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Extending the Applicability of Newton's and Secant Methods Under Regular Smoothness

Ioannis K. Argyros, Santhosh George and Shobha M. Erappa

ABSTRACT: The concept of regular smoothness has been shown to be an appropriate and powerfull tool for the convergence of iterative procedures converging to a locally unique solution of an operator equation in a Banach space setting. Motivated by earlier works, and optimization considerations, we present a tighter semi-local convergence analysis using our new idea of restricted convergence domains. Numerical examples complete this study.

Key Words: Newton's method, Secant method, Majorizing operator, Regular smoothness, Kantorovich hypothesis.

Contents

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x_∞ of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where, F is a Fréchet–differentiable operator defined on a open convex subset Ω of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$.

The most popular iterative procedures for generating a sequence approximating x_∞ are undoubtely Newton's method:

$$x_{n+1} = x_n - F'(x_n)\ F(x_n) \quad (x_0 \in \Omega), \quad (n \geq 0), \tag{1.2}$$

and the Secant method:

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1}\ F(x_n) \quad (x_{-1},\, x_0 \in \Omega), \quad (n \geq 0). \tag{1.3}$$

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Here, $F'(x) \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ denotes the Fréchet–derivative of operator F, and $[x, y; F] \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ the divided difference of order one at $x \in \Omega$ and $y \in \Omega$ [3], [9]–[18]. Newton's method requires one function evaluation, and the computation of one inverse at each step. It is self–correcting, and has a quadratic convergence under natural conditions [1]–[9].

The Secant method has some attractive properties: it is also self–correcting, it exhibits superlinear convergence, and no knowledge of the derivatives of the operators involved is required. These facts not only make the Secant iteration much cheaper than Newton's, but also makes this method applicable to equations with nondifferentiable operators [3], [9]–[18].

In the one dimensional case, the Secant method is of higher efficiency than the corresponding Newton's method. A convergence analysis for both methods has been provided under various assumptions by many authors. A survey of such results can be found in [3], [9], and the references there (see, also [1], [2], [4]–[8], [10]–[18]). In the excellent works by Galperin [6], [7], the concept of regular smoothness was introduced, which became a viable framework for the study of the convergence of iterative procedures such as Newton's method, and Secant method. This way, the applicability of these methods was extended, and in the case of Newton's method or the Secant method tighter than before error bounds on the distances involved were found.

The convergence domain for such methods is small in general. In present study, we extend the convergence domain for Newton's method and the Secant method. To achieve this goal, we first introduce the center-Lipschitz condition which determines a subset of the original domain for the operator containing the iterates. The scalar functions are then related to the subset instead of the original domain. This way, the scalar functions are more precise than if they were depending on the original domain of the mapping as in earlier studies. The new technique leads to: weaker sufficient convergence conditions tighter error bounds on the distances involved and an at least as precise information on the location of the solution. These advantages are obtained under the same computational cost as in earlier studies, since in practice the new functions are special cases of the old functions, that can be used to study other requiring inverses of linear mappings.

We also show that the sufficient convergence conditions can be weakened, indicating that the regular smoothness approach, does not necessarily lead to weakest possible or usable sufficient convergence conditions.

The rest of the paper is organized as follows. Section 2, 3 contain the semilocal convergence of Newton's method and the Secant method, respectively. The numerical examples are presented in the concluding section 4.

2. Semilocal convergence analysis for Newton's method

Let $\mathcal{T}$ denote the class of nondecreasing continuous functions $v : [0, \infty) \longrightarrow [0, \infty)$, that have convex subgraphs $\{(s, t) : s \geq 0 \text{ and } t \leq v(s)\}$, and vanish at zero, i.e., they are concave [3], [6], [7]. The functions of $\mathcal{T}$ have left– and right–hand derivatives at each $s > 0$ (being monotone), and they coincide everywhere except, perhaps, for a countable number of points.

Definition 2.1. *[9] Denote by $\underline{h}(F)$ the $\inf_{x \in \mathcal{D}} \| F'(x) \|$. Given an $\omega_1 \in \mathfrak{T}$, we say that F is ω_1–regularly smooth on Ω, or, equivalently, that ω_1 is a regular smoothness modulus of F on Ω, if there exists an $\underline{h} \in [0, \underline{h}(F)]$, such that the inequality*

$$\omega_1^{-1}(h_F(x,y) + \| F'(y) - F'(x) \|) - \omega_1^{-1}(h_F(x,y)) \leq \| y - x \|, \tag{2.1}$$

holds for all $x, y \in \Omega$, where,

$$h_F(x,y) = \min\{\| F'(x) \|, \| F'(y) \|\} - \underline{h}. \tag{2.2}$$

The operator is regularly smooth on Ω, if it is ω_1–regularly smooth there for some $\omega_1 \in \mathfrak{T}$.

