|$ absolutely converges for each fixed $n \in \mathbb{N}_0$. Since (4.3) and (4.4) hold, therefore as $m \to \infty$ in (4.9), the relation (4.10) holds. Thus by applying Minkowski's inequality and using (4.10) and (4.14) we obtain that

$$\left(\sum_n \left|\sum_k a_{nk}x_k\right|^p\right)^{1/p} = \left(\sum_n \left|\sum_k e_{nk}y_k\right|^p\right)^{1/p} \leq \sum_k |y_k| \left(\sum_n |e_{nk}|^p\right)^{1/p} < \infty$$

which means $Ax \in \ell_p$, that is, $A = (a_{nk}) \in \left(\ell_1^\lambda(\widehat{F}) : \ell_p\right)$.

Conversely let $A \in \left(\ell_1^\lambda(\widehat{F}) : \ell_p\right)$. Since $\ell_p \subset \ell_\infty$, $A \in \left(\ell_1^\lambda(\widehat{F}) : \ell_\infty\right)$. Thus Theorem 4.4 gives the necessity of (4.3) and (4.4) by the relation (4.10). Since $Ax \in \ell_p$, we deduce by (4.10) that $Ey \in \ell_p$ which means $E \in (\ell_1 : \ell_p)$. Now the necessity of (4.14) is immediate by Lemma 4.2. This step completes the proof.

Theorem 4.9. *Let $1 < p < \infty$. Then $A = (a_{nk}) \in \left(\ell_\infty^\lambda(\widehat{F}) : \ell_p\right)$ if and only if (4.3) and (4.4) hold, and*

$$\sum_k |e_{nk}| \text{ converges for all } n \in \mathbb{N}_0,$$

$$\sup_{K \in \mathcal{F}} \sum_n \left|\sum_{k \in K} e_{nk}\right|^p < \infty.$$

Proof. This is obtained in the same way as done in Theorem 4.8 by Lemma 4.3. So we omit the details.

Lemma 4.10. *[4,5] Let X and Y be any two sequence spaces, A be an infinite matrix and B be a triangle. Then $A \in (X : Y_B)$ if and only if $BA \in (X : Y)$.*

Corollary 4.11. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{\lambda_n} \sum_{i=0}^{n} (\lambda_i - \lambda_{i-1}) \left(\frac{f_i}{f_{i+1}} a_{ik} - \frac{f_{i+1}}{f_i} a_{i-1,k}\right)$$

for all $n, k \in \mathbb{N}_0$. By applying Lemma 4.10 we get A belongs to any one of the classes $\left(c_0 : \ell_p^\lambda(\widehat{F})\right)$, $\left(c : \ell_p^\lambda(\widehat{F})\right)$, $\left(\ell_\infty : \ell_p^\lambda(\widehat{F})\right)$, $\left(\ell_1 : \ell_p^\lambda(\widehat{F})\right)$, $\left(\ell_p : \ell_1^\lambda(\widehat{F})\right)$ and $\left(\ell_p : \ell_\infty^\lambda(\widehat{F})\right)$ if and only if the matrix C belongs to the classes $(c_0 : \ell_p)$, $(c : \ell_p)$, $(\ell_\infty : \ell_p)$, $(\ell_1 : \ell_p)$, $(\ell_p : \ell_1)$ and $(\ell_p : \ell_\infty)$, respectively, where $1 \leq p \leq \infty$.

Corollary 4.12. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C' = (c'_{nk})$ by*

$$c'_{nk} = \frac{1}{\lambda'_n} \sum_{i=0}^{n} (\lambda'_i - \lambda'_{i-1}) \left(\frac{f_i}{f_{i+1}} a_{ik} - \frac{f_{i+1}}{f_i} a_{i-1,k}\right); \ (n, k \in \mathbb{N}_0).$$

