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A Note on Machine Method for Root Extraction*

S. Gadtia and S. K. Padhan

ABSTRACT: The present investigation deals with the critical study of the works of Lancaster and Traub, who have developed nth root extraction methods of a real number. It is found that their developed methods are equivalent and the particular cases of Halley's and Householder's methods. Again the methods presented by them are easily obtained from simple modifications of Newton's method, which is the extension of Heron's square root iteration formula. Further, the rate of convergency of their reported methods are studied.

Key Words: Root extraction, Iterative methods, Modified iterative methods, Convergency.

Contents

1. Introduction

Throughout this paper the root extraction is considered for the real number N. In about 1500 BC the Babylonians have devised a remarkable iterative algorithm (see Burton [2], Knill [6], Osler [9]) for the calculation of square roots which was later known as the famous Heron's square root iteration formula

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right),\ i = 0, 1, 2, ... \tag{1.1}$$

In 1740, Thomas Simpson described Newton's method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},\ i = 0, 1, 2, ... \tag{1.2}$$

as an iterative method for solving general nonlinear equations using calculus. Edmond Halley in 1694 published a paper in Latin (see Gander [3]), where he presented a new method to compute roots of a polynomial, which was known as Halley's method. That was

$$x_{i+1} = x_i - \frac{2f(x_i)f'(x_i)}{2\left(f'(x_i)\right)^2 - f(x_i)f''(x_i)},\ i = 0, 1, 2, ... \tag{1.3}$$

Later, an iterative method with cubic convergence was defined by Householder (see Noor *et al.* [8]) which was

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{\left(f'(x_i)\right)^2 f''(x_i)}{2\left(f'(x_i)\right)^3},\ i = 0, 1, 2, ... \tag{1.4}$$

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Though finding iterative formula for solving nonlinear equations was the problems of 1500 BC, still many researchers (Wall [13], Hernández and Romero [5], Noor *et al.* [8], Hafiz [4], Bhagat [1], Padhan *et al.* [10,11]) are working in this area as it has wide applications in science and engineering.

The aim of this paper is to focus on the works of Lancaster [7] and Traub [12]. The root extraction methods presented by them are proved by slight modification in Newton's method. Again it is verified that Newton's method is obtained from Heron's square root iteration formula. The root extraction methods in Lancaster [7] and Traub [12] are equivalent and the particular cases of Halley's and Householder's method. Moreover, convergency of the above methods are discussed.

2. Extension of Heron's Square Root Iteration Formula

Consider

$$\begin{aligned}
& x^2 = N \\
\Rightarrow\quad & x = \frac{N}{x} \\
\Rightarrow\quad & 2x = x + \frac{N}{x} \\
\Rightarrow\quad & x = \frac{1}{2}\left(x + \frac{N}{x}\right) \\
\Rightarrow\quad & x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right),\ i = 0, 1, 2, ... \qquad (2.1)
\end{aligned}$$

which is the famous Heron's iteration formula for the square root of the real number N.
Taking $x^3 = N$, this formula can be extended for the cube root extraction of the real number N as

$$x_{i+1} = \frac{1}{3}\left(2x_i + \frac{N}{x_i^2}\right),\ i = 0, 1, 2, ... \qquad (2.2)$$

In a similar manner, let us consider

$$\begin{aligned}
& x^n = N \\
\Rightarrow\quad & x = \frac{N}{x^{n-1}} \\
\Rightarrow\quad & nx = (n-1)x + \frac{N}{x^{n-1}} \\
\Rightarrow\quad & x = \frac{1}{n}\left\{(n-1)x + \frac{N}{x^{n-1}}\right\} \\
\Rightarrow\quad & x_{i+1} = \frac{1}{n}\left\{(n-1)x_i + \frac{N}{x_i^{n-1}}\right\},\ i = 0, 1, 2, ... \qquad (2.3)
\end{aligned}$$

which is the well known Newton's iteration formula for the nth root of the real number N.

Remark 2.1. *Eqn. (2.3) shows that Newton's formula is an extension of Heron's square root iteration formula.*

3. Iterative Methods and Convergence Analysis

A connection between Newton's formula, Halley's method, Householder's method with the methods of Lancaster [7] and Traub [12] is described.

