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Existence and Decay of Solution to Coupled System of Viscoelastic Wave Equations with Strong Damping in $\mathbb{R}^n$

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ABSTRACT: In this paper, we establish a general decay rate properties of solutions for a coupled system of viscoelastic wave equations in $\mathbb{R}^n$ under some assumptions on g_1, g_2 and linear forcing terms. We exploit a density function to introduce weighted spaces for solutions and using an appropriate perturbed energy method. The questions of global existence in the nonlinear cases is also proved in Sobolev spaces using the well known Galerkin method.

Key Words: Perturbed energy, Viscoelastic, Density, Nonlinear forcing, Decay rate, Weighted spaces, Strong damping.

Contents

1. Introduction and previous results

In this paper, we consider the following problem:

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \alpha u_2 - \phi(x)\Delta\left(u_1 - \int_0^t g_1(t-s)u_1(s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + \alpha u_1 - \phi(x)\Delta\left(u_2 - \int_0^t g_2(t-s)u_2(s,x)ds + u_2'\right) = 0, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{1.1}$$

where $\alpha \neq 0, x \in \mathbb{R}^n, t \in \mathbb{R}^+_*$ where the space $D(\mathbb{R}^n)$ defined in (2.4) and $l, n \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in $(A2)$.
This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics, it was first considered by Dafermos in [6], where the general decay was discussed. The problems related to (1.1) attract a great deal of attention in the last decades and numerous results appeared on the existence

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and long time behavior of solutions but their results are by now rather developed, especially in any space dimension when it comes to nonlinear problems. The term $\int_0^t g_i(t-s)\Delta u_i(s)ds$ corresponds to the memories terms and the scalar functions $g_i(t)$ (so-called relaxation kernel) is assumed to satisfy (2.1)-(2.3). The energy of (u_1, u_2) at time t is defined by

$$\begin{aligned} E(t) &= \frac{(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|\nabla u_i\|_2^2 \\ &+ \frac{1}{2}\sum_{i=1}^{2}(g_i\circ\nabla u_i) + \alpha\int_{\mathbb{R}^n}\rho u_1 u_2 dx. \end{aligned} \tag{1.2}$$

For α small enough we use Lemma 2.3, we deduce that:

$$\begin{aligned} E(t) &\geq \frac{1}{2}(1-c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\Big[\frac{2(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho} \\ &+\sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|\nabla u_i\|_2^2 + \sum_{i=1}^{2}(g_i\circ\nabla u_i)\Big], \end{aligned} \tag{1.3}$$

and the following energy functional law holds

$$E'(t) \leq \frac{1}{2}\sum_{i=1}^{2}(g_i'\circ\nabla u_i)(t) - \sum_{i=1}^{2}\|\nabla u_i'\|_2^2, \forall t\geq 0. \tag{1.4}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories.
The following notation will be used throughout this paper

$$(g\circ\Psi)(t) = \int_0^t g(t-\tau)\left\|\Psi(t)-\Psi(\tau)\right\|_2^2 d\tau, \text{ for any } \Psi\in L^\infty(0,T;L^2(\mathbb{R}^n)) \tag{1.5}$$

In the present paper we consider the solutions in an appropriate spaces weighted by the density function $\rho(x)$ in order to compensate the lack of Poincare's inequality which play a decisive role in the proof. To motivate our work, we start with some results related to viscoelastic plate equations with strong damping in [23]:

$$u_{tt} + \Delta^2 u - \Delta_p u - \int_0^t g(t-s)\Delta u(s,x)ds - \Delta u_t + f(u) = 0, \quad x\in\Omega\times\mathbb{R}^+,$$

supplemented with the following conditions:

$$u(t,x) = \Delta u = 0, \text{ on } \partial\Omega\times\mathbb{R}^+, \quad u(0,x) = u_0, u_t(0,t) = u_1, \text{ on } \Omega. \tag{1.6}$$

In this paper, Liu and *all* extend the exponential rate result obtained in [1] to the general case and show that the rate of decay for the solution is similar to that of the memory term under the following assumption for the function g is

$$g'(t) \leq -\xi(t)g(t), \quad \text{where } \xi(t) \text{ satisfies } \xi'(t)\leq 0, \quad \int_0^t \xi(t)dt = \infty.$$

Paper [8] is concerned with a class of plate equations with memory in a history space setting and perturbations of $p-$Laplacian type

$$u_{tt} + \alpha\Delta^2 u - \Delta_p u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s,x)ds - \Delta u_t + f(u) = h,, \tag{1.7}$$

for $x \in \Omega \times \mathbb{R}^+$, and results on the well-posedness and asymptotic stability of the problem were proved.
In many existing works on this field, the following conditions on the kernel

$$g'(t) \geq -\lambda g^p(t), \quad t \geq 0, p \geq 0, \tag{1.8}$$

is crucial in the proof of the stability. For a viscoelastic systems with oscillating kernels, we mention the work by Rivera and *all* [17], the authors proved that if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, that is for $p = 1$ in (1.8), then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e. $(p > 1)$ in the inequality (1.8), then the solution also decays polynomially with the same rate of decay. Recently the problem related to (1.1) in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered as equation in [15], where they established an explicit and very general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0,$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.
For the literature, In $\mathbb{R}^n$, we quote essentially the results of [2], [3], [4], [9]- [13], [15]- [20] and the references therein. In [10], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (1.1) without strong damping in the case $l = 2, \rho(x) = 1$, is polynomial. The finite-speed propagation is used to compensate the lack of Poincare's inequality. In the case $l = 2$, in [9], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate the lack of Poincare's inequality in the absence of strong damping. The same problem treated in [9], was considered in [11], where under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. The conditions which used, on the relaxation function g and its derivative g' are different from the usual ones. Coupled systems in $\mathbb{R}^n$, we mention, for instance, the work of *[Takashi Narazaki, 2009. Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations. Discrete And Continuous Dynamical Systems, 592-601]*, where the following weakly coupled system of a damped wave equations

has considered:

$$\begin{cases} u'' - \Delta u + u' = f(v), \quad t > 0, x \in \mathbb{R}^n, \\ v'' - \Delta v + v' = f(u), \quad t > 0, x \in \mathbb{R}^n, \\ (u(0,x), v(0,x)) = (\phi_0(x), \psi_0(x)), \quad x \in \mathbb{R}^n, \\ (u'(0,x), v'(0,x)) = (\phi_1(x), \psi_1(x)), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.9}$$

Authors have shown the sufficient condition under which the Cauchy problem (1.9) admits global solutions when $n = 1, 2, 3$ provided that the initial data are sufficiently small in an associate space. Moreover, they have also shown the asymptotic behavior of the solutions and to generalize the existence result in [22] to the case $n = 1, 2, 3$ and improve time decay estimates when $n = 3$.

