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On Generalized Weakly Symmetric Kenmotsu Manifolds

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ABSTRACT: This paper aims to introduce the notions of a generalized weakly symmetric Kenmotsu manifolds and a generalized weakly Ricci-symmetric Kenmotsu manifolds. The existence of a generalized weakly symmetric Kenmotsu manifold is ensured by a non-trivial example.

Key Words: Generalized weakly symmetric Kenmotsu manifolds, Generalized weakly Ricci-symmetric Kenmotsu manifolds.

Contents

1. Introduction

The notion of weakly symmetric Riemannian manifold have been introduced by Tamássy and Binh [12]. Thereafter, a lot of research has been carried out in this topic. For details, we refer to [21], [19], [5], [6], [15], [16], [1], [20], [7] and the references there in. In the sprit of [12], a Kenmotsu manifold $(M^n, g)(n > 2)$, is said to be a weakly symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and admits the identity

$$\begin{aligned}(\nabla_X \bar{R})(Y, U, V, W) &= A_1(X)\bar{R}(Y, U, V, W) \\ &+ B_1(Y)\bar{R}(X, U, V, W) + B_1(U)\bar{R}(Y, X, V, W) \\ &+ D_1(V)\bar{R}(Y, U, X, W) + D_1(W)\bar{R}(Y, U, V, X) \quad (1.1)\end{aligned}$$

where A_1, B_1 & D_1 are non-zero 1-forms defined by $A_1(X) = g(X, \sigma_1)$, $B_1(X) = g(X, \varrho_1)$ and $D_1(X) = g(X, \pi_1)$, for all X and $\bar{R}(Y, U, V, W) = g(R(Y, U)V, W)$, ∇ being the operator of the covariant differentiation with respect to the metric tensor g. An n-dimensional Kenmotsu manifold of this kind is denoted by $(WS)_n$-manifold.

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Keeping in tune with Dubey [17], we shall call a Kenmotsu manifold of dimension n, a generalized weakly symmetric (which is abbreviated hereafter as $(GWS)_n$-manifold) if it admits the equation

$$\begin{aligned}(\nabla_X \bar{R})(Y,U,V,W) = {} & A_1(X)\bar{R}(Y,U,V,W) + B_1(Y)\bar{R}(X,U,V,W) \\ & + B_1(U)\bar{R}(Y,X,V,W) + D_1(V)\bar{R}(Y,U,X,W) \\ & + D_1(W)\bar{R}(Y,U,V,X) + A_2(X)\bar{G}(Y,U,V,W) \\ & + B_2(Y)\bar{G}(X,U,V,W) + B_2(U)\bar{G}(Y,X,V,W) \\ & + D_2(V)\bar{G}(Y,U,X,W) + D_2(W)\bar{G}(Y,U,V,X) \quad (1.2)\end{aligned}$$

where

$$\bar{G}(Y,U,V,W) = [g(U,V)g(Y,W) - g(Y,V)g(U,W)] \quad (1.3)$$

and A_i, B_i & D_i are non-zero 1-forms defined by $A_i(X) = g(X,\sigma_i)$, $B_i(X) = g(X,\varrho_i)$, and $D_i(X) = g(X,\pi_i)$, for $i = 1,2$. The beauty of such $(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space [4] (for $A_i = B_i = D_i = 0$),

(ii) recurrent space [2](for $A_1 \neq 0$, $A_2 = B_i = D_i = 0$),

(iii) generalized recurrent space [17] ($A_i \neq 0$. $B_i = D_i = 0$),

(iv) pseudo symmetric space [13] (for $\frac{A_1}{2} = B_1 = D_1 = H_1 \neq 0$, $A_2 = B_2 = D_2 = 0$),

(v) generalized pseudo symmetric space [9] (for $\frac{A_i}{2} = B_i = D_i = H_i \neq 0$),

(vi) semi-pseudo symmetric space [14] (for $A_i = B_2 = D_2 = 0, B_1 = D_1 \neq 0$),

(vii) generalized semi-pseudo symmetric space [8] (for $A_i = 0, B_i = D_i \neq 0$),

(viii) almost pseudo symmetric space [13] (for $A_1 = H_1 + K_1, B_1 = D_1 = H_1 \neq 0$ and $A_2 = B_2 = D_2 = 0$),

(ix) almost generalized pseudo symmetric space [10] (or $A_i = H_i + K_i, B_i = D_i = H_i \neq 0$),

(x) weakly symmetric space [12] (for A_1, B_1, $D_i \neq 0$, $A_2 = B_2 = D_2 = 0$).

