



## Direct Method for Solution Variational Problems by using Hermite Polynomials

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**ABSTRACT:** In this paper a new numerical method is presented for numerical approximation of variational problems. This method with variable coefficients is based on Hermite polynomials. The properties of Hermite polynomials with the operational matrices of derivative and integration are used to reduce optimal control problems to the solution of linear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

**Key Words:** Variational Problems, Hermite polynomials , Best approximating, Operational matrices of derivative and integration.

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### 1. Introduction

One of the widely used methods to solve optimal control problems is the direct method. There is a large number of research papers that employ this method to solve optimal control problems (see for example [2-4, 8, 9, 14-17, 19, 27-34, 37-40]). Razzaghi, et. al. used direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials [32]. Optimal control of switched systems based on Bezier control points presented in [15]. A new approach using linear combination property of intervals and discretization is proposed to solve a class

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of nonlinear optimal control problems, containing a nonlinear system and linear functional [37,38]. Time varying quadratic optimal control problem was solved by using Bezier control points [14]. Hybrid functions approach for nonlinear constrained optimal control problems presented by Mashayekhi et. al. [27]. The optimal control problem of a linear distributed parameter system is studied via shifted Legendre polynomials (SLPs) in [20]. An accurate method is proposed to solve problems such as identification, analysis and optimal control using the Bernstein orthonormal polynomials operational matrix of integration [36]. In [19] Jaddu and Shimemura proposed a method to solve the linear-quadratic and the nonlinear optimal control problems by using Chebyshev polynomials to parameterize some of the state variables, then the remaining state variables and the control variables are determined from the state equations. Also Razzaghi and Elnagar [33] proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state and control variables. Their approach is based on using the shifted Legendre polynomials to parameterize the derivative of each of the state variables. The approach proposed in [27] is based on approximating the state variables and control variables with hybrid functions. In [39] operational matrices with respect to Hermite polynomials and their applications is presented for solving linear differential equations with variable coefficients. Investigation of optimal control problems and solving them using Bezier polynomials has been presented [1]. In [2] Solution of optimal control problems with payoff term and fixed state endpoint by using Bezier polynomials has been presented. In this paper, we present a computational method for solving variational problems by using Hermite polynomials. The method is based on approximating the state variables with Hermite polynomials. Our method consists of reducing the variational problems into a set of linear algebraic equations by first expanding the state rate  $x(t)$  as a Hermite polynomial with unknown coefficients.

The paper is organized as follows: In Section 2 we describe the basic formulation of the Hermite functions required for our subsequent development. Section 3 is devoted to the formulation of optimal control problems. Section 4 summarizes the application of this method to the optimal control problems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed method.

## 2. Hermite Polynomials and Their Properties

Hermite polynomials are a classical orthogonal polynomial sequence that arise in probability. The explicit expression of Hermite polynomials of degree  $n$  is defined by [39]

$$H_n(t) = n! \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i (2t)^{n-2i}}{i!(n-2i)!}, \quad (2.1)$$

$\lfloor \frac{n}{2} \rfloor$  = largest integer number  $\leq \frac{n}{2}$ , where  $t$  is a real number ( $t \in R$ ), and Rodrigues formula is the following

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} (e^{-t^2}). \quad (2.2)$$

Eqs. (2.1) and (2.2) are solutions for following equation

$$x'' - 2tx' + 2nx = 0. \tag{2.3}$$

Namely  $x(t) = H_n(t)$ . The first few Hermite polynomials are  $H_0(t) = 1$ ,  $H_1(t) = 2t$ ,  $H_2(t) = 4t^2 - 2$ ,  $H_3(t) = 8t^3 - 12t$ .

