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A Three-term Recurrence Formula for the Generalized Bernoulli Polynomials

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ABSTRACT: In the present paper, we propose some new explicit formulas of the higher order Daehee polynomials in terms of the generalized r-Stirling and r-Whitney numbers of the second kind. As a consequence, we derive a three-term recurrence formula for the calculation of the generalized Bernoulli polynomials of order k.

Key Words: Bernoulli polyomials, Daehee polynomials, Stirling numbers.

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1. Introduction

The Bernoulli polynomials of order $\alpha \in \mathbb{C}$, are defined by the following generating function:

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\,\frac{z^n}{n!}.$$

When $x=0$, $B_n^{(\alpha)}(0) := B_n^{(\alpha)}$ are called the Bernoulli numbers of order α. Clearly, we have

$$B_n^{(\alpha)}(x) = \sum_{i=0}^{n} \binom{n}{i} B_i^{(\alpha)} x^{n-i}. \tag{1.1}$$

Employing the weighted Stirling numbers of the second kind $\mathcal{S}_n^k(x)$, Boutiche et *al.* [1] proved the following explicit formula for $B_n^{(\alpha)}(x)$:

$$B_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+m}{m}^{-1} \binom{n+\alpha}{n-m} \binom{\alpha+m-1}{m} \mathcal{S}_{n+m}^m(x). \tag{1.2}$$

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The classical Bernoulli polynomials $B_n(x)$ are given by

$$B_n(x) := B_n^{(1)}(x)$$

and can be computed by constructing an infinite matrix $(B_{n,p}(x))_{n,p\geq 0}$ as follows [9]: the first row of the matrix $B_{0,p}(x) = 1$, and each entry $B_{n,p}(x)$ is given recursively by

$$B_{n+1,p}(x) = (x+p)\,B_{n,p}(x) - \frac{(p+1)^2}{p+2}B_{n,p}(x)\,. \tag{1.3}$$

The first column of the matrix $B_{n,0}(x) = B_n(x)$ are the Bernoulli polynomials.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ x-\frac{1}{2} & x-\frac{1}{3} & x-\frac{1}{4} & x-\frac{1}{5} & \cdots & \\ x^2-x+\frac{1}{6} & x^2-\frac{2}{3}x & x^2-\frac{1}{2}x-\frac{1}{20} & \vdots & & \\ x^3-\frac{3}{2}x^2+\frac{x}{2} & x^3-x^2+\frac{1}{15} & \vdots & & & \\ x^4-2x^3+x^2-\frac{1}{30} & \vdots & & & & \\ \vdots & & & & & \end{pmatrix}$$

In this paper, we are interested in the three-term recurrence formula for the generalized Bernoulli polynomials of order $k \geq 1$. The paper is organized as follows. In the next section, we recall some basic definitions and some results that will be useful in the rest of the paper. Furthermore, we presents some explicit formulas for the higher order Daehee polynomials. A three-term recurrence relation for calculating the generalized Bernoulli numbers and polynomials of order k are given in section 3.

2. The Daehee polynomials of order α

Recently, Kim et *al.* in [5] defined the higher order Daehee polynomials $D_n^{(k)}(x)$ $(k \geq 1)$ by the following generating function:

$$\left(\frac{\log(1+z)}{z}\right)^k (1+z)^x = \sum_{n\geq 0} D_n^{(k)}(x)\,\frac{z^n}{n!}, \tag{2.1}$$

and proved that

$$D_n^{(k)}(x) = \sum_{i=0}^{n} s(n,i)\,B_i^{(k)}(x) \tag{2.2}$$

$$= B_n^{(n+k+1)}(x+1) \tag{2.3}$$

where $s(n,m)$ denotes the signed Stirling numbers of the first kind.

Recall that the Stirling numbers of the first kind are the coefficients in the following expansion:

$$(x)_n = \sum_{m=0}^{n} s(n,m)\, x^m,$$

where $(\lambda)_n$ denotes the falling factorial, defined by

$$(\lambda)_n = \lambda(\lambda-1)\cdots(\lambda-n+1) \text{ for } n>0,$$

with the convention $(\lambda)_0 = 1$.