Our work is structured as follows. Section 2 is concerned with Kenmotsu manifolds and some known results. In section 3, we have investigated a generalized weakly symmetric Kenmotsu manifold and it is observed that such a space is an η-Einstein manifold provided $D_1(\xi)\neq$-1. We also tabled different type of curvature restrictions for which Kenmotsu manifolds are sometimes Einstein and some other time remain η-Einstein. Section 4, is concerned with a generalized weakly Ricci-symmetric Kenmotsu manifold which is also found to be η-Einstein space. Finally, we have constructed an example of ageneralized weakly symmetric Kenmotsu manifold.

2. Kenmotsu manifolds and some known results

Let M be a n-dimensional connected differentiable manifold of class C^∞-covered by a system of coordinate neighborhoods (U, x^h) in which there are given a tensor field ϕ of type $(1,1)$, a cotravariant vector field ξ and a $1-$form η such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = 1, \qquad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

for any vector field X on M. Then the structure (ϕ, ξ, η) is called contact structure and the manifold M^n equipped with such structure is said to be an almost contact manifold, if there is given a Riemannian compatible metric g such that

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for all vector fields X and Y , then we say M is an almost contact metric manifold. An almost contact metric manifold M is called a Kenmotsu manifold if it satisfies [11],

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi \ - \ \eta(Y)\phi(X), \tag{2.5}$$

for all vector fields X and Y, where ∇ is a Levi-Civita connection of the Riemannian metric. From the above it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$
$$(\nabla_X \ \eta)Y = g(X, Y) - \eta(X)\ \eta(Y), \tag{2.7}$$

In a Kenmotsu manifold the following relations hold ([18], [3]):

$$R(X, Y)\xi = \eta(X)\ Y - \eta(Y)X, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.9}$$
$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.10}$$
$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \tag{2.11}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor of the manifold.

3. Generalized weakly symmetric Kenmotsu manifold

In this section, we consider a generalized weakly symmetric Kenmotsu manifold $(M^n, g)(n > 2)$. Now, contracting Y over W in both sides of (1.2), we get

$$\begin{aligned}(\nabla_X S)(U, V) = \ & A_1(X)S(U, V) + B_1(U)S(X, V) + B_1(R(X, U)V) \\ & + D_1(R(X, V)U) + D_1(V)S(U, X) \\ & + (n-1)[A_2(X)g(U, V) + B_2(U)g(X, V) + D_2(V)g(U, X)] \\ & + B_2(G(X, U)V) + D_2(G(X, V)U).\end{aligned}$$

As a consequence of (2.8), (2.9) and (2.10) the above equatiom yields

$$\begin{aligned}(\nabla_X S)(U,\xi) \quad = \quad & -(n-1)A_1(X)\eta(U) - (n-2)B_1(U)\eta(X) \\ & +D_1(\xi)S(U,X) - \eta(U)B_1(X) - \eta(U)D_1(X) + g(X,U)D_1(\xi) \\ & +(n-1)[A_2(X)\eta(U) + B_2(U)\eta(X) + D_2(\xi)g(U,X)] \\ & +\eta(U)B_2(X) - \eta(X)B_2(U) + \eta(U)D_2(X) - g(U,X)D_2(\xi)\end{aligned}$$

for $V = \xi$. Again, replacing V by ξ, in the following identity

$$(\nabla_X S)(U,V) = \nabla_X S(U,V) - S(\nabla_X U,V) - S(U,\nabla_X V) \tag{3.1}$$