We denote by ω_1^{-1}, a function whose closed epigraph $cl\,\{(s,t) : s \geq 0 \text{ and } t \geq \omega_1^{-1}(s)\}$ is symmetrical to closure of the subgraph of ω_1 with respect to the axis $t = s$. Clearly, ω_1^{-1} is a convex function on $[0, \infty)$ vanishing at zero, continuously increasing in $[0, \omega_1(\infty))$, and equal to ∞ for all $s > \omega_1(\infty)$. In view of the convexity of ω_1^{-1}, each ω_1–regularly smooth operator is also ω_1–smooth, in the sense that:

$$\| F'(x) - F'(y) \| \leq \omega_1(\| x - y \|), \quad \text{for all } x, y \in \Omega. \tag{2.3}$$

However, the converse is not true [7].

Definition 2.2. *Let $x_0 \in \Omega$ be fixed. Given an $\omega_0 \in T$, we say that ω_0 is a center-regular smoothness modulus of F on Ω at x_0, if there exists $\underline{h} \in [0, \underline{h}(F)]$ such that:*

$$\omega_0^{-1}(h_F(x,x_0) + \| F'(x_0) - F'(x) \|) - \omega_0^{-1}(h_F(x,x_0)) \leq \| x_0 - x \|, \tag{2.4}$$

for all $x \in \Omega$.

Clearly, as in (2.1) and (2.3), we have by (2.4) that

$$\| F'(x_0) - F'(x) \| \leq \omega_0(\| x_0 - x \|) \tag{2.5}$$

for all $x \in \Omega$.

Note that in general

$$\omega_0(t) \leq \omega_1(t) \qquad t \geq 0, \tag{2.6}$$

holds, $\dfrac{\omega_1(t)}{\omega_0(t)}$ can be arbitrarily large [1]–[5] and (2.1) implies (2.4) but not necessarily vice versa.

It turns out that when upper bounds on the norms $\| F'(x)^{-1} \|$ are to be found the stronger condition (2.1) (or (2.3)) is used in the literature [1]–[2], [6]–[18], instead of the needed condition (2.4) (or (2.5)), which is weaker, and tighter.

Let $U(z,\rho)$, $\overline{U}(z,\rho)$, stand, respectively for the open and closed balls in X with center $z \in X$ and of radius $\rho > 0$.

Define parameter r_0 and the set Ω_0, respectively by

$$r_0 = sup\{t \geq 0 : w_0(t) < 1\} \tag{2.7}$$

and

$$\Omega_0 = \Omega \cap U(x_0, r_0) \tag{2.8}$$

Definition 2.3. *Given an $w \in T$, we say that w is a regular smoothness modulus of F on Ω_0, if there exists an $\underline{h}_0 \in [0, \underline{h}_0(F)]$, such that the inequality*

$$\omega^{-1}(\underline{h}_F^0(x,y) + \|F'(y) - F'(x)\|) - \omega^{-1}(\underline{h}_F^0(x,y)) \leq \|y - x\| \tag{2.9}$$

holds for all $x, y \in \Omega_0$, where $\underline{h}_0(F)$ denotes the $\inf\limits_{x\in\Omega_0} \|F'(x)\|$ and

$$\underline{h}_F^0(x,y) = min\{\|F'(x)\|, \|F'(y)\|\} - \underline{h}_0 \tag{2.10}$$

holds for all $x, y \in \Omega_0$.

We have that

$$\omega(t) \leq \omega_1(t), \qquad t \geq 0, \tag{2.11}$$

since $\Omega_0 \subseteq \Omega$.

The construction of function ω depends on function ω_0. The creation of function ω was not possible before in the studies using only function ω_1 [1], [2], [5]–[18]. Clearly, in these studies ω can replace ω_1 leading to the advantages as stated previously, when strict inequality holds in (2.1).