Then the necessary and sufficient conditions such that A belongs to any one of the classes $\left(\ell_p^\lambda(\widehat{F}) : \ell_\infty^{\lambda'}(\widehat{F})\right)$, $\left(\ell_p^\lambda(\widehat{F}) : c_0^{\lambda'}(\widehat{F})\right)$, $\left(\ell_p^\lambda(\widehat{F}) : c^{\lambda'}(\widehat{F})\right)$, $\left(\ell_p^\lambda(\widehat{F}) : \ell_1^{\lambda'}(\widehat{F})\right)$, $\left(\ell_1^\lambda(\widehat{F}) : \ell_p^{\lambda'}(\widehat{F})\right)$ and $\left(\ell_\infty^\lambda(\widehat{F}) : \ell_p^{\lambda'}(\widehat{F})\right)$, where $1 \leq p \leq \infty$ are obtained from the respective Theorems 4.4 to 4.9 by replacing the entries of matrix A by those of C and $\lambda' = \left(\lambda'_k\right)$ is a strictly increasing sequence of positive reals tending to infinity and $E = (e_{nk})$ is a triangle defined by (2.2) with λ' instead of λ.

5. Some geometric properties of $\ell_p^\lambda(\widehat{F})$, $(1 < p < \infty)$

In this section we study some geometric properties of the space $\ell_p^\lambda(\widehat{F})$, where $1 < p < \infty$.
A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits

a subsequence (z_n) such that the sequence $C_1(z)$ is convergent in the norm of X, (see [11]), where $C_1(z) = (t_n)$ defined by

$$t_n = \frac{1}{n+1}\left(z_0 + z_1 + \cdots + z_n\right)$$

for all $n \in \mathbb{N}_0$.
A Banach space X is said to have the *weak Banach-Saks property* whenever, given any weakly null sequence (x_n) in X, there exists a subsequence (z_n) of (x_n) such that the sequence (t_n) is strongly convergent to zero.

García-Falset [14] introduced the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : (x_n) \subset B(X),\ x_n \to 0 \text{ weakly},\ x \in B(X)\right\},$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. *[15] A Banach space X with $R(X) < 2$ has the weak fixed point property.*

Let $1 < p < \infty$. A Banach space is said to have the Banach-Saks type p if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that for some $C > 0$,

$$\left\|\sum_{l=0}^{n} x_{k_l}\right\| < C(n+1)^{1/n}$$

for all $n \in \mathbb{N}_0$ (see [23]).

Theorem 5.2. *Let $1 < p < \infty$. The space $\ell_p^{\lambda}(\widehat{F})$ has Banach-Saks type p.*

Proof. Let (ϵ_n) be a sequence of positive numbers for which $\sum_{n=1}^{\infty} \epsilon_n \leq 1/2$. Let (x_n) be a weakly null sequence in $B\left(\ell_p^{\lambda}(\widehat{F})\right)$. Let $u_0 = x_0$ and $u_1 = x_{n_1}$. Then there exists $t_1 \in \mathbb{N}_0$ such that

$$\left\|\sum_{i=t_1+1}^{\infty} u_1(i)e^{(i)}\right\|_{\ell_p^{\lambda}(\widehat{F})} < \epsilon_1.$$

Since (x_n) is a weakly null sequence implies $x_n \to 0$ (coordinatewise), there exists $n_2 \in \mathbb{N}_0$ such that

$$\left\|\sum_{i=0}^{t_1} x_n(i)e^{(i)}\right\|_{\ell_p^{\lambda}(\widehat{F})} < \epsilon_1,$$

where $n \geq n_2$. Set $u_2 = x_{n_2}$. Then there exists $t_2 > t_1$ such that

$$\left\|\sum_{i=t_2+1}^{\infty} u_2(i)e^{(i)}\right\|_{\ell_p^{\lambda}(\widehat{F})} < \epsilon_2.$$

By using the fact that $x_n \to 0$ with respect to coordinatewise, there exists $n_3 > n_2$ such that

$$\left\|\sum_{i=0}^{t_2} x_n(i)e^{(i)}\right\|_{\ell_p^{\lambda}(\widehat{F})} < \epsilon_2,$$

where $n \geq n_3$. If we continue this process, we can find two increasing sequences (t_i) and (n_i) of natural numbers such that

$$\left\|\sum_{i=0}^{t_j} x_n(i)e^{(i)}\right\|_{\ell_p^{\lambda}(\widehat{F})} < \epsilon_j$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=t_j+1}^{\infty} u_j(i)e^{(i)} \right\|_{\ell_p^\lambda(\widehat{F})} < \epsilon_j,$$