and then making use of (2.1), (2.6), (2.9), we find

$$(\nabla_X S)(U,\xi) = -(n-1)g(X,U) - S(U,X). \tag{3.2}$$

Now, using (3.2) in (3.1), we have

$$\begin{aligned} & -(n-1)g(X,U) - S(U,X) \\ = \; & -(n-1)A_1(X)\eta(U) - (n-2)B_1(U)\eta(X) \\ & +D_1(\xi)S(U,X) - \eta(U)B_1(X) + g(X,U)D_1(\xi) - \eta(U)D_1(X) \\ & +(n-1)[A_2(X)\eta(U) + B_2(U)\eta(X) + D_2(\xi)g(U,X)] \\ & +\eta(U)B_2(X) - \eta(X)B_2(U) + \eta(U)D_2(X) - g(U,X)D_2(\xi)\end{aligned} \tag{3.3}$$

which leaves

$$[A_1(\xi) + B_1(\xi) + D_1(\xi)] = [A_2(\xi) + B_2(\xi) + D_2(\xi)] \tag{3.4}$$

for $X = U = \xi$. In particular, if $A_2(\xi) = B_2(\xi) = B_2(\xi) = D_2(\xi) = 0$, (3.4) turns into

$$A_1(\xi) + B_1(\xi) + D_1(\xi) = 0. \tag{3.5}$$

This leads to the following

Theorem 3.1. *In a generalized weakly symmetric Kenmotsu manifold $(M^n, g)(n > 2)$, the relation (3.4) hold good.*

In a similar manner, we can have

$$\begin{aligned} & -(n-1)g(X,V) - S(V,X) \\ = \; & -(n-1)A_1(X)\eta(V) - (n-2)D_1(V)\eta(X)] \\ & +B_1(\xi)S(X,V) + g(X,V)B_1(\xi) - \eta(V)B_1(X) - \eta(V)D_1(X) \\ & +(n-1)[\{A_2(X)\}\eta(V) + B_2(\xi)g(X,V) + D_2(V)\eta(X)] \\ & +\eta(V)B_2(X) - g(X,V)B_2(\xi) + \eta(V)D_2(X) - \eta(X)D_2(V).\end{aligned} \tag{3.6}$$

Now, putting $V = \xi$ in (3.6) and using (2.1), (2.9), we obtain

$$\begin{aligned} & (n-1)A_1(X) + B_1(X) + D_1(X) + (n-2)[B_1(\xi) + D_1(\xi)]\eta(X) \\ = \; & [(n-1)A_2(X) + (n-2)\{B_2(\xi) + D_2(\xi)\}\eta(X)] + B_2(X) + D_2(X).\end{aligned} \tag{3.7}$$

Putting $X = \xi$ in (3.6) and using (2.1), (2.2), (2.9), we obtain

$$\begin{aligned} & (n-1)[A_1(\xi) + B_1(\xi)]\eta(V) + (n-2)D_1(V) + \eta(V)D_1(\xi) \\ = \; & (n-1)[\{A_2(\xi) + B_2(\xi)\}\eta(V) + D_2(V)] + \eta(V)D_2(\xi) - D_2(V). \end{aligned} \tag{3.8}$$

Replacing V by X in the above equation and using (3.4), we get

$$D_1(X) - D_1(\xi)\eta(X) = D_2(X) - D_2(\xi)\eta(X) \tag{3.9}$$

Moreover, in view of (3.4), (3.7) and (3.9), we get

$$B_1(X) - B_1(\xi)\eta(X) = B_2(X) - B_2(\xi)\eta(X). \tag{3.10}$$

Subtracting (3.9), (3.10) from (3.7), we get

$$A_1(X) + [B_1(\xi) + D_1(\xi)]\eta(X) = A_2(X) + \{B_2(\xi) + D_2(\xi)\}\eta(X)]. \tag{3.11}$$

Again, adding (3.9), (3.10) and (3.11), we get

$$A_1(X) + B_1(X) + D_1(X) = [A_2(X) + B_2(X) + D_2(X)]. \tag{3.12}$$

Next, for the choice of $A_2 = B_2 = C_2 = D_2 = 0$, the relation (3.12) yields

$$A_1(X) + B_1(X) + D_1(X) = 0. \tag{3.13}$$

This motivates us to state the followings

Theorem 3.2. *In a generalized weakly symmetric Kenmotsu manifold $(M^n, g)(n > 2)$, the sum of the associated 1-forms is given by (3.12).*

Theorem 3.3. *There does not exist a Kenmotsu manifold which is*
(i) recurrent,
(ii) generalized recurrent provided the 1-forms associated to the vector fields are collinear,
(iii) pseudo symmetric,
(iv) generalized semi-pseudo symmetric provided the 1-forms associated to the vector fields are collinear.