**2.1. Some Properties of Hermite Polynomials**

Hermite polynomials obey the recurrence relation

$$H_{i+1}(t) = 2tH_i(t) - 2iH_{i-1}(t). \tag{2.4}$$

An important property of the Hermite polynomials is the following derivative relation [39]

$$H'_i(t) = 2iH_{i-1}(t). \tag{2.5}$$

Weher  $i = 0, \dots, n$  and  $H'_i(t)$  is drevation Hermite polynomials of degree  $i$ . Further,  $H_i(t)$  are orthogonal in  $L^2_w(\Lambda)$ , weher  $\Lambda = (-\infty, +\infty)$  with respect to the weight function  $w(t) = e^{-t^2}$  and satisfy in the following relation

$$\int_{-\infty}^{+\infty} H_i(t)H_j(t)w(t)dt = 2^i i! \sqrt{\pi} \delta_{i,j}. \tag{2.6}$$

Weher  $\delta_{i,j}$  is kronecker delta function. some property for Hermite polynomials are  $H_i(-t) = (-1)^i H_i(t)$ ,  $H_{2i}(0) = (-1)^i \frac{(2i)!}{i!}$ ,  $H_{2i+1}(0) = 0$ ,  $H'_{2i}(0) = 0$ ,  $H'_{2i+1}(0) = (-1)^i \frac{(2i+2)!}{(i+1)!}$ .

**2.2. The operational matrices for the Hermite Polynomials**

A function  $x(t) \in L^2_w(\Lambda)$ , can be expressed in terms of Hermite polynomials as

$$x(t) = \sum_{-\infty}^{+\infty} a_i H_i(t), \tag{2.7}$$

where the coefficients  $a_i$  is given by

$$a_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{+\infty} H_i(t)x(t)w(t)dt. \tag{2.8}$$

In practice, only the frst  $n + 1$  term of the Hermite polynomials are considered. Then we have:

$$x_n(t) = \sum_{i=0}^n a_i H_i(t) = A\Phi_n(t), \tag{2.9}$$

where Hermite coefficients vector  $A$  and Hermite vector  $\Phi(t)$  are given by

$$A = [a_0, \dots, a_n],$$

$$\Phi_n(t) = [H_0(t), \dots, H_n(t)]^T, \tag{2.10}$$

where  $T$  denotes transposition.

**The operational matrix of derivative:** The differentiation of vector  $\Phi_n(t)$  can be expressed as

$$\Phi'_n(t) = D_\phi \Phi_n(t), \tag{2.11}$$

where  $D_\phi$  is the  $(n + 1)(n + 1)$  operational matrix of derivative for the Hermite polynomials and it is given as following:

$$D_\phi = (d_{i,j}) = \begin{cases} 2i, & j = i - 1, \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

**The operational matrix of integration:** The integration of vector  $\Phi_n(t)$  can be expressed as

$$\int_a^t \Phi_n(x)dx = P_\phi \Phi_n(t) \tag{2.13}$$

where  $P_\phi$  is the  $(n + 1)(n + 1)$  operational matrix of integration for the Hermite polynomials. The integration of  $H_i(x)$  of order  $i$  can be obtained as following formula:

$$\int_a^t H_i(x)dx = \frac{1}{2(i + 1)} [H_{i+1}(t) - H_{i+1}(a)H_0] \tag{2.14}$$

where for  $a = 0$  we get:

$$H_{i+1}(0) = \begin{cases} (-1)^{\frac{i+1}{2}} \frac{(i+1)!}{(\frac{i+1}{2})!}, & \text{if } i \text{ odd,} \\ 0, & \text{if } i \text{ even} \end{cases} \tag{2.15}$$

finally we can written  $P_\phi$  matrix as:

$$P_\phi = \begin{bmatrix} -\frac{1}{2}H_1(a) & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ -\frac{1}{4}H_2(a) & 0 & \frac{1}{4} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{2(i+1)}H_{i+1}(a) & & \dots & & \frac{1}{2(i+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ -\frac{1}{2(n+1)}H_{n+1}(a) & & \dots & & 0 & 0 & 0 \end{bmatrix} \tag{2.16}$$