2. Function spaces and statements

In this section we introduce some notation and results needed for our work. We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denotes $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real-valued. Here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. We denote by B_R the open ball of $\mathbb{R}^n$ with center 0 and radius R.
First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:
$(A1)$ We assume that the function $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$(for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \int_0^\infty g_i(t)dt \geq k_i > 0, g_i(0) = g_{i0} > 0, \tag{2.1}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\xi'(t) \leq 0, \quad \forall t > 0, \quad \int_0^\infty \xi(t) = \infty, \quad \xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \tag{2.2}$$

where

$$g_i'(t) + \xi(t)g_i(t) \leq 0. \tag{2.3}$$

$(A2)$ The function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n - qn + 2q}$.

Definition 2.1 ([9], [19])**.** *We define the function spaces of our problem and its norm as follows:*

$$D(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla f \in (L^2(\mathbb{R}^n))^n \right\}, \tag{2.4}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product:

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{2.5}$$

For $1 < l < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^l_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^l dx \right)^{1/l}. \tag{2.6}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.
So, we are able to construct the necessary *evolution triple* for the space setting of our problem, which is:

$$D(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1}(\mathbb{R}^n), \tag{2.7}$$

where all the embedding are compact and dense.
The following technical Lemma will play an important role in the sequel.

Lemma 2.2. *[5] (Lemma1.1) For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$\begin{aligned} v'(t) \int_0^t g(t-s)v(s)ds \quad &= \quad -\frac{1}{2}\frac{d}{dt} \int_0^t g(t-s)|v(t)-v(s)|^2 ds \\ &+\frac{1}{2}\frac{d}{dt} \left(\int_0^t g(s)ds \right) |v(t)|^2 \\ &+\frac{1}{2} \int_0^t g'(t-s)|v(t)-v(s)|^2 ds \\ &-\frac{1}{2} g(t)|v(t)|^2. \end{aligned}$$

and

$$\left| \int_0^t g(t-s)(v(t)-v(s))ds \right|^2 \leq \left(\int_0^t |g(s)|^{2(1-\theta)} ds \right) \int_0^t |g(t-s)|^{2\theta} |v(t)-v(s)|^2 ds.$$

Lemma 2.3. *[4] Let ρ satisfies (A2), then for any $u \in D(\nabla)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)}, \tag{2.8}$$

with,

$$s = \frac{2n}{2n - qn + 2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

Corollary 2.4. *If $q = 2$, the Lemma 2.3. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)},$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = c > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^2(\mathbb{R}^n)}. \tag{2.9}$$

To study the properties of the operator $\phi\Delta$, we consider as in [13], the equation

$$\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^n, \tag{2.10}$$

without boundary conditions. Since for every u, v in $C_0^\infty(\mathbb{R}^n)$

$$(\phi\Delta u, v)_{L^2_\rho} = \int_{\mathbb{R}^n} \nabla u \nabla v dx, \tag{2.11}$$

and $L^2_\rho(\mathbb{R}^n)$ are defined with respect to the inner product (2.5), we may consider equation (2.10) as operator equation:

$$\Delta_0 u = \eta, \quad \Delta_0 : D(\Delta_0) \subseteq L^2_\rho(\mathbb{R}^n) \to L^2_\rho(\mathbb{R}^n), \quad \eta \in L^2_\rho(\mathbb{R}^n).$$

The relations (2.11) implies that the operators $\phi\Delta$ with domain of definition $D(\Delta_0) = C_0^\infty(\mathbb{R}^n)$ being symmetric. Let us note that the operator $\phi\Delta$ is not symmetric in the standard Lebesgue space $L^2(\mathbb{R}^n)$, because of the appearance of $\phi(x)$ (see [[21], pages 185-187]). From (2.9) and (2.11) we have

$$\|u\|_{L^2_\rho} \leq c(\Delta_0 u, u)_{L^2_\rho}, \quad \text{for all } u \in D(\Delta_0). \tag{2.12}$$

From (2.11) and (2.12) we conclude that Δ_0 is a symmetric, strongly monotone operator on $L^2_\rho(\mathbb{R}^n)$. The energy scalar product is given by:

$$(u, v)_E = \int_{\mathbb{R}^n} \nabla u \nabla v dx,$$

and the energy space is the completion of $D(\Delta_0)$ with respect to $(u, v)_E$. It is obvious that the energy space X_E is the homogeneous Sobolev space $D(\mathbb{R}^n)$. The energy extension Δ_E, namely

$$\phi\Delta : D(\mathbb{R}^n) \to D^{-1}(\mathbb{R}^n),$$

is defined to be the duality mapping of $D(\mathbb{R}^n)$. For every $\eta \in D^{-1}(\mathbb{R}^n)$ the equation (2.10), has a unique solution. Define $D(\Delta_1)$ to be the set of all solutions of the equation (2.10) for arbitrary $\eta \in L^2_\rho(\mathbb{R}^n)$. The operator extension Δ_1 of Δ_0, [see [24], Theorem 19.C] is the restriction of the energy extension Δ_E to the set $D(\Delta_1)$. The operator Δ_1 is self-adjoint and therefore graph-closed. Its domain is a Hilbert space with respect to the graph scalar product

$$(u, v)_{D(\Delta_1)} = (u, v)_{L^2_\rho} + (\Delta_1 u, \Delta_1 v)_{L^2_\rho}, \quad \text{for all } u, v \in D(\Delta_1).$$

The norm induced by the scalar product $(u, v)_{D(\Delta_1)}$ is

$$\|u\|_{D(\Delta_1)} = \left\{ \int_{\mathbb{R}^n} \rho |u|^2 dx + \int_{\mathbb{R}^n} \phi |\Delta u|^2 dx \right\}^{\frac{1}{2}}.$$

which is equivalent to the norm

$$\|\Delta_1 u\|_{L^2_\rho} = \left\{ \int_{\mathbb{R}^n} \phi |\Delta u|^2 dx \right\}^{\frac{1}{2}}.$$

So, we have established the *evolution quartet*

$$D(\Delta_1) \subset D(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1}(\mathbb{R}^n), \tag{2.13}$$

where all the embedding are dense and compact. A consequence of the compactness of the embedding in (2.13) is that the eigenvalue problem

$$-\Delta u = \mu u, x \in \mathbb{R}^n, \tag{2.14}$$

has a complete system of eigenfunctions $\{w_n, \mu_n\}$ with the following properties:

$$\begin{cases} -\Delta w_j = \mu w_j, & j = 1, 2\cdots, \quad w_j \in D(\mathbb{R}^n), \\ 0 < \mu_1 \leq \mu_2 \leq \cdots, & \mu_j \to \infty, \qquad as \quad j \to \infty. \end{cases} \tag{2.15}$$

It can be shown, as in [4], that every solution of (2.14) is such that

$$u(x) \longrightarrow 0, \quad as \quad |x| \longrightarrow \infty, \tag{2.16}$$

uniformly with respect to x. Finally, we give the definition of *weak solutions* for the problem (1.1).