Again from (3.3), putting $X = \xi$, we have

$$\begin{aligned} & (n-1)[-\{A_1(\xi) - A_2(\xi)\} - \{D_1(\xi) - D_2(\xi)\}]\eta(U) \\ = \; & \{B_1(\xi) - B_2(\xi)\}\eta(U) + (n-2)\{B_1(U) - B_2(U)\} \end{aligned} \tag{3.14}$$

Using (3.4), above equation becomes

$$\{B_1(\xi) - B_2(\xi)\}\eta(U) = B_1(U) - B_2(U) \tag{3.15}$$

Setting $U = \xi$, we have

$$\begin{aligned} & -(n-1)\{A_1(X) - A_2(X)\} - \{B_1(X) - B_2(X)\} - \{D_1(X) - D_2(X)\} \\ = \; & -(n-2)\{A_1(\xi) - A_2(\xi)\} \end{aligned} \tag{3.16}$$

Using (3.4) in (3.16), we obtain

$$\{A_1(\xi) - A_2(\xi)\}\eta(X) = A_1(X) - A_2(X) \tag{3.17}$$

Again from (3.3), we have

$$\begin{aligned} S(U,X) = &\frac{[(n-1)\{A_1(X) - A_2(X)\}+\{B_1(X) - B_2(X)\}+\{D_1(X) - D_2(X)\}]}{[1+D_1(\xi)]}\eta(U) \\ &-\frac{[(n-1)\{1 + D_2(\xi)\} + D_1(\xi) - D_2(\xi)]}{[1+D_1(\xi)]}g(X,U) \\ &-\frac{(n-2)[B_1(U)-B_2(U)]}{[1+D_1(\xi)]}\eta(X) \end{aligned} \tag{3.18}$$

In view of (3.15), (3.17) and (3.18), we have

$$\begin{aligned} S(U,X) = &-\frac{[(n-1)\{1 + D_2(\xi)\} + D_1(\xi) - D_2(\xi)]}{[1+D_1(\xi)]}g(X,U) \\ &+\frac{(n-2)[\{A_1(\xi) - A_2(\xi)\}-\{B_1(\xi) - B_2(\xi)\}]}{[1+D_1(\xi)]}\eta(U)\eta(X) \end{aligned} \tag{3.19}$$

This leads to the followings

Theorem 3.4. *A generalized weakly symmetric Kenmotsu manifold is an η-Einstein space provided $D_1(\xi) \neq -1$.*

Theorem 3.5. *In a Kenmotsu manifold the following table hold good*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
locally symmetric space	*Einstein space*
locally recurrent space	*η-Einstein space*
generalized recurrent space	*η-Einstein space*
pseudo symmetric space	*η-Einstein space*
generalized pseudo symmetric space	*η-Einstein space*
semi-pseudo symmetric space	*η-Einstein space*
generalized semi-pseudo symmetric space	*η-Einstein space*
almost pseudo symmetric space	*η-Einstein space*
almost generalized pseudo symmetric space	*η-Einstein space*
weakly symmetric space	*η-Einstein space*

4. Generalized weakly Ricci-symmetric Kenmotsu manifold

A Kenmotsu manifold $(M^n, g)(n > 2)$, is said to be a generalized weakly Ricci-symmetric if there exist 1-forms $\bar{A}_i$, $\bar{B}_i$ and $\bar{D}_i$ which satisfy the condition

$$\begin{aligned}(\nabla_X S)(U,V) &= \bar{A}_1(X)S(U,V) + \bar{B}_1(U)S(X,V) + \bar{D}_1(V)S(U,X) \\ &\quad + \bar{A}_2(X)g(U,V) + \bar{B}_2(U)g(X,V) + \bar{D}_2(V)g(U,X).\end{aligned} \tag{4.1}$$