**2.3. Approximations by Hermite polynomials**

Now in this section, we present some useful theorems which show the approximations of functions by Hermite polynomials. For this purpose, let us define  $S_n = span\{H_0(t), H_1(t), \dots, H_n(t)\}$ . Any polynomial  $h(t)$  of degree  $m$  can be expanded in terms of  $H_i(t), i = 0 \dots n$  as follows

$$h(t) = \sum_{i=0}^n c_i H_i(t). \tag{2.17}$$

Also the  $L^2(\Lambda)$ -orthogonal projection  $p_n : L^2(\Lambda) \rightarrow S_n$  is a mapping in a way that for any  $y(t) \in L^2(\Lambda)$ , we have:

$$\langle p_n(y) - y, \phi \rangle = 0, \quad \forall \phi \in S_n.$$

Due to the orthogonality, we can write

$$p_n(y) = \sum_{i=0}^{n-1} c_i H_i(t), \tag{2.18}$$

where  $c_i$  are constants in the following form

$$c_i = \frac{1}{\gamma_i} \langle y(t), H_i(t) \rangle_{L_w^2},$$

where  $\gamma_i = 2^i i! \sqrt{\pi}$ . In the literature of spectral methods,  $p_n(y)$  is named as Hermite expansion of  $y(t)$  and approximates  $y(t)$  on  $(-\infty, +\infty)$ . Also estimating the distance between  $y(t)$  and its Hermite expansion as measured in the weighted norm  $\| \cdot \|_w$  is an important problem in numerical analysis. The following theorem provide the basic approximation results for Hermite expansion.

**Theorem 2.1.** *we have*

$$\| \frac{d^l}{dt^l} (p_n(y) - y) \|_{w(t)} \leq n^{(l-m)/2} \| \frac{d^m}{dt^m} y(t) \|_{w(t)},$$

$$0 \leq l \leq m, \quad \forall y \in B^m(\Lambda),$$

where

$$B^m(\Lambda) = \{ \forall y \in L_w^2 : \frac{d^l}{dt^l} y \in L_w^2(\Lambda), 0 \leq l \leq m \}.$$

*Proof:* see [41].

### 3. Variational Problems

Consider the following variational problem:

$$Z(x(t)) = \int_a^b F(t, x(t), \dot{x}(t), \dots, x^{(n)}(t)) dt, \tag{3.1}$$

with the boundary conditions

$$x(a) = a_0, \dot{x}(a) = a_1, \dots, x^{(n-1)}(a) = a_{n-1}, \tag{3.2}$$

$$x(b) = b_1, \dot{x}(b) = b_1, \dots, x^{(n-1)}(b) = b_{n-1}, \tag{3.3}$$

where  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ . The problem is to find the extremum of Eq. (3.1), subject to boundary conditions (3.2) and (3.3). The method consists of reducing the variational problem into a set of algebraic equations by first expanding  $x(t)$  in terms of Hermite polynomials with unknown coefficients.

### 4. The Proposed Method

Let

$$x_i(t) \simeq X^i \Phi_n(t), \tag{4.1}$$

where  $X^i, i = 1, \dots, n$ , are state coefficient vectors respectively. Then using (2.4) we get

$$\dot{x}_i(t) \simeq X^i [D_\phi \Phi_n(t)], \quad (4.2)$$

by  $j$  times drevating of Eq. (4.2) we have

$$x_i^{(j)}(t) \simeq X^i [D_\phi^j \Phi_n(t)], \quad (4.3)$$

Using Eq. (4.1) we have

$$x(t) \simeq X \Phi_n(t) = \left[ \sum_{j=0}^n X_j^1 H_j(t), \dots, \sum_{j=0}^n X_j^n H_j(t) \right], \quad (4.4)$$

where  $X = (X_i^k)_{n \times (n+1)}$  is state coefficient matrice. The boundary conditions in Eqs. (3.2) and (3.3) can be rewritten as

$$x^{(j)}(a) = a_j = a_j \otimes E \Phi_n(t), \quad (4.5)$$

$$x^{(j)}(b) = b_j = b_j \otimes E \Phi_n(t). \quad (4.6)$$

where  $j = 0, 1, \dots, n$  and  $E = [1, 0, \dots, 0]$  is  $1 \times (n+1)$  constant vector, and the symbol ' $\otimes$ ' denotes Kronecker product [23]. If  $x(a)$  or  $x(b)$  is unknown in Eqs. (3.2) and (3.3), then we put

$$x(a) \simeq X \Phi_n^T(a) = \sum_{j=0}^n X_j^1 H_j(a), \dots, \sum_{j=0}^n X_j^n H_j(a), \quad (4.7)$$

$$x(b) \simeq X \Phi_n^T(b) = \sum_{j=0}^n X_j^1 H_j(b), \dots, \sum_{j=0}^n X_j^n H_j(b). \quad (4.8)$$