Definition 2.5. *A weak solution of (1.1) is (u_1, u_2) such that*

- $(u_1, u_2) \in (L^2[0, T; D(\mathbb{R}^n)])^2$, $\quad (u_1', u_2') \in (L^2[0, T; L^l_\rho(\mathbb{R}^n)])^2$ *and* $(u_1'', u_2'') \in (L^2[0, T; D^{-1}(\mathbb{R}^n)])^2$,
- *For all* $(v, w) \in (C_0^\infty([0, T] \times \mathbb{R}^n))^2$, (u_1, u_2) *satisfies the generalized formula:*

$$\begin{cases} \int_0^T \left(\left(|u_1'|^{l-2} u_1' \right)', v \right)_{L^l_\rho} ds + \alpha \int_0^T (u_2, v)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} \nabla u_1 \nabla v dx ds \\ + \int_0^T \int_{\mathbb{R}^n} \nabla u_1' \nabla v dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_1(s-\tau) \nabla u_1(\tau) d\tau \nabla v(s) dx ds = 0, \\ \int_0^T \left(\left(|u_2'|^{l-2} u_2' \right)', w \right)_{L^l_\rho} ds + \alpha \int_0^T (u_1, w)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} \nabla u_2 \nabla w dx ds \\ + \int_0^T \int_{\mathbb{R}^n} \nabla u_2' \nabla w dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_2(s-\tau) \nabla u_2(\tau) d\tau \nabla w(s) dx ds = 0. \end{cases}$$

- (u_1, u_2) *satisfies the initial conditions* $(u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2$, $\quad (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2$.

We are now ready to state and prove our existence results.

3. Well-posedness result for nonlinear case

This section is devoted to prove the existence and uniqueness of solutions to the system (1.1) taking account the nonlinear case in the terms responsible on the relation between tow equations, that is replacing $\alpha u_1, \alpha u_2$ by $f_1(u_1, u_2), f_2(u_1, u_2)$ introduced in the last section. First, we prove the existence of the unique solution of the restricted problem on B_R, the main ingredient used here is the Galerkin approximations introduced in [14].

Lemma 3.1. *Assume that (A1), (A2), (5.2)-(5.6) are satisfied. Suppose that the constants* $T>0,\ R>0$ *and the initial conditions*

$$(u_{10},u_{20})\in (D(B_R))^2,(u_{11},u_{21})\in (L^l_\rho(B_R))^2,$$

are given. Then there exists a unique solution for the problem (1.1) such that

$$u_i\in C[0,T;D(B_R)] \quad and \quad u_i'\in C[0,T;L^l_\rho(B_R)].$$

Proof: The existence is proved by using the Galerkin method, which consists in constructing approximations of the solution, then we obtain a priori estimates necessary to guarantee the convergence of these approximations. So, we take $\{w_i\}_{i=1}^\infty$ be the eigen-functions of the operator $-\Delta$. Then $\{w_i\}_{i=1}^\infty$ is an orthogonal basis of $D(B_R)$ which is orthonormal in $L^2_\rho(B_R)$.
Let

$$V_m=span\{w_1,w_2,\cdots,w_m\},$$

and the projection of the initial data on the finite dimensional subspace V_m is given by:

$$u_{10}^m=\sum_{j=0}^m a_jw_j,\quad u_{20}^m=\sum_{j=0}^m b_jw_j,\quad u_{11}^m=\sum_{j=0}^m c_jw_j,\quad u_{21}^m=\sum_{j=0}^m d_jw_j,$$

We search approximate solutions

$$u_1^m(x,t):=\sum_{j=0}^m h_j^m(t)w_j(x),\quad u_2^m(x,t):=\sum_{j=0}^m k_j^m(t)w_j(x),$$

of the approximate problem in V_m

$$\begin{cases} \int_{B_R}\left(\rho(x)\left(|u_1'^m|^{l-2}u_1'^m\right)'w-\int_0^t g_1(t-s)\nabla u_1^m(s,x)\nabla w ds\right)dx \\ +\int_{B_R}\left(\rho(x)f_1(u_1^m,u_2^m)w+\nabla u_1^m\nabla w+\nabla u_1'^m\nabla w\right)dx=0, \\ \\ \int_{B_R}\left(\rho(x)\left(|u_2'^m|^{l-2}u_2'^m\right)'w-\int_0^t g_2(t-s)\nabla u_2^m(s,x)\nabla w ds\right)dx \\ +\int_{B_R}\left(\rho(x)f_2(u_1^m,u_2^m)w+\nabla u_2^m\nabla w+\nabla u_2'^m\nabla w\right)dx=0, \\ \\ u_1^m(0)=u_{10}^m,u_1'^m(0)=u_{11}^m,u_2^m(0)=u_{20}^m,u_2'^m(0)=u_{21}^m. \end{cases} \tag{3.1}$$

Based on standard existence theory for differential equations, one can conclude the existence of solution (u_1^m,u_2^m) of (3.1) on a maximal time interval $[0,t_m)$, for each $m\in\mathbb{N}$.