These advantages are obtained under the same computational cost, since in practice the computation of function ω_1 requires the computation of functions ω_0 and ω_1 as special cases. From now on we assume that

$$\omega_0(t) < \omega(t) \text{ for all } t \in [0, r_0). \tag{2.12}$$

We shall show the advantages first for Newton's method. It is convenient for us to adopt some of the new standard notation in [3,6,7], so we can make the comparison between the two works easier to carry out.

Let $\omega \in \mathcal{T}$, and $\Omega(t) = \displaystyle\int_0^t \omega(\tau)\, d\tau$.

Denote:

$$\theta(\sigma,\tau) := \min\{\tau,\, \sigma - \tau\}, \quad m(u,v,\tau) = \min\{u,\, (u - \theta(u - v, \tau))^+\},$$

and

$$\psi(u,v,w) = \int_0^w (\omega(m(u,v,\tau) + \tau) - \omega(m(u,v,\tau)))\, d\tau, \qquad u, w > 0, \tag{2.13}$$

where,

$$a^+ = \max\{a,\, 0\}. \tag{2.14}$$

Denote also by ψ_0 the function given in (2.13) with ω_0 replacing ω.
Set

$$F_0 = F'(x_0)^{-1}\ F \tag{2.15}$$

for some $x_0 \in \Omega$, such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.
Clearly, the Newton iterations for F and F_0 are identical.
Let $\underline{h}_0$ be a lower bound for $\underline{h}_0(F_0)$:

$$0 \leq \underline{h}_0 \leq \underline{h}_0(F_0), \tag{2.16}$$

and let $\omega_0, \omega \in \mathcal{T}$ satisfy (2.4), (2.9), respectively with F_0 replacing F.
Define the constant:

$$\kappa = \omega_0^{-1}(1 - \underline{h}), \tag{2.17}$$

and consider an upper bound a for the norm:

$$\| F_0(x_0) \| \leq a. \tag{2.18}$$

Moreover, define the sequence of triplets $(\alpha_n, \gamma_n, \delta_n)$ $(n \geq 0)$ by:

$$\begin{aligned}
\alpha_0 = \kappa,\ \gamma_0 = 1,\ \delta_0 = a,\\
\alpha_n = (\alpha_{n-1} - \delta_{n-1})^+,\\
\gamma_n = 1 - \omega_0(\alpha_n + t_n) + \omega_0(\alpha_n),\\
\delta_n = \gamma_n^{-1}\ \psi(\alpha_{n-1}, \alpha_{n-1} - \delta_{n-1}, \delta_{n-1}),
\end{aligned} \tag{2.19}$$

where,

$$t_n = \sum_{i=0}^{n-1} \delta_i.$$

If $\omega(t) = c\ t$, and $\omega_0(t) = c_0\ t$, then, we have the sequence:

$$t_{n+1} = t_n + \frac{c\ (t_n - t_{n-1})^2}{2\ (1 - c_0\ t_n)} \tag{2.20}$$

and the triplet is:

$$\alpha_n = \kappa - t_n, \quad \gamma_n = 1 - c_0\ t_n, \quad \delta_n = t_{n+1} - t_n. \tag{2.21}$$

Sequence $\{t_n\}$ further reduces to the one considered by Kantorovich for $c = c_0$:

$$s_{n+1} = s_n + \frac{c\ (s_n - s_{n-1})^2}{2\ (1 - c\ s_n)} = \frac{.5\ c\ s_n^2 - s_n + a}{1 - c\ s_n}. \tag{2.22}$$

The analog of the Kantorovich's majorant function $t \longrightarrow .5\ c\ t^2 - t + a$ is given by:

$$\phi_{\underline{h}}(t) = a - t + \psi(\kappa, (\kappa - t)^+ - t, t), \quad t > 0. \tag{2.23}$$

Denote also by $\phi_{\underline{h}}^0$, function $\phi_{\underline{h}}$, with ψ replaced by ψ_0.

The triple $(\alpha_n, \gamma_n, \delta_n)$ given in (2.13) is well defined, if $\gamma_n > 0$. This is true if:

$$t_n < \omega_0^{-1}(1) \qquad (n \geq 0). \tag{2.24}$$

In this case, $\delta_n \geq 0$, and $\alpha_n = (\kappa - t_n)^+$.