#### 4.1. Performance Index Approximation for the Variational Problem

By expanding  $x^{(n)}(t)$  using the Bezier polynomials we have

$$x^{(n)}(t) = X^T \Phi_n(t), \quad (4.9)$$

where  $X^T$  is vector of order  $1 \times (n+1)$ , By integrating Eq.(4.9) from 0 to  $t$  we get

$$x^{(n-1)}(t) - x^{(n-1)}(a) = \int_a^t X^T \Phi_n(t) = X^T P_\phi \Phi_n(t), \quad (4.10)$$

where  $P_\phi$  is operational matrix of integration given in Eq. (2.5). By using Eqs. (3.2) and (4.10) we get

$$x^{(n-1)}(t) = a_{n-1} + X^T P_\phi \Phi_n(t). \quad (4.11)$$

By  $n-1$  times integrating of Eq.(4.11) from 0 to  $t$  and using the boundary conditions given in Eq.(3.2) we have

$$\begin{aligned} x^{(n-2)}(t) &= a_{n-2} + a_{n-1}t + X^T P_\phi^2 \Phi_n(t), \\ &\vdots \\ \dot{x}(t) &= a_1 + a_2t + \frac{a_3}{2!}t^2 + \dots + \frac{a_{n-1}}{(n-1)!}t^{n-2} + X^T P_\phi^{n-1} \Phi_n(t), \\ x(t) &= a_0 + a_1t + \frac{a_2}{2!}t^2 + \dots + \frac{a_{n-1}}{(n-1)!}t^{n-1} + X^T P_\phi^n \Phi_n(t). \end{aligned} \tag{4.12}$$

Where  $P_\phi$  is obtained in Eq.(2.16).

By expanding  $t^j, j = 0, 1, \dots, n-1$  in term of Hermite polynomials we have

$$t^j = \frac{j!}{2^j} \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{H_{j-2i}(t)}{i!(j-2i)!}. \tag{4.13}$$

We can write Eq.(4.13) as following:

$$t^j = d_j \Phi_n(t), \tag{4.14}$$

where

$$d_j = \frac{j!}{2^j} \left\{ \begin{array}{l} [\frac{1}{2!}, 0, \frac{1}{(\frac{j}{2}-1)!2!}, 0, \dots, 0, \frac{1}{(\frac{j}{2}-i)!(2i)!}, 0, 0, \dots, 0], \text{ if } j \text{ even,} \\ [0, \frac{1}{[\frac{j}{2}]!}, 0, \frac{1}{([\frac{j}{2}]-1)!(2+1)!}, 0, \dots, 0, \frac{1}{([\frac{j}{2}]-i)!(2i+1)!}, 0, 0, \dots, 0], \text{ if } j \text{ odd.} \end{array} \right\} \tag{4.15}$$

By substituting Eq. (4.14) in Eq. (4.12) we obtain

$$x(t) = [a_0d_0 + a_1d_1 + \frac{a_2}{2!}d_2 + \dots + \frac{a_{n-1}}{(n-1)!}d_{n-1} + X^T P_\phi^n] \Phi_n(t).$$

So Eq. (3.1) can be rewritten as

$$Z[x(t)] = Z[X]. \tag{4.16}$$

The boundary conditions in Eq.(3.3) can be expressed as

$$q_k^1 = x^{(k)}(a) - a_k = 0, \tag{4.17}$$

$$q_k^2 = x^{(k)}(b) - b_k = 0, \tag{4.18}$$

where  $k = 0, \dots, n-1$ . We now find the extremum of Eq.(4.16) subject to Eqs. (4.17) and (4.19) using the Lagrange multiplier technique. Let

$$Z[x, \lambda_1, \lambda_2] = Z[X] + \lambda_1 Q_1 + \lambda_2 Q_2, \tag{4.19}$$

where  $Q_1$  and  $Q_2$  are of order  $(n \times 1)$  constant matrices. The necessary conditions for the extremum of (4.19) are

$$\nabla Z[X, \lambda_1, \lambda_2] = 0. \tag{4.20}$$

### 5. Illustrative Examples

This section is devoted to numerical examples. We implemented the proposed method in last section with MALAB (2017) in personal computer. To illustrate our technique, we present four numerical examples, and make a comparison with some of the results in the literatures.