- (A priori estimate 1): In (3.1), let $w=(u_1^m)'$ in the first equation and $w=(u_2^m)'$ in the second equation, add the resulting equations and integrate by parts to obtain

$$\frac{d}{dt}E^m(t)=\frac{1}{2}\sum_{i=1}^2(g_i'\circ\nabla u_i^m)(t)-\frac{1}{2}\sum_{i=1}^2 g_i(t)\|\nabla u_i^m(t)\|_2^2-\sum_{i=1}^2\|\nabla u_i'^m\|_2^2, \tag{3.2}$$

This means, using $(A1)$, that for some positive constant C independent of t and m, we have

$$E^m(t) \leq E^m(0) \leq C. \tag{3.3}$$

• (A priori estimate 2): In (3.1), let $w = -\Delta u_1'^m$ in the first equation and $w = -\Delta u_2'^m$ in the second equation, add the resulting equations, integrate by parts and use $(A1)$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\Delta u_i'^m\|_{L_\rho^l}^l + \frac{1}{2}\left(1-\int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2 + \frac{1}{2}(g_i \circ \Delta u_i^m)\right) \\
= &\sum_{i=1}^{2}\left(\frac{1}{2}(g_i' \circ \Delta u_i^m) - \frac{1}{2}g_i(t)\|\Delta u_i^m\|_2^2 + \|\Delta u_i'^m\|_2^2\right) \\
- &\sum_{i=1}^{2}\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i'^m dx \\
\leq & -\sum_{i=1}^{2}\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i'^m dx.
\end{aligned} \tag{3.4}$$

Then, integrating over $(0,t)$ yields

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\Delta u_i'^m\|_{L_\rho^l}^l + \frac{1}{2}\left(1-\int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2 + \frac{1}{2}(g_i \circ \Delta u_i^m)\right) \\
\leq &\sum_{i=1}^{2}\left(\|\Delta u_{i1}^m\|_{L_\rho^l}^l + \|\Delta u_{i0}^m\|_2^2 - \int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i^m dx\right) \\
+ &\sum_{i=1}^{2}\int_{B_R}\rho(x)\left(f_i(u_{10}^m,u_{20}^m)\Delta u_{i0}^m\right)dx \\
+ &\int_0^t\int_{B_R}\rho(x)\left(\frac{\partial f_1}{\partial u_2}u_2'^m\Delta u_1^m + \frac{\partial f_2}{\partial u_1}u_1'^m\Delta u_2^m\right)dxds.
\end{aligned} \tag{3.5}$$

To estimate the terms on the right hand side of (3.6), we use (5.2)-(5.4), Young's inequality and (2.9) and take (3.3) into account to get

$$\begin{aligned}
&\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i^m \leq k\int_{B_R}\rho(x)\left(|u_1^m|+|u_2^m|+|u_1^m|^{\beta_{i1}}+|u_2^m|^{\beta_{i2}}\right)\Delta u_i^m, \\
&\leq \delta\|\Delta u_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\int_{B_R}\rho(x)\left(|u_1^m|^2+|u_2^m|^2+|u_1^m|^{2\beta_{i1}}+|u_2^m|^{2\beta_{i2}}\right), \\
&\leq \delta\|\Delta u_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\left(\|u_1^m\|_{L_\rho^2}^2+\|u_2^m\|_{L_\rho^2}^2+\|u_1^m\|_{L_\rho^2}^{2\beta_{i1}}+\|u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right), \\
&\leq \delta\|\Delta u_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}\left(\|\nabla u_1^m\|_{L_\rho^2}^2+\|\nabla u_2^m\|_{L_\rho^2}^2+\|\nabla u_1^m\|_{L_\rho^2}^{2\beta_{i1}}+\|\nabla u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right), \\
&\leq \delta\|\Delta u_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}E^m(0)E^m(t), \\
&\leq \delta\|\Delta u_i^m\|_{L_\rho^2}^2 + \frac{c}{\delta}.
\end{aligned} \tag{3.6}$$

Since $1 \leq \beta_{ij}, i, j = 1, 2$. Now, we estimate

$$I := \int_{B_R} \rho(x) \frac{\partial f_i}{\partial u_1} u_i'^m \Delta u_i^m.$$

First, we observe that

$$\frac{\beta_{1j} - 1}{2\beta_{1j}} + \frac{1}{2\beta_{1j}} + \frac{1}{2} = 1,$$

and use $(A2)$ and the generalized Hölde's inequality to infer

$$\begin{aligned} |I| &\leq d \int_{B_R} \rho(x) \left(1 + |u_1^m|^{\beta_{11}-1} + |u_2^m|^{\beta_{12}-1}\right) u_i'^m \Delta u_i^m, \\ &\leq d \left(\|u_i'^m\|_{L^2_\rho} + \|u_i'^m\|_{L^{2\beta_{11}}_\rho} \|u_1^m\|^{\beta_{11}-1}_{L^{2\beta_{11}}_\rho} + \|u_i'^m\|_{L^{2\beta_{12}}_\rho} \|u_2^m\|^{\beta_{12}-1}_{L^{2\beta_{12}}_\rho} \right) \|\Delta u_i^m\|_{L^2_\rho}. \end{aligned}$$

Then, by (2.9), (3.3) and Young's inequality, we arrive at

$$\begin{aligned} |I| &\leq c \left(1 + \|\nabla u_1^m\|_2^{\beta_{11}-1} + \|\nabla u_2^m\|_2^{\beta_{12}-1}\right) \|\nabla u_i'^m\|_{L^2_\rho} \|\Delta u_i^m\|_{L^2_\rho}, \\ &\leq c \left(\|\nabla u_i'^m\|_{L^2_\rho} \cdot \|\Delta u_i^m\|_{L^2_\rho} \right) \leq c \|\nabla u_i'^m\|^2_{L^2_\rho} + c \|\Delta u_i^m\|^2_{L^2_\rho}. \end{aligned} \tag{3.7}$$

Since the other terms in (3.6) can be similarly treated and the norms of the initial data are uniformly bounded, we combine (3.6), (3.7), use $(A1)$ and take δ small enough to end up with

$$\sum_{i=1}^{2} \left(\|\nabla u_i'^m\|^l_{L^l_\rho} + \|\Delta u_i^m\|_2^2 \right) \leq c + c \sum_{i=1}^{2} \int_0^t \left(\|\nabla u_i'^m\|^l_{L^l_\rho} + \|\Delta u_i^m\|_2^2 \right) ds.$$

Using Gronwall's inequality, this implies that

$$\sum_{i=1}^{2} \left(\|\nabla u_i'^m\|^l_{L^l_\rho} + \|\Delta u_i^m\|_2^2 \right) \leq C, \quad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \tag{3.8}$$

• (A priori estimate 3): In (3.1), let $w = (u_1^m)''$ in the first equation and $w = (u_2^m)''$ in the second equation. Then, by exploiting the previous estimates and using similar arguments, we find

$$\sum_{i=1}^{2} \|u_i''^m\|_2^2 \leq C, \quad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \tag{3.9}$$