Putting $V = \xi$ in (4.1), we obtain

$$\begin{aligned}(\nabla_X S)(U,\xi) &= (n-2)[\bar{A}_1(X)\eta(U) + \bar{B}_1(U)\eta(X)] + \bar{D}_1(\xi)S(U,X) \\ &\quad + \bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U,X).\end{aligned} \tag{4.2}$$

In view of (3.2), the relation (4.2) becomes

$$\begin{aligned}&-(n-1)g(X,U) - S(U,X) \\ =\ &-(n-1)[\{\bar{A}_1(X) + \bar{B}_1(X)\}\eta(U) + \bar{B}_1(U)\eta(X)] + \bar{D}_1(\xi)S(U,X) \\ &+\bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U,X).\end{aligned} \tag{4.3}$$

Setting $X = U = \xi$ in (4.3) and using (2.1), (2.2) and (2.9), we get

$$\begin{aligned}&(n-1)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + B_1(\xi) + \bar{D}_1(\xi)] \\ =\ &[\bar{A}_2(\xi) + \bar{B}_2(\xi) + B_2(\xi) + \bar{D}_2(\xi)].\end{aligned} \tag{4.4}$$

Again, putting $X = \xi$ in (4.3), we get

$$\begin{aligned}&(n-1)[\{\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)\}\eta(U) + \bar{B}_1(U)] \\ =\ &[\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{D}_2(\xi)]\eta(U) + \bar{B}_2(U).\end{aligned} \tag{4.5}$$

Setting $U = \xi$ in (4.3) and then using (2.1), (2.2) and (2.9), we obtain

$$\begin{aligned}&(n-1)[\{\bar{A}_1(X) + \bar{B}_1(X)\} + \{B_1(\xi) + \bar{D}_1(\xi)\}\eta(X)] \\ =\ &\bar{A}_2(X) + B_2(\xi)\eta(X) + \bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.6}$$

Replacing U by X in (4.5) and then adding the resultant with (4.6), we have

$$\begin{aligned}&(n-1)[\bar{A}_1(X) + \bar{B}_1(X)] - [\bar{A}_2(X) + \bar{B}_2(X)] \\ =\ &-(n-1)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)]\eta(X) + [\bar{A}_2(\xi) + \bar{B}_2(\xi) \\ &\bar{D}_2(\xi)]\eta(X) - (n-1)\bar{D}_1(\xi)\eta(X) + \bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.7}$$

By virtue of (4.4), the above equation becomes

$$\begin{aligned}&(n-1)[\bar{A}_1(X) + \bar{B}_1(X)] + (n-1)\bar{D}_1(\xi)\eta(X) \\ =\ &[\bar{A}_2(X) + \bar{B}_2(X)] + \bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.8}$$

Next, putting $X = U = \xi$ in (4.1), we get

$$\begin{aligned}&(n-1)[\bar{A}_1(\xi) + \bar{B}_1(\xi)]\eta(V) + (n-1)\bar{D}_1(V) \\ =\ &[\bar{A}_2(\xi) + \bar{B}_2(\xi)]\eta(V) + \bar{D}_2(V).\end{aligned} \tag{4.9}$$

Replacing V by X in (4.9) and adding with (4.8), we obtain

$$\begin{aligned} & (n-1)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{D}_1(X)] \\ & (n-1)[\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{D}_1(\xi)]\eta(V) \\ = & [\bar{A}_2(X)+\bar{B}_2(X)+\bar{D}_1(X)]+[\bar{A}_2(\xi)+\bar{B}_2(\xi)+\bar{D}_2(\xi)]\eta(V). \end{aligned} \tag{4.10}$$

By virtue of (4.4), the above equatin becomes

$$(n-1)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{D}_1(X)] = [\bar{A}_2(X)+\bar{B}_2(X)+\bar{D}_1(X)]. \tag{4.11}$$

This leads to the followings

Theorem 4.1. *In a generalized weakly Ricci symmetric Kenmotsu manifold* (M^n, g) $(n > 2)$*, the sum of the associated 1-forms are related by (4.11).*