**Example 1.** Consider the problem of finding the external of the functional [5]

$$Z = \int_0^1 [\dot{x}^2(t) + t\dot{x}(t)]dt, \quad (5.1)$$

The boundary conditions

$$x(0) = 0, x(1) = \frac{1}{4}. \quad (5.2)$$

The exact solution is obtained by using the Euler equation ( $\frac{\partial F}{\partial x} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}}) = 0$ ) as following:

$$Z_e = 0.16666666666666667, \quad x_e(t) = -\frac{1}{4}t^2 + \frac{1}{2}t, \quad (5.3)$$

where  $F(t, x(t), \dot{x}(t)) = \dot{x}^2(t) + t\dot{x}(t)$ . Here we solve this problem with Hermite polynomials by choosing  $n = 2$ . Let

$$x(t) = X\Phi_2(t), \quad (5.4)$$

$$t = d\Phi_2(t), \quad (5.5)$$

where  $X = [X_0, X_1, X_2]$  is unknown and  $d = [0, \frac{1}{2}, 0]$ . Using Eqs. (2.11) and (5.4) we get

$$\dot{x}(t) = X[D_\phi\Phi_2(t)], \quad (5.6)$$

where  $D_\phi$  is the operational matrix of derivative given in Eq. (2.12). By substituting Eqs. (5.5)-(5.6) in Eqs. (5.1) we obtain

$$\begin{aligned} Z(t, x(t)) &= \int_0^1 [\dot{x}^2(t) + t\dot{x}(t)]dt \\ &= \int_0^1 [(XD\Phi_2(t))(XD\Phi_2(t))^T + (d\Phi_2(t))(XD\Phi_2(t))^T]dt \\ &= XD[\int_0^1 \Phi_2(t)\Phi_2^T(t)dt]D^T X^T + d[\int_0^1 \Phi_2(t)\Phi_2^T(t)dt]D^T X^T \\ &= XDV D^T X^T + dV D^T X^T \\ &= Z(X), \end{aligned} \quad (5.7)$$

where  $V = \int_0^1 \Phi_2(t)\Phi_2^T(t)dt$ . Using the lagrange multiplier technigue to find extremum of Eq.(5.1) with boundary conditions Eq.(5.2) we have

$$Z(X, \lambda_1, \lambda_2) = Z(X) + \lambda_1 Q_1 + \lambda_2 Q_2, \quad (5.8)$$

where  $Q_1 = X\Phi_2(0) - 0$  and  $Q_2 = X\Phi_2(1) - \frac{1}{4}$ .  
 The necessary conditions are

$$\nabla Z(X, \lambda_1, \lambda_2) = 0. \tag{5.9}$$

We obtain the approximate solution as following

$$X = [-\frac{1}{8}, \frac{1}{4}, -\frac{1}{16}], Z = \frac{1}{6} \tag{5.10}$$

$$\begin{aligned} x_{app}(t) &= -\frac{1}{8}H_0(t) + \frac{1}{4}H_1(t) - \frac{1}{16}H_2(t) = -\frac{1}{8} + \frac{1}{4}(2t) - \frac{1}{16}(4t^2 - 2) \\ &= \frac{t}{2} - \frac{t^2}{4}, \end{aligned} \tag{5.11}$$

which is the exact solution.