From (3.3), (3.8) and (3.9), we conclude that

$$\begin{aligned} u_i^m &\text{ are uniformly bounded in } L^\infty(0, T; D(B_R)), \\ u_i^{m'} &\text{ are uniformly bounded in } L^\infty(0, T; L^l_\rho(B_R)), \\ u_i^{m''} &\text{ are uniformly bounded in } L^2(0, T; D^{-1}(B_R)), \end{aligned}$$

which implies that there exists subsequences of $\{u_i^m\}$, which we still denote in the same way, such that

$$\begin{aligned} u_i^m &\xrightarrow{*} \textit{weak } u_i \text{ in } L^\infty(0,T;D(B_R)), \\ u_i^{m'} &\xrightarrow{*} \textit{weak } u_i' \text{ in } L^\infty(0,T;L^l_\rho(B_R)), \\ u_i^{m''} &\xrightarrow{*} \textit{weak } u_i'' \text{ in } L^2(0,T;D^{-1}(B_R)). \end{aligned} \tag{3.10}$$

In the sequel, we will deal with the nonlinear term. By Aubin's Lemma (see [14]), we find, up to a subsequence, that

$$u_i^m \to u_i \text{ strongly in } L^2(0,T;L^l_\rho(B_R)). \tag{3.11}$$

Then,

$$u_i^m \to u_i \text{ almost everywhere in } (0,T)\times B_R, \tag{3.12}$$

and therefore, from (5.5), (5.6),

$$f_i(u_1^m,u_2^m) \to f_i(u_1,u_2) \;\text{ almost everywhere in }\; (0,T)\times B_R, \text{ for } i=1,2. \tag{3.13}$$

Also, as u_i^m are bounded in $L^\infty(0,T;L^2_\rho(B_R))$, then the use of (5.2)-(5.6) gives that $f_i(u_1^m,u_2^m)$ is bounded in $L^\infty(0,T;L^2_\rho(B_R))$. From (3.13), we can deduce that

$$f_i(u_1^m,u_2^m) \rightharpoonup f_i(u_1,u_2) \text{ in } L^2(0,T;L^2_\rho(B_R)), \quad \text{for } i=1,2.$$

Combining the results obtained above, we can pass to the limit and conclude that (u_1,u_2) is the solution of system (1.1) restricted un B_R.

□

In the next result, we will extend our solutions to $\mathbb{R}^n$.

Theorem 3.2. *Assume that (A1), (A2), (5.2)-(5.6) are satisfied. Suppose that the initial conditions*

$$(u_{10},u_{11}) \in (C_0^\infty(B_R)^2, (u_{20},u_{21}) \in (C_0^\infty(B_R))^2,$$

are given. Then for the problem (1.1), there exists a unique solution such that

$$(u_1,u_2) \in (C[0,T;D(\mathbb{R}^n)])^2 \quad \textit{and} \quad (u_1',u_2') \in (C[0,T;L^l_\rho(\mathbb{R}^n)])^2.$$

Proof: (a) **Existence.** Let $R_0>0$ such that $supp(u_{10},u_{20}) \subset B_{R_0}$ and $supp(u_{11},u_{21}) \subset B_{R_0}$. Then, for $R \geq R_0$, $R\in\mathbb{N}$, we consider the approximating problem

$$\begin{cases} \left(|u_1^{\prime R}|^{l-2}u_1^{\prime R}\right)' + f_1(u_1^R,u_2^R) \\ -\phi(x)\Delta\left(u_1^R + \int_0^t g_1(s)u_1^R(s-t,x)ds + u_1^{\prime R}\right) = 0, x\in B_R\times\mathbb{R}^+, \\ \left(|u_2^{\prime R}|^{l-2}u_2^{\prime R}\right)' + f_2(u_1^R,u_2^R) \\ -\phi(x)\Delta\left(u_2^R + \int_0^t g_2(s)u_2^R(s-t,x)ds + u_2^{\prime R}\right) = 0, x\in B_R\times\mathbb{R}^+, \\ (u_1^R(0,x),u_2^R(0,x)) = (u_1^0(x),u_2^0(x)) \in (C_0^\infty(B_R))^2, \\ (u_1^{\prime R}(0,x),u_2^{\prime R}(0,x)) = (u_1^1(x),u_2^1(x)) \in (C_0^\infty(B_R))^2. \end{cases} \tag{3.14}$$

By Lemma 3.1, problem (3.14) has a unique solution u_i^R such that

$$(u_1^R, u_2^R) \in (C[0,T; D(B_R)])^2 \quad \text{and} \quad ((u_1^R)', (u_2^R)') \in (C[0,T; L_\rho^l(B_R)])^2.$$

We extend the solution of the problem (3.14) as

$$(\widetilde{u}_1^R, \widetilde{u}_2^R) =: \begin{cases} (u_1^R, u_2^R), & \textit{if } |x| \leq R, \\ 0, & \textit{otherwise.} \end{cases} \tag{3.15}$$

The solution (u_1^R, u_2^R) satisfies the estimates

$$\begin{array}{ll} \|\widetilde{u}_i^R\|_{L^\infty(0,T;D(\mathbb{R}^n))} \leq K, & \|f(\widetilde{u}_i^R)\|_{L^\infty(0,T;D(\mathbb{R}^n))} \leq K, \\ \|(\widetilde{u}_i^R)'\|_{L^\infty(0,T;L_\rho^l(\mathbb{R}^n))} \leq K, & \|(\widetilde{u}_i^R)''\|_{L^\infty(0,T;D^{-1}(\mathbb{R}^n))} \leq K, \end{array} \tag{3.16}$$

where the constant K is independent of R. The estimates (3.16) imply that

$$\widetilde{u}_i^R \text{ is relatively compact in } C([0,T]; L_\rho^2(\mathbb{R}^n)). \tag{3.17}$$

Next using relations (3.16) and (3.17), the continuity of the embedding

$$C([0,T]; L_\rho^2(\mathbb{R}^n)) \subset L^2([0,T]; L_\rho^2(\mathbb{R}^n)),$$

and the continuity of f_i we may extract a subsequence of $\widetilde{u}_i^R$, denoted by $\widetilde{u}_i^{R_m}$, such that as $R_m \to \infty$ we get

$$\begin{aligned} \widetilde{u}_i^{R_m} &\overset{*}{\longrightarrow} \widetilde{u}_i \text{ in } L^\infty(0,T; D(B_R)), \\ (\widetilde{u}_i^{R_m})' &\overset{*}{\longrightarrow} u_i' \text{ in } L^\infty(0,T; L_\rho^l(B_R)), \\ (\widetilde{u}_i^{R_m})'' &\overset{*}{\longrightarrow} u_i'' \text{ in } L^\infty(0,T; D^{-1}(B_R)), \\ f(\widetilde{u}_i^{R_m}) &\overset{*}{\longrightarrow} f(\widetilde{u}_i) \text{ in } L^\infty(0,T; D(B_R)). \end{aligned} \tag{3.18}$$