where $u_j = x_{n_j}$. Hence

$$\begin{aligned} \left\| \sum_{j=0}^{n} u_j \right\|_{\ell_p^\lambda(\widehat{F})} &= \left\| \sum_{j=0}^{n} \left[\sum_{i=0}^{t_{j-1}} u_j(i)e^{(i)} + \sum_{i=t_{j-1}+1}^{t_j} u_j(i)e^{(i)} + \sum_{i=t_j+1}^{\infty} u_j(i)e^{(i)} \right] \right\|_{\ell_p^\lambda(\widehat{F})} \\ &\leq \left\| \sum_{j=0}^{n} \left[\sum_{i=t_{j-1}+1}^{t_j} u_j(i)e^{(i)} \right] \right\|_{\ell_p^\lambda(\widehat{F})} + 2\sum_{j=0}^{n} \epsilon_j. \end{aligned}$$

On the other hand, we have $\|x\|_{\ell_p^\lambda(\widehat{F})} < 1$. Thus $\|x\|^p_{\ell_p^\lambda(\widehat{F})} < 1$ and we have

$$\begin{aligned} \left\| \sum_{j=0}^{n} \left(\sum_{i=t_{j-1}+1}^{t_j} u_j(i)e^{(i)} \right) \right\|_{\ell_p^\lambda(\widehat{F})} &\leq \sum_{j=0}^{n} \sum_{i=t_{j-1}+1}^{t_j} |E_i(u_i)|^p \\ &\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} |E_i(u_i)|^p \leq (n+1). \end{aligned}$$

Therefore

$$\left\| \sum_{j=0}^{n} \left(\sum_{i=t_{j-1}+1}^{t_j} u_j(i)e^{(i)} \right) \right\|_{\ell_p^\lambda(\widehat{F})} \leq (n+1)^{1/p}.$$

By using the fact that $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}_0$ and $1 < p < \infty$, we have

$$\left\| \sum_{j=0}^{n} u_j \right\|_{\ell_p^\lambda(\widehat{F})} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}.$$

Therefore the space $\ell_p^\lambda(\widehat{F})$ has *Banach-Saks* type p.

Remark 5.3. *Note that* $R\left(\ell_p^\lambda(\widehat{F})\right) = R(\ell_p) = 2^{1/p}$, *since* $\ell_p^\lambda(\widehat{F})$ *is linearly isomorphic to* ℓ_p.

Theorem 5.4. *The space* $\ell_p^\lambda(\widehat{F})$ *has the weak fixed point property, where* $1 < p < \infty$.

6. Compact operators on the spaces $\ell_p^\lambda(\widehat{F})$, $(1 \leq p \leq \infty)$

In this section we establish some estimates for the operator norms and the Hausdorff measures of non-compactness of certain matrix operators on the spaces $\ell_p^\lambda(\widehat{F})$ and $\ell_\infty^\lambda(\widehat{F})$. Further by using the Hausdorff measure of non-compactness, we characterize some classes of compact operators on these spaces.
For our investigations we need the following results:

Theorem 6.1. *[26,39] Let* X *and* Y *be FK spaces. Then* $(X:Y) \subset B(X:Y)$, *that is, every* $A \in (X:Y)$ *defines a linear operator* $L_A \in B(X:Y)$, *where* $L_A x = Ax$ *and* $x \in X$.

Theorem 6.2. *[27] Let* $X \supset \phi$ *and* Y *be BK-spaces. Then* $A \in (X:\ell_\infty)$ *if and only if*

$$\|A\|_X^* = \sup_{n \in \mathbb{N}_0} \|A_n\|_X^* < \infty.$$

Furthermore, if $A \in (X:\ell_\infty)$ *then it follows that* $\|L_A\| = \|A\|_X^*$.

Theorem 6.3. *[25] Let X be a BK-space. Then $A \in (X : \ell_1)$ if and only if*

$$\|A\|^*_{(X:\ell_1)} = \sup_{N \subset \mathbb{N}_0} \left\| \left(\sum_{n \in N} a_{nk} \right)_{k \in \mathbb{N}_0} \right\|^*_X < \infty,$$

where N is finite. Moreover if $A \in (X : \ell_1)$ then $\|A\|^*_{(X:\ell_1)} \leq \|L_A\| \leq 4\|A\|^*_{(X:\ell_1)}$.