**Example 2.** Consider the problem of finding the external of the functional [5]

$$Z = \int_0^1 [x^2(t) + t\dot{x}(t) + x^2(t)]dt, \tag{5.12}$$

The boundary conditions

$$x(0) = 0, x(1) = \frac{1}{4}. \tag{5.13}$$

The exact solutions of this problem are obtained by

$$x_e(t) = -\frac{e^{-t}(e^t - 1)(e - 2e^2 - 2e^t + e^{1+t})}{4(e^2 - 1)}, \tag{5.14}$$

$$Z_e = 0.1975939946587107. \tag{5.15}$$

For  $n = 3$  we get

$$X = [-\frac{219}{1892}, \frac{1899}{7568}, -\frac{219}{3784}, \frac{7}{1376}] \tag{5.16}$$

$$\begin{aligned} x_{app}(t) &= -\frac{219}{1892}H_0(t) + \frac{1899}{7568}H_1(t) - \frac{219}{3784}H_2(t) + \frac{7}{1376}H_3(t) \\ &= \frac{417}{946}t - \frac{219}{946}t^2 + \frac{7}{172}t^3. \end{aligned} \tag{5.17}$$

Table 1 and Figure 1 show respectively error of exact and approximate values of  $Z$  and plots of errors for state function for example 2.

Table 1: The approximate values of  $Z$ , for Example 2.

n	Presented Method	error
3	0.1975951374207188	1.1428e-006
4	0.1975939970022613	2.3435e-009
5	0.1975939946699765	1.1266e-011
6	0.1975939946587209	1.0186e-014
7	0.1975939946587107	5.4668e-018

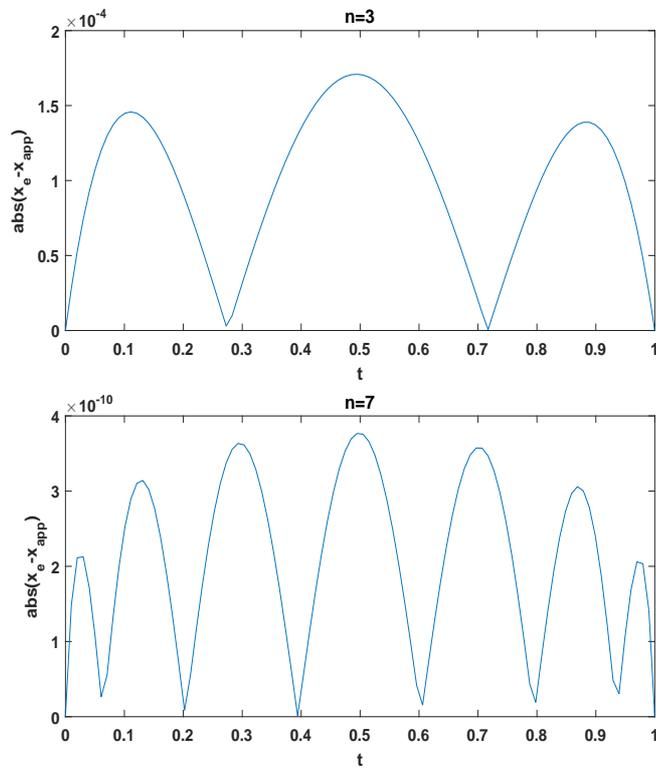


Figure 1: Plots of errors for exact and approximate solutions for Example 2( $n=3,7$ )

**Example 3.** Find the extremum of the functional [10]

$$Z = \int_0^1 \left[ \frac{1}{2} \ddot{x}^2(t) + 4(1-t)\dot{x}(t) \right] dt \tag{5.18}$$

with the conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \tag{5.19}$$

where the values of  $x(1)$  and  $\dot{x}(1)$  are unspecified. The exact solutions using Euler equation are

$$x_e(t) = -\frac{1}{6}t^4 + \frac{2}{3}t^3 - t^2, \quad Z_e = -0.4. \tag{5.20}$$

For  $n = 4$  we obtain

$$X = [-\frac{5}{8}, \frac{1}{2}, -\frac{3}{8}, \frac{1}{12}, -\frac{1}{96}], \tag{5.21}$$

$$\begin{aligned} x_{app}(t) &= -\frac{5}{8}H_0(t) + \frac{1}{2}H_1(t) - \frac{3}{8}H_2(t) + \frac{1}{12}H_3(t) - \frac{1}{96}H_4(t) \\ &= -\frac{5}{8} + \frac{1}{2}(2t) - \frac{3}{8}(4t^2 - 2) + \frac{1}{12}(8t^3 - 12t) - \frac{1}{96}(16t^4 - 48t^2 + 12) \\ &= -\frac{1}{6}t^4 + \frac{2}{3}t^3 - t^2 \end{aligned} \tag{5.22}$$

which is the exact solution, and also not required to Euler equation and natural boundary conditions.