For fixed $R = R_m$, let L_m denote the operator of restriction

$$L_m : [0,T] \times \mathbb{R}^n \to [0,T] \times B_R.$$

It is clear that the restricted subsequence $L_m \widetilde{u}_i^{R_m}$ satisfies the estimates obtained in Lemma 3.1. Therefore there exists a subsequence $\widetilde{u}_i^{R_{m_j}} = \widetilde{u}_i^j$ for which it can be shown by following the procedure of Lemma 3.1, that $L_m \widetilde{u}_i^j$ converges weakly to solution $\widetilde{u}_i^m$. We have

$$\begin{cases} \int_0^T \left(L_m \left(|\widetilde{u}_1^{j'}|^{l-2}\widetilde{u}_1^{j'} \right)', v \right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_1(L_m\widetilde{u}_1^j, L_m\widetilde{u}_2^j), v \right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} \nabla L_m \widetilde{u}_1^j \nabla v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{B_R} \nabla \widetilde{u}_1^j \nabla v dx ds \\ + \int_0^T \int_{B_R} \nabla L_m \widetilde{u}_1'^j \nabla v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_1'^j|^{l-2}\widetilde{u}_1^{j'} \right)', v \right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_1(\widetilde{u}_1^j, \widetilde{u}_2^j), v \right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} \nabla \widetilde{u}_1^j \nabla v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{\mathbb{R}^n} \nabla \widetilde{u}_1^j \nabla v dx ds, \\ \int_0^T \left(L_m \left(|\widetilde{u}_2^{j'}|^{l-2}\widetilde{u}_2^{j'} \right)', v \right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_2(L_m\widetilde{u}_1^j, L_m\widetilde{u}_2^j), v \right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} \nabla L_m \widetilde{u}_2^j \nabla v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{B_R} \nabla \widetilde{u}_2^j \nabla v dx ds \\ + \int_0^T \int_{B_R} \nabla L_m \widetilde{u}_2'^j \nabla v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_2'^j|^{l-2}\widetilde{u}_2^{j'} \right)', v \right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_2(\widetilde{u}_1^j, \widetilde{u}_2^j), v \right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} \nabla \widetilde{u}_2^j \nabla v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{\mathbb{R}^n} \nabla \widetilde{u}_2^j \nabla v dx ds, \end{cases} \tag{3.19}$$

for every $v \in C_0^\infty([0,T] \times B_R)$. Passing to the limit in (3.19) as $j \longrightarrow \infty$, we obtain that $L_m\widetilde{u}_i = \widetilde{u}_i^m$. The equalities (3.19) hold for any $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$ since the radius R is arbitrarily chosen. Therefore $\widetilde{u}_i$ is a solution of the problem (3.14).
(b) **Uniqueness.** Let us assume that $(u_{11}, u_{21}), (u_{12}, u_{22})$ are two strong solutions of (1.1). Then, $(z_1, z_2) = (u_{11} - u_{12}, u_{21} - u_{22})$ satisfies, for all $w \in D(\mathbb{R}^n)$

$$\begin{cases} \int_{\mathbb{R}^n} \left(\rho(x) \left(|z_1'|^{l-2} z_1' \right)' w + \nabla z_1 \nabla w + \int_0^t g_1(s) \nabla z_1(s-t, x) \nabla w ds \right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_1(z_1, z_2) w dx + \nabla z_1' \nabla w = 0, \\ \int_{\mathbb{R}^n} \left(\rho(x) \left(|z_2'|^{l-2} z_2' \right)' w + \nabla z_2 \nabla w + \int_0^t g_2(s) \nabla z_2(s-t, x) \nabla w ds \right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_2(z_1, z_2) w dx + \nabla z_2' \nabla w = 0. \end{cases} \tag{3.20}$$

Substituting $w = z_1'$ in the first equation and $w = z_2'$ in the second equation, adding the resulting equations, integrating by parts and using $(A1)$, yield

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^{2} \left(\frac{l-1}{l} \|z_i'\|_{L^l_\rho}^l + \frac{1}{2} \left(1 - \int_0^t g_i(s) ds \right) \|\nabla z_i\|_2^2 + \frac{1}{2} (g_i \circ \nabla z_i) \right) \\ \leq & \int_{\mathbb{R}^n} \left([f_1(u_{21}, u_{22}) + f_1(u_{11}, u_{12})] \, z_1' + [f_2(u_{21}, u_{22}) + f_2(u_{11}, u_{12})] \, z_2' \right) dx. \end{aligned}$$

Making use of (5.6) and following similar arguments that used to obtain (3.7), we find

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \left([f_1(u_{21},u_{22}) + f_1(u_{11},u_{12})]\, z_1' + [f_2(u_{21},u_{22}) + f_2(u_{11},u_{12})]\, z_2' \right) dx \\
\leq \; & k \int_{\mathbb{R}^n} \left(1 + |u_{11}|^{\beta_{11}-1} + |u_{12}|^{\beta_{11}-1} + |u_{21}|^{\beta_{12}-1} + |u_{22}|^{\beta_{12}-1}\right) (|z_1| + |z_2|) z_1' dx \\
+ \; & k \int_{B_R} \left(1 + |u_{11}|^{\beta_{21}-1} + |u_{12}|^{\beta_{21}-1} + |u_{21}|^{\beta_{22}-1} + |u_{22}|^{\beta_{22}-1}\right) (|z_1| + |z_2|) z_2' dx, \\
\leq \; & c \sum_{i=1}^{2} \left(\|z_i'\|_{L^l_\rho}^l + \|\nabla z_i\|_2^2 . \right) \qquad (3.21)
\end{aligned}
$$

Combining (3.20)- (3.21), integrating over $(0,t)$ and using Gronwall's Lemma, then we deduce that

$$
\sum_{i=1}^{2} \left(\|z_i'\|_{L^l_\rho}^l + \|z_i\|_2^2 \right) = 0, \qquad (3.22)
$$

which means that $(u_{11}, u_{21}) = (u_{12}, u_{22})$. This completes the proof. □

We can now state and prove the asymptotic behavior of the solution of (1.1).

4. Decay rate for linear cases

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is similar to that of the memory terms, making some small perturbation in the associate energy, for this purpose, we introduce the functional

$$
\psi(t) = \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) u_i |u_i'|^{l-2} u_i' dx. \qquad (4.1)
$$

The following Lemma will be useful in the proof of our next result.