3.1. Modification of Newton's Formula

From eqn. (2.3), we know that Newton's formula for cube root extraction is

$$\begin{aligned} x_{i+1} &= \frac{1}{3}\left(2x_i + \frac{N}{x_i^2}\right), \ i = 0, 1, 2, ... \\ \Rightarrow \quad x_{i+1} &= \frac{2x_i^3 + N}{3x_i^2} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i(2x_i^3 + N)}{3x_i^3} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i(x_i^3 + x_i^3 + N)}{2x_i^3 + x_i^3} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i(x_i^3 + 2N)}{2x_i^3 + N}, \end{aligned}$$

which is the machine method for cube root extraction as proposed by Lancaster [7].

The Newton's formula for cube root extraction in general can be expressed as

$$x_{i+1} = \frac{x_i(2x_i^3 + 4N)}{4x_i^3 + 2N}, \ i = 0, 1, 2, ...$$

Let us try to generalize it in the following way. We know that Newton's formula for nth root extraction is

$$\begin{aligned} x_{i+1} &= \frac{1}{n}\left\{(n-1)x_i + \frac{N}{x_i^{n-1}}\right\}, \ i = 0, 1, 2, ... \\ \Rightarrow \quad x_{i+1} &= \frac{(n-1)x_i^n + N}{nx_i^{n-1}} \\ \Rightarrow \quad x_{i+1} &= \frac{2x_i[(n-1)x_i^n + N]}{2nx_i^n} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i[2(n-1)x_i^n + 2N]}{2nx_i^n} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i[(n-1)x_i^n + (n-1)x_i^n + 2N]}{(n+1)x_i^n + (n-1)x_i^n} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i[(n-1)x_i^n + (n-1)N + 2N]}{(n+1)x_i^n + (n-1)N} \\ \Rightarrow \quad x_{i+1} &= \frac{x_i[(n-1)x_i^n + (n+1)N]}{(n+1)x_i^n + (n-1)N}, \end{aligned}$$

which is the machine method for nth root extraction as proposed by Lancaster [7] and Traub [12].

Remark 3.1. From the above discussions, one can easily observe that the methods proposed by Lancaster [7] and Traub [12] are obtained from Newton's method.

3.2. Comparison with Halley's Method

Let us consider the Halley's method for cube root extraction. In this case

$$\begin{aligned} &f(x) = x^3 - N, f'(x) = 3x^2, f''(x) = 6x \\ &x = x - \frac{2(x^3 - N)3x^2}{2(9x^4) - (x^3 - N)6x} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad x = x - \frac{6x^5 - 6Nx^2}{18x^4 - 6x^4 + 6Nx} \\
&\Rightarrow \quad x = \frac{6x^5 + 12Nx^2}{12x^4 + 6Nx} \\
&\Rightarrow \quad x = \frac{x(x^3 + 2N)}{2x^3 + N} \\
&\Rightarrow \quad x_{i+1} = \frac{x_i(x_i^3 + 2N)}{2x_i^3 + N},
\end{aligned}$$

which is the machine method for cube root extraction as proposed by Lancaster [7].

Remark 3.2. When $n = 3$, Halley's formula for nth root extraction becomes the cube root extraction formula of Lancaster [7].

Again consider the Halley's method for nth root extraction, in this case

$$\begin{aligned}
& f(x) = x^n - N, f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2} \\
& x = x - \frac{2(x^n - N)nx^{n-1}}{2\left(nx^{n-1}\right)^2 - (x^n - N)n(n-1)x^{n-2}} \\
\Rightarrow \quad & x = \frac{(n^2 - n)x^{2n-1} + (Nn^2 + Nn)x^{n-1}}{(n^2 + n)x^{2n-2} + (Nn^2 - Nn)x^{n-2}} \\
\Rightarrow \quad & x = \frac{n(n-1)x^{2n-1} + Nn(n+1)x^{n-1}}{n(n+1)x^{2n-2} + Nn(n-1)x^{n-2}} \\
\Rightarrow \quad & x = \frac{nx^{n-1}\{(n-1)x^n + (n+1)N\}}{nx^{n-2}\{(n+1)x^n + (n-1)N\}} \\
\Rightarrow \quad & x = \frac{x\{(n-1)x^n + (n+1)N\}}{(n+1)x^n + (n-1)N} \\
\Rightarrow \quad & x_{i+1} = \frac{x_i\{(n-1)x_i^n + (n+1)N\}}{(n+1)x_i^n + (n-1)N},
\end{aligned}$$

which is the machine method for nth root extraction as proposed by Lancaster [7] and Traub [12].

Remark 3.3. The nth root extraction methods in Lancaster [7] and Traub [12] are the particular cases of Halley's formula.