Again from(4.3), we have

$$\begin{aligned} S(U,X) = & \frac{1}{-[1+\bar{D}_1(\xi)]}[\bar{D}_2(\xi)+(n-1)]g(X,U) \\ & +\frac{1}{[1+\bar{D}_1(\xi)]}[(n-1)\{\bar{A}_1(X)+\bar{B}_1(X)\}-\bar{A}_2(X)]\eta(U) \\ & +\frac{1}{[1+\bar{D}_1(\xi)]}[(n-1)\bar{B}_1(U)-\bar{B}_2(U)]\eta(X). \end{aligned} \tag{4.12}$$

From (4.6), we have

$$\begin{aligned} & (n-1)[\{\bar{A}_1(X)+\bar{B}_1(X)\}-\bar{A}_2(X)] \\ = & [-(n-1)\{B_1(\xi)+\bar{D}_1(\xi)\}+B_2(\xi)+\bar{D}_2(\xi)]\eta(X). \end{aligned} \tag{4.13}$$

Using (4.4) in (4.5), we have

$$(n-1)\bar{B}_1(U)-\bar{B}_2(U) = -(n-1)\bar{B}_1(\xi)+\bar{B}_2(\xi) \tag{4.14}$$

In view of (4.12), (4.13) and (4.14), we have

$$\begin{aligned} & S(U,X) \\ = & \frac{[\bar{D}_2(\xi)+(n-1)]}{-[1+\bar{D}_1(\xi)]}g(X,U) \\ & +\frac{[-2\{(n-1)B_1(\xi)-B_2(\xi)\}-\{(n-1)\bar{D}_1(\xi)-\bar{D}_2(\xi)\}]}{[1+\bar{D}_1(\xi)]}\eta(U)\eta(X). \end{aligned} \tag{4.15}$$

This leads to the followings

Theorem 4.2. *A generalized weakly Ricci symmetric Kenmotsu manifold is an* η*-Einstein space provided* $D_1(\xi) \neq -1$.

Theorem 4.3. *In a Kenmotsu manifold the following table hold good*

Type of curvature restriction	*Nature of the space corresponding to curvature restriction*
locally Ricci symmetric space	*Einstein space*
locally Ricci recurrent space	η*-Einstein space*
generalized Ricci recurrent space	η*-Einstein space*
pseudo Ricci symmetric space	η*-Einstein space*
generalized pseudo Ricci symmetric space	η*-Einstein space*
Ricci semi-pseudo symmetric space	η*-Einstein space*
generalized semi-pseudo Ricci symmetric space	η*-Einstein space*
almost pseudo Ricci symmetric space	η*-Einstein space*
almost generalized pseudo Ricci symmetric space	η*-Einstein space*
weakly Ricci symmetric space	η*-Einstein space*

5. Example of an $(\mathbf{GWS})_3$ Kenmotsu manifold

(see [18], page 21-22) Let $M^3(\phi, \xi, \eta, g)$ be a Kenmotsu manifold (M^3, g) with a ϕ-basis

$$e_1 = e^{-z}\frac{\partial}{\partial x},\ e_2 = e^{-z}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Then from Koszul's formula for Riemannian metric g, we can obtain the Levi-Civita connection as follows

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\bar{R}(e_1, e_3, e_1, e_3) = \bar{R}(e_2, e_3, e_2, e_3) = 1 = \bar{R}(e_1, e_2, e_1, e_2).$$

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X, Y, U,\ V \in \chi(M)$ can be written as