As shown in [7] $\phi_{\underline{h}}$ is decreasing on $[0, \omega^{-1}(1)]$, and is increasing on $[\omega^{-1}(1), +\infty)$. Denote by $\phi_{\underline{h},1}^{-1}$, $\phi_{\underline{h},2}^{-1}$ the inverse of the restrictions of $\phi_{\underline{h}}$ to $[0, \omega^{-1}(1)]$, and to $[\omega^{-1}(1), \infty)$, respectively. Similarly, for the inverses of $\phi_{\underline{h},1}^0$, and $\phi_{\underline{h},2}^0$, which are defined on $[0, \omega_0^{-1}(1)]$, $[\omega_0^{-1}(1), \infty)$, respectively.

We can show the main semilocal convergence result for Newton's method.

Theorem 2.4. *Let $F_0 : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a ω_0 and ω–regularly smooth.*

Assume: Conditions (2.12), (2.24)

and

$$\overline{U}(x_0, t_\infty) \subseteq \mathcal{D} \tag{2.25}$$

hold, where

$$t_\infty = \lim_{n\to\infty} t_n. \tag{2.26}$$

Then, the sequence $\{x_n\}$ $(n \geq 0)$, generated by Newton's method (1.2) is well defined, remains in $\overline{U}(x_0, t_\infty)$ for all $n \geq 0$, and converges to a solution x_∞ of equation $F(x) = 0$.

Moreover, the following estimates hold:

$$\| F_0'(x_n) \| \geq \omega_0(\alpha_n) + \underline{h}, \tag{2.27}$$

$$\overline{\gamma_n}^{-1} \leq \gamma_n^{-1}, \tag{2.28}$$

$$\overline{\delta_n} \leq \gamma_n^{-1}\ \psi(\alpha_{n-1}, \alpha_n - \delta_{n-1}, \delta_{n-1}), \tag{2.29}$$

$$\Delta_{n+1} = \| x_{n+1} - x_\infty \| \leq \overline{\gamma_n}^{-1}\ \psi(\overline{\alpha_n}, \omega_0^{-1}(\| F_0'(x_\infty) \| - \underline{h}) - \Delta_n, \Delta_n), \tag{2.30}$$

and

$$\| x_n - x_\infty \| \leq t_\infty - t_n, \tag{2.31}$$

where, $\overline{\gamma_n}$ and $\overline{\delta_n}$ are defined in (2.32).

Furthermore, if a is such that $t_\infty \leq \kappa$, then the solution x_∞ is unique in $\overline{U}(x_0, \phi_{\underline{h},2}^{0,-1}(0))$.

Proof. It is convenient for us to set:

$$\overline{\alpha_n} = \omega_0^{-1}(\| F_0'(x_n) \| - \underline{h}_0), \quad \overline{\gamma_n} = \| F_0'(x_n)^{-1} \|^{-1}, \quad \overline{\delta_n} = \| x_{n+1} - x_n \|. \tag{2.32}$$

These numbers are well defined, and the relations:

$$\overline{\alpha_0} = \kappa = \alpha_0, \quad \overline{\gamma_0} = 1 = \gamma_0, \quad \overline{\delta_0} \leq a = \delta_0 \tag{2.33}$$

hold.

Suppose that $n \geq 1$, and for all $k \leq n-1$, the following statements hold: $F_0'(x_k)$ exists, $F_0'(x_k)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and

$$\overline{\alpha_k} \geq \alpha_k, \quad \overline{\gamma_k} \geq \gamma_k, \quad \overline{\delta_k} \leq \delta_k. \tag{2.34}$$

We then have:

$$\| x_n - x_0 \| \leq \sum_{k=0}^{n-1} \overline{\delta_k} \leq \sum_{k=0}^{n-1} \delta_k = t_n, \tag{2.35}$$

so that $F_0'(x_n)$ exists.

Moreover, due to ω_0–regular smoothness of F_0 on $U(x_0, t_\infty)$, we have in turn:

$$\begin{array}{rcl} \| F_0'(x_n) - F_0'(x_0) \| & \leq & \omega_0(\omega_0^{-1}(\min\{\| F_0'(x_0) \|, \| F_0'(x_n) \|\} - \underline{h}_0) \\ & & + \| x_n - x_0 \|) - \min\{\| F_0'(x_0) \|, \| F_0'(x_n) \|\} + \underline{h}_0 \\ & = & \omega_0(\min\{\kappa, \overline{\alpha_n}\} + \| x_n - x_0 \|) - \omega_0(\min\{\kappa, \overline{\alpha_n}\}) \\ & \leq & \omega_0(\min\{\kappa, \alpha_n\} + \| x_n - x_0 \|) - \omega_0(\min\{\kappa, \alpha_n\}) \\ & \leq & \omega_0(\alpha_n + t_n) - \omega_0(\alpha_n). \end{array} \tag{2.36}$$