3.3. Comparison with Householder's Method

Let us discuss the above proposed methods using Householder's method as below

For cube root extraction

$$\begin{aligned}
& f(x) = x^3 - N, f'(x) = 3x^2, f''(x) = 6x \\
& x = x - \frac{x^3 - N}{3x^2} - \frac{(x^3 - N)^2 6x}{2(3x^2)^3} \\
\Rightarrow & x = \frac{30x^7 + 30Nx^4 - 6N^2x}{54x^6} \\
\Rightarrow & x = \frac{6x(5x^6 + 5Nx^3 - N^2)}{54x^6} \\
\Rightarrow & x = \frac{x(5Nx^3 + 5Nx^3 - N^2)}{9Nx^3}
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow x = \frac{x(10x^3 - N)}{9x^3} \\
&\Rightarrow x = \frac{x(3x^3 + 6N)}{6x^3 + 3N} \\
&\Rightarrow x = \frac{x(x^3 + 2N)}{2x^3 + N} \\
&\Rightarrow x_{i+1} = \frac{x_i(x_i^3 + 2N)}{2x_i^3 + N},
\end{aligned}
$$

which is the machine method for cube root extraction as proposed by Lancaster [7].

For nth root extraction

$$
\begin{aligned}
&f(x) = x^n - N, f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2} \\
&x = x - \frac{x^n - N}{nx^{n-1}} - \frac{(x^n - N)^2 n(n-1)x^{n-2}}{2(nx^{n-1})^3} \\
\Rightarrow\quad &x = \frac{n(n-1)(2n-1)x^{3n-2} + 2Nn(2n-1)x^{2n-2} - N^2 n(n-1)x^{n-2}}{2n^3 x^{3n-3}} \\
\Rightarrow\quad &x = \frac{nx^{n-2}\left\{(n-1)(2n-1)x^{2n} + 2N(2n-1)x^n - N^2(n-1)\right\}}{2n^3 x^{3n-3}}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad &x = \frac{N\left\{(n-1)(2n-1)N + 2(2n-1)N - N(n-1)\right\}}{2n^2 x^{n-1} x^n} \\
\Rightarrow\quad &x = \frac{x(2n^2 N)}{2n^2 x^n} \\
\Rightarrow\quad &x = \frac{x(2nN)}{2nx^n} \\
\Rightarrow\quad &x = \frac{x\{(n-1)x^n + (n+1)N\}}{(n+1)x^n + (n-1)N} \\
\Rightarrow\quad &x_{i+1} = \frac{x_i\{(n-1)x_i^n + (n+1)N\}}{(n+1)x_i^n + (n-1)N},
\end{aligned}
$$

which is the machine method for nth root extraction as proposed by Lancaster [7] and Traub [12].

Remark 3.4. The root extraction formulae in Lancaster [7] and Traub [12] are obtained from Householder's method as particular cases.

3.4. Convergency Test

The recurrence relation for the nth root extraction of a real number N is

$$
x_{i+1} = \frac{x_i\{(n-1)x_i^n + (n+1)N\}}{(n+1)x_i^n + (n-1)N},\ i = 0, 1, 2, ... \tag{3.1}
$$

From eqn. (3.1) we have

$$
\Phi(x) = \frac{x\{(n-1)x^n + (n+1)N\}}{(n+1)x^n + (n-1)N}.
$$

One can easily check that $\Phi(x) = x$, $\Phi'(x) = 0$, $\Phi''(x) = 0$ but $\Phi'''(x) \neq 0$. Hence this method converges cubically.

Remark 3.5. The recurrence relation for square root extraction is expressed as

$$
x_{i+1} = \frac{x_i(x_i^2 + 3N)}{3x_i^2 + N},\ i = 0, 1, 2, ...
$$

which has cubic convergence.

4. Concluding Remarks

(i) Newton's method is an extension of Heron's square root iteration formula.

(ii) The methods proposed by Lancaster [7] and Traub [12] can be obtained by modifying Newton's method.

(iii) The above machine methods are the particular cases of Halley's and Householder's methods.

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S. Gadtia,
Department of Mathematics,
Veer Surendra Sai University of Technology, Burla - 768018, India
Department of Mathematics,
Agalpur Panchayat Samiti College, Roth, Balangir - 767061, India.
E-mail address: subigyan101@gmail.com

and

S. K. Padhan,
Department of Mathematics,
Veer Surendra Sai University of Technology, Burla - 768018
India.
E-mail address: skpadhan_math@vssut.ac.in