Throughout let $T = (t_{nk})_{k,n \in \mathbb{N}_0}$ be a triangle, that is, $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}_0$, S its inverse and $R = S^t$, the transpose of S. We recall the following known results.

Theorem 6.4. *[17,39] Let $(X, \|\cdot\|)$ be a BK-space. Then X_T is a BK-space with $\|\cdot\|_T = \|T \cdot\|$.*

Remark 6.5. *[17] The matrix domain X_T of a normed sequence space X has a basis if and only if X has a basis.*

Theorem 6.6. *[17] Let X be a BK-space with AK and $R = S^t$. If $a = (a_k) \in (X_T)^\beta$, then $\sum_k a_k x_k = \sum_k R_k(a) T_k(x)$ for all $x = (x_k) \in X_T$.*

Remark 6.7. *[17] The conclusion of Theorem 6.6 holds for $X = c$ and $X = \ell_\infty$.*

Theorem 6.8. *[26] Let X and Y be Banach spaces, $S_X = \{x \in X : \|x\| = 1\}$, $K_X = \{x \in X : \|x\| \leq 1\}$ and $A \in B(X : Y)$. Then the Hausdorff measure of non-compactness $\|A\|_\chi$ of a compact operator A is given by $\|A\|_\chi = \chi(AK) = \chi(AS)$.*

Furthermore A is compact if and only if $\|A\|_\chi = 0$, (see [26]). The Hausdorff measure of non-compactness satisfies the inequality $\|A\|_\chi \leq \|A\|$, (see [26]).

Theorem 6.9. *[26] Let X be a Banach space with Schauder basis $\{e_1, e_2, \ldots\}$, Q be a bounded subset of X and $P_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then*

$$\frac{1}{a} \limsup_{n \to \infty} \left[\sup_{x \in Q} \|(I - P_n)x\| \right] \leq \chi(Q) \leq \limsup_{n \to \infty} \left[\sup_{x \in Q} \|(I - P_n)x\| \right],$$

where $a = \limsup_{n \to \infty} \|I - P_n\|$.

Theorem 6.10. *[34] Let X be any of the spaces ℓ_p or c_0 and Q be a bounded subset of a normed space X. If $P_n : X \to X$ is an operator defined by $P_n(x) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$, then*

$$\chi(Q) = \lim_{n \to \infty} \left(\sup_{x \in Q} \|(I - P_n)x\| \right).$$

Theorem 6.11. *[12] Let X be a normed sequence space and χ_T and χ denote the Hausdorff measures of non-compactness on M_{X_T} and M_X, the collection of all bounded sets in X_T and X, respectively. Then $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.*

Lemma 6.12. *[26] Let X denote any of the spaces c_0, c or ℓ_∞. Then $X^\beta = \ell_1$ and $\|a\|^*_X = \|a\|_1$ for all $a \in \ell_1$.*

If $A = (a_{nk})_{k,n \in \mathbb{N}}$ is an infinite matrix, and N is any finite subset of $\mathbb{N}_0$, we write $b^{(N)} = \left(b_k^{(N)} \right)_{k \in \mathbb{N}_0} = (\sum_{n \in N} a_{nk})_{k \in \mathbb{N}_0}$. Also, we have $\widehat{a}_{nk} = R_k A_n$ for all $n, k \in \mathbb{N}_0$.

Theorem 6.13. *[13] Let X be any of the spaces ℓ_p with $1 \leq p \leq \infty$ or c_0. Then the following statements hold:*

(a) Let $Y \in \{c_0, c, \ell_\infty\}$. If $A \in (X_T : Y)$, then we put

$$\begin{aligned}\|A\|_{(X_T:\ell_\infty)} &= \sup_{n\in\mathbb{N}_0} \|E_n\|_q \\ &= \begin{cases} \sup\limits_{n\in\mathbb{N}_0} \sum\limits_k |\widehat{a}_{nk}| & , \quad (X = c_0, \ell_\infty), \\ \sup\limits_{n\in\mathbb{N}_0} \left(\sum\limits_k |\widehat{a}_{nk}|^q\right)^{1/q} & , \quad (X = \ell_p \ for\ 1 < p < \infty), \\ \sup\limits_{k,n\in\mathbb{N}_0} |\widehat{a}_{nk}| & , \quad (X = \ell_1). \end{cases}\end{aligned}$$

Therefore we have $\|L_A\| = \|A\|_{(X_T:\ell_\infty)}$.