Lemma 4.1. *Under the assumptions (A1), (A2), the functional ψ satisfies, along the solution of (1.1),*

$$
\psi'(t) \leq \sum_{i=1}^{2} \|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l - (k - 1 - \delta + |\alpha| c) \sum_{i=1}^{2} \|\nabla u_i\|_2^2 + c \sum_{i=1}^{2} (g_i \circ \nabla u_i), \qquad (4.2)
$$

for positive constants c.

Proof: From (4.1), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi'(t) &= \int_{\mathbb{R}^n} \rho(x)u_1'^l dx + \int_{\mathbb{R}^n} \rho(x)u_1 \left(|u_1'|^{l-2}u_1'\right)' dx \\
&+ \int_{\mathbb{R}^n} \rho(x)u_2'^l dx + \int_{\mathbb{R}^n} \rho(x)u_2 \left(|u_2'|^{l-2}u_2'\right)' dx, \\
&= \int_{\mathbb{R}^n} \left(\rho(x)u_1'^l - u_1\Delta u_1 - u_1\Delta u_1'\right) dx \\
&= \int_{\mathbb{R}^n} \left(-\alpha\rho(x)u_1u_2 + u_1\int_0^t g_1(t-s)\Delta u_1(s,x)ds\right) dx \\
&+ \int_{\mathbb{R}^n} \left(-\alpha\rho(x)u_1u_2 + u_2\int_0^t g_2(t-s)\Delta u_2(s,x)ds\right) dx, \\
&+ \int_{\mathbb{R}^n} \left(-\alpha\rho(x)u_1u_2 + u_2\int_0^t g_2(t-s)\Delta u_2(s,x)ds\right) dx, \\
&= \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \left(1 - \int_0^t g_i(s)ds\right)\sum_{i=1}^{2}\|\nabla u_i\|_2^2 \\
&- \sum_{i=1}^{2}\|\nabla u_i'\|_2^2 - 2\alpha\int_{\mathbb{R}^n}\rho(x)u_1u_2dx \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\nabla u_i\int_0^t g_i(t-s)(\nabla u_i(s) - \nabla u_i(t))dsdx.
\end{aligned}$$

Recalling that $\int_0^t g_i(s)ds \le \int_0^\infty g_i(s)ds = 1 - k_i$, using Young's inequality, Lemma 2.3 and Lemma 2.2, we obtain

$$\begin{aligned}
\psi'(t) &\le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}\|\nabla u_i'\|_2^2 - (k_i - 1 + |\alpha|\|\rho\|^{-1}_{L^s(\mathbb{R}^n)})\sum_{i=1}^{2}\|\nabla u_i\|_2^2 \\
&+ \delta\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{1}{4\delta}\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla u_i(s) - \nabla u_i(t)|ds\right)^2 dx, \\
&\le \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}\|\nabla u_i'\|_2^2 - (k - 1 - \delta + |\alpha|c)\sum_{i=1}^{2}\|\nabla u_i\|_2^2 \\
&+ \frac{(1-k)}{4\delta}\sum_{i=1}^{2}(g_i \circ \nabla u_i).
\end{aligned}$$

For α small enough and $k = \min\{k_1, k_2\}$. □

Our main result reads as follows.

Theorem 4.2. *Let $(u_{10}, u_{11}), (u_{20}, u_{21}) \in D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)$ and suppose that (A1), (A2) hold. Then there exist positive constants W, ω such that the energy of solution* *given by (1.1) satisfies,*

$$E(t) \leq WE(0) \exp\left(-\omega \int_0^t \xi(s)ds\right), \forall t \geq 0. \tag{4.3}$$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \varepsilon\psi(t), \quad \forall \varepsilon > 0. \tag{4.4}$$

for $N_1 > 1$, we need the next lemma, which means that there is equivace between the perturbed energy and energy functions.

Lemma 4.3. *For $N_1 > 1$, we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \quad \forall t \geq 0, \tag{4.5}$$

holds for some positive constants β_1 and β_2.

Proof: By (4.1) and (4.4), we have

$$\begin{aligned} |L(t) - N_1 E(t)| &\leq \varepsilon|\psi_1(t)|, \\ &\leq \varepsilon \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx. \end{aligned}$$

Thanks to Hölder's and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 2.3

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right)^{1/l} \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l}, \\ &\leq \frac{1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right), \\ &\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)} \|\nabla u_i\|_2^l. \end{aligned} \tag{4.6}$$

Then, since $l \geq 2$, we have by using (1.4)

$$\begin{aligned} |L(t) - N_1 E(t)| &\leq \varepsilon c \sum_{i=1}^{2} \left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|\nabla u_i\|_2^l\right), \\ &\leq \varepsilon c(E(t) + E^{l/2}(t)), \\ &\leq \varepsilon c E(t)(1 + E^{[(l/2)-1]}(t)), \\ &\leq \varepsilon c E(t)(1 + E^{[(l/2)-1]}(0)), \\ &\leq \varepsilon c E(t). \end{aligned}$$

Consequently, (4.5) follows. □

Proof of Theorem 4.2 From (1.4), results of Lemma 4.1, we have

$$\begin{aligned} L'(t) &= N_1 E'(t) + \varepsilon \psi'(t), \\ &\le N_1 \Big(\frac{1}{2} \sum_{i=1}^{2} (g_i' \circ \nabla u_i)(t) - \sum_{i=1}^{2} \|\nabla u_i'\|_2^2 \Big) \\ &+ \varepsilon \sum_{i=1}^{2} \Big(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k - 1 - \delta + |\alpha| c) \|\nabla u_i\|_2^2 + c(g_i \circ \nabla u_i) \Big), \end{aligned}$$

At this point, we choose N_1 large and ε so small such that

$$L'(t) \le M_0 \sum_{i=1}^{2} (g_i \circ \nabla u_i) - \varepsilon E(t), \quad \forall t \ge 0. \tag{4.7}$$

Multiplying (4.7) by $\xi(t)$ gives

$$\xi(t) L'(t) \le -\varepsilon \xi(t) E(t) + M_0 \xi(t) \sum_{i=1}^{2} (g_i \circ \nabla u_i). \tag{4.8}$$

The last term can be estimated, using $(A1)$ as follows

$$\begin{aligned} \xi(t) \sum_{i=1}^{2} (g_i \circ \nabla u_i) &\le \sum_{i=1}^{2} \xi_i(t) \int_{\mathbb{R}^n} \int_0^t g_i(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le \sum_{i=1}^{2} \int_{\mathbb{R}^n} \int_0^t \xi_i(t-s) g_i(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le - \sum_{i=1}^{2} \int_{\mathbb{R}^n} \int_0^t g_i'(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le - \sum_{i=1}^{2} (g_i' \circ \nabla u_i) \le -E'(t). \end{aligned} \tag{4.9}$$