**Example 4.** It has been studied by using bezier parameterization [34] and also bezier polynomials [1] for optimal control by differential evolution

$$\min Z = \int_0^1 [3x^2(t) + u^2(t)]dt \tag{5.23}$$

with boundary conditions

$$\dot{x} = x + u, x(0) = 1, \tag{5.24}$$

where  $x$  and  $u$  respectively are state and control functions. To solve it by presented method, we reducte optimal control problems to variational problems, therefore let  $u = \dot{x} - x$ . then Eqs. (5.23) and (5.24) can be writing as following

$$\min Z = \int_0^1 [3x^2(t) + (\dot{x}(t) - x(t))^2]dt, \tag{5.25}$$

with boundary conditions

$$x(0) = 1. \tag{5.26}$$

The exact solutions are obtained

$$x_e(t) = \frac{3e^{-4}}{3e^{-4} + 1}e^{2t} + \frac{1}{3e^{-4} + 1}e^{-2t}, Z_e = 2.791659975310063. \tag{5.27}$$

Let  $n = 3$  then we have

$$X = [\frac{104}{59}, -\frac{255}{236}, \frac{45}{118}, -\frac{35}{944}], \tag{5.28}$$

$$\begin{aligned}
 x_{app}(t) &= \frac{104}{59}H_0(t) - \frac{255}{236}H_1(t) + \frac{45}{118}H_2(t) - \frac{35}{944}H_3(t) \\
 &= \frac{104}{59} - \frac{255}{236}(2t) + \frac{45}{118}(4t^2 - 2) - \frac{35}{944}(8t^3 - 12t) \\
 &= -\frac{35}{118}t^3 + \frac{90}{59}t^2 - \frac{405}{236}t + 1.
 \end{aligned} \tag{5.29}$$

Table 1 and figure 2 show respectively error of exact and approximate values of  $Z$  and plots of errors for state function for example 4.

Table 2: Error and the approximate values of  $Z$ , for Example 4.

n	Presented Method	error
3	2.792372881355932	7.1290e-004
4	2.791662024685567	2.0516e-006
5	2.791660082922831	1.0761e-007
6	2.791659975445970	1.3590e-010
7	2.791659975313821	3.7578e-012
8	2.791659975310065	2e-015

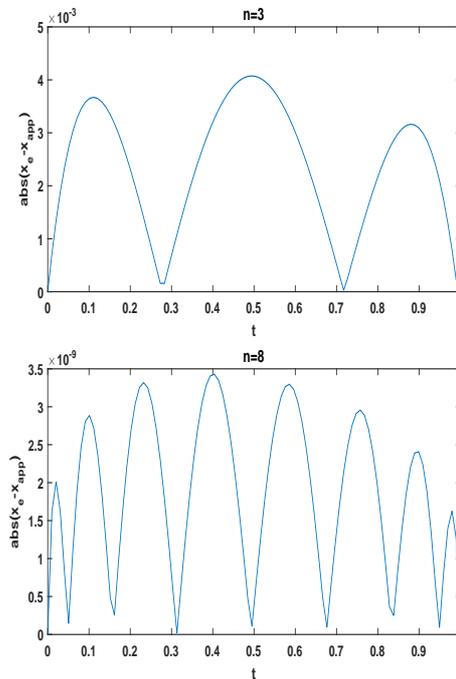


Figure 2: Plots of errors for exact and approximate solutions for Example 4( $n=3,8$ )

## 6. Conclusion

In this paper we presented a numerical scheme for solving variational problems. The Hermite polynomials were employed. Also several test problems were used to see the applicability and efficiency of the method. The obtained results show that the new approach can solve the problem effectively.

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