(b) Let $Y = \ell_1$. If $A \in (X_T : \ell_1)$. Then we put

$$\begin{aligned}\|A\|_{(X_T:\ell_1)} &= \sup_N \|\widehat{b}^{(N)}\|_q \\ &= \begin{cases} \sup\limits_{N\in\mathcal{F}} \sum\limits_k \left|\sum_{n\in N} \widehat{a}_{nk}\right| & , \quad (X = c_0, \ell_\infty), \\ \sup\limits_{N\in\mathcal{F}} \left(\sum\limits_k \left|\sum\limits_{n\in N} \widehat{a}_{nk}\right|^q\right)^{1/q} & , \quad (X = \ell_p \ \ for \ \ 1 < p < \infty) \end{cases}\end{aligned}$$

and

$$\|A\|_{((\ell_1)_T:\ell_1)} = \sup_{k\in\mathbb{N}_0} \|E^k\|_1 = \sup_{k\in\mathbb{N}_0} \sum_{n=0}^{\infty} |\widehat{a}_{nk}|.$$

If $X = \ell_1$, then $\|L_A\| = \|A\|_{((\ell_1)_T:\ell_1)}$ holds, otherwise we have $\|A\|_{(X_T:\ell_1)} \le \|L_A\| \le 4\|A\|_{(X_T:\ell_1)}$.

By $\mathbb{N}_r$, we denote the subset of $\mathbb{N}_0$ with the elements that are greater than or equal to $r \in \mathbb{N}_0$ and $\sup_{\mathbb{N}_r}$ for the supremum taken over finite subset of $\mathbb{N}_r$.

Theorem 6.14. *[13] Let $A = (a_{nk})$ be an infinite matrix and $1 \le p \le \infty$. Then the following statements hold:*

(a) If $A \in ((\ell_p)_T : c_0)$ or $A \in ((c_0)_T : c_0)$, then we have

$$\begin{aligned}\|L_A\|_\chi &= \lim_{r\to\infty} \left(\sup_{n\in\mathbb{N}_r} \|E_n\|_q\right) \\ &= \begin{cases} \lim\limits_{r\to\infty} \left(\sup\limits_{n\in\mathbb{N}_r} \sum\limits_k |\widehat{a}_{nk}|\right) & , \quad (X = c_0, \ell_\infty), \\ \lim\limits_{r\to\infty} \left[\sup\limits_{n\in\mathbb{N}_r} \left(\sum\limits_k |\widehat{a}_{nk}|^q\right)^{1/q}\right] & , \quad (X = \ell_p \ for\ 1 < p < \infty), \\ \lim\limits_{r\to\infty} \left(\sup\limits_{n\in\mathbb{N}_r, k\in\mathbb{N}} |\widehat{a}_{nk}|\right) & , \quad (X = \ell_1). \end{cases}\end{aligned}$$

(b) If $A \in ((\ell_p)_T : \ell_1)$ $(1 < p \le \infty)$ or $A \in ((c_0)_T : \ell_1)$, then we have

$$\lim_{r\to\infty} \left(\sup_{\mathbb{N}_r} \left\|\sum_{n\in\mathbb{N}_r} E_n\right\|_q\right) \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \left(\sup_{\mathbb{N}_r} \left\|\sum_{n\in\mathbb{N}_r} E_n\right\|_q\right),$$

if $A \in ((\ell_1)_T : \ell_1)$, then we have

$$\|L_A\|_\chi = \lim_{r\to\infty} \left[\sup_{k\in\mathbb{N}_0} \left(\sum_{n=r}^{\infty} |\widehat{a}_{nk}|\right)\right].$$

(c) If $A \in ((\ell_p)_T : c)$ or $A \in ((c_0)_T : c)$, then we have

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\|E_n-\widehat{\alpha}\|_q\right)\leq\|L_A\|_\chi\leq\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\|E_n-\widehat{\alpha}\|_q\right),$$

where $\widehat{\alpha}=(\widehat{\alpha}_k)_{k\in\mathbb{N}_0}$ with $\widehat{a}_{nk}\to\widehat{\alpha}_k$, as $n\to\infty$, for every $k\in\mathbb{N}_0$.