Thus, (4.7) becomes

$$\xi(t) L'(t) + M_0 E'(t) \le -\varepsilon \xi(t) E(t) \quad \forall t \ge 0. \tag{4.10}$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing and so ξ is differentiable, with $\xi'(t) \le 0$ for a.e t, then

$$(\xi(t) L(t) + M_0 E(t))' \le \xi(t) L'(t) + M_0 E'(t) \le -\varepsilon \xi(t) E(t) \quad \forall t \ge 0. \tag{4.11}$$

Since, using (4.5)

$$F = \xi L + M_0 E \sim E, \tag{4.12}$$

we obtain, for some positive constant ω

$$F'(t) \leq -\omega\xi(t)F(t) \quad \forall t \geq 0. \tag{4.13}$$

Integration over $(0,t)$ leads to, for some constant $\omega > 0$ such that

$$F(t) \leq WF(0)\exp\left(-\omega\int_0^t \xi(s)ds\right), \forall t \geq 0. \tag{4.14}$$

Recalling (4.12), estimate (4.14) yields the desired result (4.3). This completes the proof of Theorem (4.2).

5. Concluding comments

1- One can easily obtain the same result in Theorem (4.2) in the nonlinear case

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) - \phi(x)\Delta\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) - \phi(x)\Delta\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x),u_2(0,x)) = (u_{10}(x),u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_{11}(x),u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{5.1}$$

where our nonlinearity is given by the functions f_1, f_2 satisfying the next assumptions:

(***hyp1***) The functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for i=1,2) is of class C^1 and there exists a function F such that

$$f_1(x,y) = \frac{\partial F}{\partial x}, \qquad f_2(x,y) = \frac{\partial F}{\partial y}, \tag{5.2}$$

$$F \geq 0,\ xf_1(x,y) + yf_2(x,y) - F(x,y) \geq 0. \tag{5.3}$$

and

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}) \quad \forall (x,y) \in \mathbb{R}^2, \tag{5.4}$$

for some constant $d > 0$ and $1 \leq \beta_{ij} \leq \frac{n}{n-2}$ for $i,j = 1,2$.

(***hyp2***) There exists a positive constant k such that

$$|f_i(x,y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}), \tag{5.5}$$

and

$$\begin{aligned} &|f_i(x,y) - f_i(r,s)| \\ &\leq k(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1} + |r|^{\beta_{i1}-1} + |s|^{\beta_{i2}-1})(|x-r| + |y-s|), \end{aligned} \tag{5.6}$$

for all $(x,y),(r,s) \in \mathbb{R}^2$ and $i = 1,2$. Noting that we follow the same steps in the linear cases with the same perturbed function and some calculations related with the presence of f_1, f_2.

2. Let us remark that, it is similar to study the question of existence and decay of solution of the same problem with the presence of weak-viscoelasticity in the form

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) - \phi(x)\Delta\left(u_1 + \alpha_1(t)\int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) - \phi(x)\Delta\left(u_2 + \alpha_2(t)\int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x),u_2(0,x)) = (u_{10}(x),u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_{11}(x),u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{5.7}$$

where we should need additional, conditions on α as follows

$$1-\alpha_i(t)\int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \alpha_i(t) > 0, \tag{5.8}$$

$$\lim_{t\to+\infty}\frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0 \tag{5.9}$$

where

$$\alpha(t) = \min\{\alpha_1(t),\alpha_2(t)\}, \quad \forall t \geq 0.$$

For the reader we shall develop here the next important technical Lemma.

Lemma 5.1. *For any $v \in C^1\left(0,T,H^1(\mathbb{R}^n)\right)$ we have*

$$\begin{aligned} & -\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ = \; & \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g\circ A^{1/2}v\right)(t) \\ - \; & \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2dxds\right] \\ & -\frac{1}{2}\alpha(t)\left(g'\circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2dxds \\ & -\frac{1}{2}\alpha'(t)\left(g\circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2dxds. \end{aligned}$$

Proof: It's not hard to see

$$\begin{aligned} & \int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ = \; & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)A^{1/2}v(s)dxds \\ = \; & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)\left[A^{1/2}v(s)-A^{1/2}v(t)\right]dxds \\ & +\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)A^{1/2}v(t)dxds. \end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) Av(s)v'(t) ds dx \\
= & -\frac{1}{2}\alpha(t) \int_0^t g(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} \left| A^{1/2}v(s) - A^{1/2}v(t) \right|^2 dx ds \\
& + \alpha(t) \int_0^t g(s) \left(\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} \left| A^{1/2}v(t) \right|^2 dx \right) ds
\end{aligned}$$

which implies,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) Av(s)v'(t) ds dx \\
= & -\frac{1}{2}\frac{d}{dt} \left[\alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2}v(s) - A^{1/2}v(t) \right|^2 dx ds \right] \\
& + \frac{1}{2}\frac{d}{dt} \left[\alpha(t) \int_0^t g(s) \int_{\mathbb{R}^n} \left| A^{1/2}v(t) \right|^2 dx ds \right] \\
& + \frac{1}{2}\alpha(t) \int_0^t g'(t-s) \int_{\mathbb{R}^n} \left| A^{1/2}v(s) - A^{1/2}v(t) \right|^2 dx ds \\
& - \frac{1}{2}\alpha(t) g(t) \int_{\mathbb{R}^n} \left| A^{1/2}v(t) \right|^2 dx ds. \\
& + \frac{1}{2}\alpha'(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} \left| A^{1/2}v(s) - A^{1/2}v(t) \right|^2 dx ds \\
& - \frac{1}{2}\alpha'(t) \int_0^s g(s) ds \int_{\mathbb{R}^n} \left| A^{1/2}v(t) \right|^2 dx ds.
\end{aligned}$$

This completes the proof. □

Under this additional conditions on α, the decay of energy associate with problem (5.7) is given in the next result

Theorem 5.2. *Let $(u_{i0}, u_{i1}) \in (D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)), i = 1, 2$ and suppose that (A1), (A2), (5.2)-(5.6) hold. Then there exist positive constants W, ω such that the energy of solution given by (5.7) satisfies,*

$$E(t) \leq W E(t_0) \exp\left(-\omega \int_{t_0}^t \alpha(s)\xi(s) ds \right), \tag{5.10}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq t_0 \geq 0$.

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