These numbers satisfy the recurrence relation given by

$$s(n+1,m) = s(n,m-1) - ns(n,m) \qquad (1 \le m \le n), \tag{2.4}$$

Now, for $x=0$ in (2.1), $D_n^{(k)} := D_n^{(k)}(0)$ are called the Daehee numbers of order k and given explicitly by

$$D_n^{(k)} = \binom{n+k}{k}^{-1} s(n+k,k).$$

The weighted Stirling numbers of the second kind $\mathcal{S}_n^p(x)$ is defined by (see [3,4])

$$\mathcal{S}_n^p(x) = \frac{1}{p!}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}(x+j)^n,$$

or by the exponential generating function

$$\sum_{n\ge p}\mathcal{S}_n^p(x)\frac{z^n}{n!} = \frac{1}{p!}e^{xz}(e^z-1)^p. \tag{2.5}$$

For appropriate choices of x, the weighted Stirling numbers of the second kind reduce to some special combinatorial sequences. The Stirling numbers of the second kind, denoted by $S(n,p) = \mathcal{S}_n^p(0)$, are the coefficients in the following expansion:

$$x^n = \sum_{p=0}^{n} p!\binom{x}{p} S(n,p).$$

The Stirling numbers of the second kind $S(n,p)$ count the number of ways to partition a set of n elements into exactly p nonempty subsets.

For any nonnegative integer r, the r-Stirling numbers of the second kind

$$S_r(n+r,p+r) = \mathcal{S}_n^p(r),$$

are obviously a generalization of the Stirling numbers of the second kind [2]. $S_r(n,p)$ count the number of partitions of a set of n objects into exactly p nonempty and disjoint subsets such that the first r elements are in distinct subsets.

Now, by the Stirling transform [7], one can generalize (2.2) as:

Proposition 2.1. *For $n, m \geq 0$, we have*

$$\sum_{i=0}^{m} s(m,i) B_{n+i}^{(k)}(x) = \sum_{i=0}^{n} S_m(n+m, i+m) D_{m+i}^{(k)}(x).$$

For any positive integer m, The r-Whitney numbers of the second kind

$$W_{m,r}(n,p) = m^{n-p} \mathcal{S}_n^p \left(\frac{r}{m}\right),$$

are the coefficients in the following expansion [6,8]:

$$(mx+r)^n = \sum_{p=0}^{n} m^k W_{m,r}(n,p) (x)_p$$

and are given by their generating function as follows:

$$\sum_{n=p}^{\infty} W_{m,r}(n,p) \frac{z^n}{n!} = \frac{1}{m^p p!} e^{rz} \left(e^{mz} - 1\right)^p.$$

Clearly, we have

$$W_{1,0}(n,p) = S(n,p) \qquad \text{and} \qquad W_{1,r}(n,p) = S_r(n+r, p+r).$$

An explicit formula for $D_n^{(k)}(x)$ in terms of weighted Stirling numbers of the second kind is easily derived from (1.2) and (2.3).

Since the generalized binomial coefficient is defined as

$$\binom{x}{n} = \begin{cases} \frac{(x)_n}{n!}, & \text{if } n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then, the $D_n^{(k)}(x)$ can be naturally extended to the Daehee polynomials $D_n^{(\alpha)}(x)$ of order α, which can be defined by

$$D_n^{(\alpha)}(x) = B_n^{(n+\alpha+1)}(x+1).$$

Theorem 2.1. *The following relationship holds true:*

$$D_n^{(\alpha)}(x) = \sum_{i=0}^{n} (-1)^i \binom{n+i}{i}^{-1} \binom{2n+\alpha+1}{n-i} \binom{n+\alpha+i}{i} \mathcal{S}_{n+i}^i (x+1). \tag{2.6}$$

In particular, we obtain an explicit representation for the Daehee polynomials of order α at rational arguments involving the r-Whitney numbers of the second kind.

$$D_n^{(\alpha)}\left(\frac{r}{m}\right) = \sum_{i=0}^{n} (-1)^i \frac{\binom{2n+\alpha+1}{n-i}\binom{n+\alpha+i}{i}}{m^n \binom{n+i}{i}} W_{m,r+m}(n+i, i).$$

If we set $x = r$ and $m = 1$ in (2.6), we get

$$D_n^{(\alpha)}(r) = \sum_{i=0}^{n} \frac{(-1)^i \binom{2n+\alpha+1}{n-i}\binom{n+\alpha+i}{i}}{\binom{n+i}{i}} S_{r+1}(n+i+r+1, i+r+1).$$

3. Recurrence relation for the generalized Bernoulli polynomials of order k

In the next Theorem, we propose an algorithm which is based on three-term recurrence relation for calculating the generalized Bernoulli numbers of order k.