$$
\begin{aligned}
X &= \sum_1^3 a_i e_i,\ Y = \sum_1^3 b_i e_i,\ U = \sum_1^3 c_i e_i,\ \ V = \sum_1^3 d_i e_i, \\
\bar{R}(X,Y,U,V) &= (a_1b_2 \ - a_2b_1)(c_1d_2 \ - c_2d_1) + (a_1b_3 \ - a_3b_1)(c_1d_3 \\
&\quad - c_3d_1) + (a_2b_3 \ - a_3b_2)(c_2d_3 \ - c_3d_2) \\
&= T_1 \text{ (say)} \\
\bar{R}(e_1,Y,U,V) &= b_3(c_1d_3 \ - c_3d_1) + b_2\ (c_1d_2 \ - c_2d_1) = \lambda_1 \text{ (say)} \\
\bar{R}(e_2,Y,U,V) &= b_3(c_2d_3 \ - c_3d_2) - b_1\ (c_1d_2 \ - c_2d_1) = \lambda_2 \text{ (say)} \\
\bar{R}(e_3,Y,U,V) &= b_1(c_3d_{1-}c_1d_3\ ) + b_2\ (c_3d_2 - c_2d_3) = \lambda_3 \text{ (say)} \\
\bar{R}(X,e_1,U,V) &= a_3(c_1d_3 \ - c_3d_1) + a_2\ (c_1d_2 \ - c_2d_1) = \lambda_4 \text{ (say)} \\
\bar{R}(X,e_2,U,V) &= a_3(c_2d_3 \ - c_3d_2) + a_1\ (c_2d_1 \ - c_1d_2) = \lambda_5 \text{ (say)} \\
\bar{R}(X,e_3,U,V) &= a_1(c_3d_{1-}c_1d_3\ ) + a_2\ (c_3d_2 - c_2d_3\ ) = \lambda_6 \text{ (say)} \\
\bar{R}(X,Y,e_1,V) &= d_3(a_1b_3 \ - a_3b_1) + d_2\ (a_1b_2 - a_2b_1\ )(= \lambda_7 \text{ (say)} \\
\bar{R}(X,Y,e_2,V) &= d_3(a_2b_3 \ - a_3b_2) + d_1\ (a_2b_1 - a_1b_2\ ) = \lambda_8 \text{ (say)} \\
\bar{R}(X,Y,e_3,V) &= d_1(a_3b_1 - a_1b_3\ ) + d_2\ (a_3b_2 - a_2b_3) = \lambda_9 \text{ (say)} \\
\bar{R}(X,Y,U,e_1) &= c_3(a_1b_3 \ - a_3b_1) + c_2\ (a_1b_2 - a_2b_1\ ) = \lambda_{10} \text{ (say)} \\
\bar{R}(X,Y,U,e_2) &= c_3(a_2b_3 - a_3b_2\ ) + c_1\ (a_2b_1 - a_1b_2\ ) = \lambda_{11} \text{ (say)} \\
\bar{R}(X,Y,U,e_3) &= c_1(a_3b_1 - a_1b_3) + c_2\ (a_3b_2 - a_2b_3\ ) = \lambda_{12} \text{ (say)} \\
\bar{G}(X,Y,U,V) &= (b_1c_1 + b_2c_2 - b_3c_3)(a_1d_1 \ + a_2d_2 - a_3d_3) \\
&\quad -(a_1c_1 + a_2c_2 - a_3c_3)(b_1d_1 \ + b_2d_2 - b_3d_3) = T_2 \text{ (say)} \\
\bar{G}(e_1,Y,U,V) &= (b_2c_2 - b_3c_3)d_1 - (b_2d_2 - b_3d_3)c_1 = \omega_1 \text{ (say)} \\
\bar{G}(e_2,Y,U,V) &= (b_1c_1 - b_3c_3)d_2 - (b_1d_1 \ - b_3d_3)c_2 = \omega_2 \text{ (say)} \\
\bar{G}(e_3,Y,U,V) &= (b_1c_1 - b_2c_2)d_3 - (b_1d_1 \ - b_2d_2)c_3 = \omega_3 \text{ (say)} \\
\bar{G}(X,e_1,U,V) &= (a_2d_2 - a_3d_3)c_1 - (a_2c_2 - a_3c_3)d_1 = \omega_4 \text{ (say)} \\
\bar{G}(X,e_2,U,V) &= (a_1d_1 - a_3d_3)c_2 - (a_1c_1 - a_3c_3)d_2 = \omega_5 \text{ (say)} \\
\bar{G}(X,e_3,U,V) &= (a_1d_1 \ - a_2d_2)c_3 - (a_1c_1 - a_2c_2)d_3 = \omega_6 \text{ (say)} \\
\bar{G}(X,Y,e_1,V) &= (a_2d_2 - a_3d_3)b_1 - (b_2d_2 - b_3d_3)a_1 = \omega_7 \text{ (say)} \\
\bar{G}(X,Y,e_2,V) &= (a_1d_1 \ - a_3d_3)b_2 - (b_1d_1 - b_3d_3)a_2 = \omega_8 \text{ (say)} \\
\bar{G}(X,Y,e_3,V) &= (b_1d_1 - b_2d_2)a_3 - (a_1d_1 \ - a_2d_2)b_3 = \omega_9 \text{ (say)} \\
\bar{G}(X,Y,U,e_1) &= (b_2c_2 - b_3c_3)a_1 - (a_2c_2 - a_3c_3)b_1, = \omega_{10} \text{ (say)} \\
\bar{G}(X,Y,U,e_2) &= (b_1c_1 - b_3c_3)a_2 - (a_1c_1 - a_3c_3)b_2 = \omega_{11} \text{ (say)} \\
\bar{G}(X,Y,U,e_3) &= (b_1c_1 - -b_2c_2)a_3 - (a_1c_1 + a_2c_2)b_3 = \omega_{12} \text{ (say)}
\end{aligned}
$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$\begin{aligned}(\nabla_{e_1}\bar{R})(X,Y,U,V) &= -a_1\lambda_3 \ + a_3\lambda_2 \ - b_1\lambda_6 \ + b_3\lambda_5 \\ &\quad -c_1\lambda_9 + \ c_3\lambda_8 - d_1\lambda_{12} \ + d_3\lambda_{11}, \\ (\nabla_{e_2}\bar{R})(X,Y,U,V) &= = -a_2\lambda_3 + a_3\lambda_2 - b_2\lambda_6 + b_3\lambda_5 \\ &\quad -c_2\lambda_9 + c_3\lambda_8 d_3\lambda_{11} - d_2\lambda_{12}+, \\ (\nabla_{e_3}\bar{R})(X,Y,U,V) &= 0.\end{aligned}$$