Theorem 6.15. *Let $A=(a_{nk})$ be an infinite matrix and define the matrix $E=(e_{nk})_{k,n\in\mathbb{N}}$ by (4.1). Then the following statements hold:*

(a) Let $Y\in\{c_0,c,\ell_\infty\}$. If $A\in\left(\ell_p^\lambda(\widehat{F}):Y\right)$ with $1\leq p\leq\infty$, then we have

$$\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_\infty)}=\begin{cases}\sup\limits_{n\in\mathbb{N}_0}\sum\limits_k|\widehat{a}_{nk}| & ,\quad(p=\infty),\\ \sup\limits_{n\in\mathbb{N}_0}\left(\sum\limits_k|\widehat{a}_{nk}|^q\right)^{1/q} & ,\quad(1<p<\infty),\\ \sup\limits_{k,n\in\mathbb{N}_0}|\widehat{a}_{nk}| & ,\quad(p=1).\end{cases}$$

Then we have $\|L_A\|=\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_\infty)}$.

(b) If $A\in\left(\ell_p^\lambda(\widehat{F}):\ell_1\right)$ with $1\leq p\leq\infty$, then we have

$$\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_1)}=\begin{cases}\sup\limits_{N\in\mathcal{F}}\sum\limits_k\left|\sum\limits_{n\in N}\widehat{a}_{nk}\right| & ,\quad(p=\infty),\\ \sup\limits_{N\in\mathcal{F}}\left(\sum\limits_k\left|\sum\limits_{n\in N}\widehat{a}_{nk}\right|^q\right)^{1/q} & ,\quad(1<p<\infty),\\ \sup\limits_{k\in\mathbb{N}_0}\sum\limits_{n=0}^{\infty}|\widehat{a}_{nk}| & ,\quad(p=1).\end{cases}$$

Then for $p=1$, $\|L_A\|=\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_1)}$ holds, otherwise $\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_1)}\leq\|L_A\|\leq 4\|A\|_{(\ell_p^\lambda(\widehat{F}):\ell_1)}$.

Proof. Suppose that $A\in\left(\ell_p^\lambda(\widehat{F}):Y\right)$. Then we have $A_n\in\left[\ell_p^\lambda(\widehat{F})\right]^\beta$ for all $n\in\mathbb{N}_0$ and it follows from Theorem 6.6 that

$$A_n(x)=\sum_k a_{nk}x_k=\sum_k R_k(A_n)T_k(x)$$

for all $x\in c^\lambda(\widehat{F})$ and $n\in\mathbb{N}_0$, where $R_k(A_n)=\sum\limits_j r_{kj}a_{nj}=\sum\limits_j s_{jk}a_{nj}$. Here $T=E$ and $S=E^{-1}$. Therefore we have $R_k(A_n)=e_{nk}$ for all $k,n\in\mathbb{N}_0$.

Proof of Part (a) can be obtained by applying Theorem 6.13. Since the proof of Part (b) is similar to the proof of Part (a), we omit the details.

Theorem 6.16. *Let $A=(a_{nk})$ be an infinite matrix and $E=(e_{nk})_{k,n\in\mathbb{N}_0}$ be defined by (4.1). Then, for $1\leq p\leq\infty$, we have*

(a) If $A\in\left(\ell_p^\lambda(\widehat{F}):c_0\right)$, then we have

$$\|L_A\|_\chi=\begin{cases}\lim\limits_{r\to\infty}\left(\sup\limits_{n\in\mathbb{N}_r}\sum\limits_k|\widehat{a}_{nk}|\right) & ,\quad(p=\infty),\\ \lim\limits_{r\to\infty}\left(\sup\limits_{n\in\mathbb{N}_r}\left(\sum\limits_k|\widehat{a}_{nk}|^q\right)^{1/q}\right) & ,\quad(1<p<\infty),\\ \lim\limits_{r\to\infty}\left(\sup\limits_{n\geq r,k\geq 0}|\widehat{a}_{nk}|\right) & ,\quad(p=1).\end{cases}$$