It is convenient to introduce the following sequence $\mathfrak{B}_{n,m}^{(k)}$ with two indices by

$$\mathfrak{B}_{n,m} := \mathfrak{B}_{n,m}^{(k)} = \frac{\binom{m+k}{k}}{s(m+k,k)} \sum_{i=0}^{n} S_m(n+m, i+m) D_{m+i}^{(k)} \tag{3.1}$$

with $\mathfrak{B}_{0,m} = 1$ and $\mathfrak{B}_{n,0} = B_n^{(k)}$.

Theorem 3.1. *The $\mathfrak{B}_{n,m}^{(k)}$ satisfies the following three-term recurrence relation:*

$$\mathfrak{B}_{n+1,m} = \frac{m+1}{m+k+1} \frac{s(m+k+1,k)}{s(m+k,k)} \mathfrak{B}_{n,m+1} + m\mathfrak{B}_{n,m} \tag{3.2}$$

with the initial sequence given by

$$\mathfrak{B}_{0,m} = 1.$$

Proof: From Proposition 2.1 and (2.4), we have

$$\mathfrak{B}_{n,m+1} = \frac{\binom{m+k+1}{k}}{s(m+k+1,k)} \sum_{i=0}^{m+1} (s(m,i-1) - ms(m,i)) B_{n+i}^{(k)}.$$

After some rearrangement, we find that

$$\mathfrak{B}_{n,m+1} = \frac{m+k+1}{m+1} \frac{s(m+k,k)}{s(m+k+1,k)} (\mathfrak{B}_{n+1,m} - m\mathfrak{B}_{n,m}).$$

This evidently completes the proof of Theorem 3.1. □

Finally, we consider the polynomials $\mathfrak{B}_{n,m}^{(k)}(x)$ defined by

$$\mathfrak{B}_{n,m}(x) := \mathfrak{B}_{n,m}^{(k)}(x) = \sum_{i=0}^{n} \binom{n}{i} \mathfrak{B}_{i,m}^{(k)} x^{n-i}.$$

It is obviously from (1.1) that $\mathfrak{B}_{0,m}^{(k)}(x) = 1$ and $\mathfrak{B}_{n,0}^{(k)}(x) = B_n^{(k)}(x)$.

Theorem 3.2. *The polynomials $\mathfrak{B}_{n,m}^{(k)}(x)$ satisfy the following three-term recurrence relation:*

$$\mathfrak{B}_{n+1,m}(x) = \frac{m+1}{m+k+1}\frac{s(m+k+1,k)}{s(m+k,k)}\mathfrak{B}_{n,m+1}(x) + (m+x)\mathfrak{B}_{n,m}(x). \quad (3.3)$$

Proof: It is readily seen that

$$x\frac{d}{dx}\mathfrak{B}_{n,m}(x) = n\sum_{i=0}^{n}\binom{n}{i}\mathfrak{B}_{i,m}^{(k)}x^{n-i} - n\sum_{i=0}^{n-1}\binom{n-1}{i}\mathfrak{B}_{i+1,m}^{(\alpha)}x^{n-i-1}.$$

By using (3.2), we obtain

$$x\frac{d}{dx}\mathfrak{B}_{n,m}(x) = n\mathfrak{B}_{n,m}(x) - n\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-i-1}\left(\frac{m+1}{m+k+1}\frac{s(m+k+1,k)}{s(m+k,k)}\mathfrak{B}_{i,m+1} + m\mathfrak{B}_{i,m}\right).$$

After some manipulations, we find (3.3). □

Now, for $k = 1$ and from $s(n,1) = (-1)^{n+1}(n-1)!$, we get (1.3).

4. Conclusion

In our present research, we have investigated higher order Daehee polynomials. We have derived some explicit formulas for these polynomials. We have also given a recursive method for the calculation of the generalized Bernoulli polynomials of order k.

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