Depending on the following choice of the the 1-forms

$$\begin{aligned}A_1(e_1) &= \frac{a_3\lambda_2 \ - a_1\lambda_3 + b_3\lambda_5 \ - b_1\lambda_6}{T_1}, \\ A_2(e_1) &= \frac{c_3\lambda_8 \ - c_1\lambda_9 + d_3\lambda_{11} - d_1\lambda_{12}}{T_2}, \\ A_1(e_2) &= \frac{a_3\lambda_2 - a_2\lambda_3 + b_3\lambda_5 - b_2\lambda_6}{T_1}, \\ A_2(e_2) &= \frac{c_3\lambda_8 - c_2\lambda_9 + d_3\lambda_{11} - d_2\lambda_{12}}{T_2}, \\ B_1(e_3) &= \frac{1}{a_3\lambda_3 + b_3\lambda_6}, \ B_2(e_3) = \frac{1}{a_3\theta_3 + b_3\theta_6}, \\ D_1(e_3) &= -\frac{1}{c_3\lambda_9 + d_3\lambda_{12}}, D_2(e_3) = -\frac{1}{c_3\theta_9 + d_3\theta_{12}},\end{aligned}$$

one can easily verify the relations

$$\begin{aligned}(\nabla_{e_i}\bar{R})(X,Y,U,V) &= A_1(e_i)\bar{R}(X,Y,U,V) + B_1(X)\bar{R}(e_i,Y,U,V) \\ &\quad +B_1(Y)\bar{R}(X,e_i,U,V) + D_1(U)\bar{R}(X,Y,e_i,V) \\ &\quad +D_1(V)\bar{R}(X,Y,U,e_i) + A_2(e_i)\bar{G}(X,Y,U,V) \\ &\quad +B_2(X)\bar{G}(e_i,Y,U,V) + B_2(Y)\bar{G}(X,e_i,U,V) \\ &\quad +D_2(U)\bar{G}(X,Y,e_i,V) + D_2(V)\bar{G}(X,Y,U,e_i)\end{aligned}$$

for$1, 2, 3$. From the above, we can state that

Theorem 5.1. *There exixt a Kenmotsu manifold* (M^3, g) *which is a generalized weakly symmetry Kenmotsu manifold.*

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