(b) If $A \in \left(\ell_p^{\lambda}(\widehat{F}) : \ell_1\right)$ *with* $1 < p \leq \infty$, *then we have*

$$\lim_{r\to\infty}\left(\sup_{\mathbb{N}_r}\left\|\sum_{n\in\mathbb{N}_r} E_n\right\|_q\right) \leq \|L_A\|_{\chi} \leq 4\lim_{r\to\infty}\left(\sup_{\mathbb{N}_r}\left\|\sum_{n\in\mathbb{N}_r} E_n\right\|_q\right).$$

If $A \in \left(\ell_1^{\lambda}(\widehat{F}) : \ell_1\right)$, *then we have*

$$\|L_A\|_{\chi} = \lim_{r\to\infty}\left[\sup_{k\in\mathbb{N}_0}\left(\sum_{n=r}^{\infty}|\widehat{a}_{nk}|\right)\right].$$

(c) If $A \in \left(\ell_p^{\lambda}(\widehat{F}) : c\right)$, *then we have*

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\|E_n - \widehat{\alpha}\|_q\right) \leq \|L_A\|_{\chi} \leq \lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\|E_n - \widehat{\alpha}\|_q\right),$$

where $\widehat{\alpha} = (\widehat{\alpha}_k)_{k\in\mathbb{N}}$ *with* $\widehat{\alpha}_k = \lim_{n\to\infty}\widehat{a}_{nk}$ *for all* $k \in \mathbb{N}_0$.

Proof. Proof of Theorem 6.16 can be given in the same way as that of Theorem 6.15 by applying Theorem 6.14 instead of Theorem 6.13.

Corollary 6.17. *Let* $1 \leq p \leq \infty$. *Then the following statements hold:*

(a) If $A \in \left(\ell_p^{\lambda}(\widehat{F}) : c_0\right)$, *then* L_A *is compact if and only if*

(i) for $p = \infty$,

$$\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\sum_k|\widehat{a}_{nk}|\right) = 0,$$

(ii) for $1 < p < \infty$,

$$\lim_{r\to\infty}\left[\sup_{n\in\mathbb{N}_r}\left(\sum_k|\widehat{a}_{nk}|^q\right)^{1/q}\right] = 0,$$

(iii) for $p = 1$

$$\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r,k\in\mathbb{N}}|\widehat{a}_{nk}|\right) = 0.$$

(b) If $A \in \left(\ell_p^{\lambda}(\widehat{F}) : \ell_1\right)$, *then* L_A *is compact if and only if*

(i) for $1 < p \leq \infty$,

$$\lim_{r\to\infty}\left(\sup_{\mathbb{N}_r}\left\|\sum_{n\in\mathbb{N}_r} E_n\right\|_q\right) = 0,$$

(ii) for $p = 1$,

$$\lim_{r\to\infty}\left(\sup_{k\in\mathbb{N}_0}\sum_{n=r}^{\infty}|\widehat{a}_{nk}|\right) = 0.$$

(c) If $A \in \left(\ell_p^{\lambda}(\widehat{F}) : c\right)$, *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{n\in\mathbb{N}_r}\|E_n - \widehat{\alpha}\|_q\right) = 0,$$

where $\widehat{\alpha} = (\widehat{\alpha}_k)_{k\in\mathbb{N}_0}$ *with* $\widehat{\alpha}_k = \lim_{n\to\infty}\widehat{a}_{nk}$ *for all* k *and the matrix* $E = (e_{nk})_{k,n\in\mathbb{N}}$ *is defined by (4.1).*

7. Conclusion

We should state that although the domains of the matrices Λ and $\widehat{F}$ in the classical sequence spaces ℓ_p and ℓ_∞ are investigated by Mursaleen and Noman [30], and Kara [18], since we employ the composition of the triangles Λ and $\widehat{F}$ the main results of the present paper are much more general than the corresponding results obtained by Mursaleen and Noman [30], and Kara [18]. It is worth mentioning here that in spite of the domain of the matrix E in the space ℓ_p of absolutely p-summable sequences has been studied in the present paper for the case $1 \leq p \leq \infty$, one can derive the similar results concerning the domain of the matrix E in the space ℓ_p for $0 < p < 1$ which are new and are also complementary of our contribution.

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