We also have $\alpha_n \leq \alpha_0 = \kappa$, and $\sum_{k=0}^{n-1} \overline{\delta_k} \leq \delta_k$, which lead to

$$\begin{array}{rcl} \overline{\gamma_n} = d(F_0'(x_n)) & \geq & d(F_0'(x_0)) - \| F_0'(x_n) - F_0'(x_0) \| \\ & \geq & 1 - \omega_0(\alpha_n + t_n) + \omega_0(\alpha_n) \\ & \geq & 1 - \omega_0(t_n) > 0 \qquad (\text{by } (2.18)). \end{array} \tag{2.37}$$

It follows from (2.37), and the Banach lemma of invertible operators [11], [2], [4], [5], that $F_0'(x_n)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, so that (2.27), and (2.28) hold.

Note that in [7], less precise estimates were obtained with ω replacing ω_0 in estimates (2.36), and (2.37).

Using the identity

$$x_{n+1} - x_n = F_0'(x_n)^{-1} \, (F_0(x_n) - F_0(x_{n-1}) - F_0'(x_{n-1}) \, (x_n - x_{n-1})), \tag{2.38}$$

and (2.28), we obtain:

$$\begin{array}{rcl} \overline{\delta_n} & \leq & \overline{\gamma_n}^{-1} \, \psi(\overline{\alpha_{n-1}}, \overline{\alpha_n} - \overline{\delta_{n-1}}, \overline{\delta_{n-1}}) \\ & \leq & \psi(\alpha_{n-1}, \alpha_n - \delta_{n-1}, \delta_{n-1}) = \delta_n, \end{array} \tag{2.39}$$

which completes the induction, and also show (2.29).

In view of the estimate:

$$\| x_{n+m} - x_n \| \leq \sum_{k=n}^{n+m-1} \overline{\delta_k} \leq \sum_{k=n}^{n+m-1} \delta_k < \sum_{k=n}^{\infty} \overline{\delta_k} = t_\infty - t_n, \tag{2.40}$$

we deduce that sequence $\{x_n\}$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x_\infty \in \overline{U}(x_0, t_\infty)$ (since $\overline{U}(x_0, t_\infty)$ is a closed set). We also have:

$$\| F_0(x_n) \| \leq \psi(\alpha_{n-1}, \alpha_n - \delta_{n-1}, \delta_{n-1}) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{2.41}$$

That is $F(x_\infty) = 0$.

The rest of the proof follows by simply replacing ω by ω_0 in the corresponding proof in [7] for the estimates involving the computation of the inverses.

Remark 2.5. *If $\omega_0(t) = \omega(t) = \omega_1(t)$ $(t > 0)$, then our Theorem 2.4 reduces to Theorem 4.3 in [7]. Otherwise, it constitutes an improvement with advantages as stated previously.*

Note also that a posteriori bounds given in [7] are also becoming tighter with our approach.

Note that in Application 3.6, we show how to replace delicate condition (2.24).

3. Semilocal convergence of the Secant method

We need the analogous definitions 2.1–2.3 of regularly continuity for the Secant method, respectively.

Definition 3.1. *[8] The dd $[x, y; F]$ is said to be ω_1–regularly continuous on Ω if there exist an $\omega_1 \in \mathfrak{T}$ (call it regularity modulus), and a constant $\underline{h} \in [0, \underline{h}([x, y; F])]$ such that for all $x, y, u, v \in \Omega$*

$$\begin{aligned} &\omega_1^{-1}\Bigg(\min\{\| [x, y; F] \|, \| [u, v; F] \|\} - \underline{h} + \| [x, y; F] - [u, v; F] \| \Bigg) \\ &-\omega_1^{-1}\Bigg(\min\{\| [x, y; F] \|, \| [u, v; F] \|\} - \underline{h} \Bigg) \leq \| x - u \| + \| y - v \|, \end{aligned} \tag{3.1}$$

where,

$$\underline{h}([x, y; F]) = \inf_{x, y \in \Omega^2} \{\| [x, y; F] \|\}. \tag{3.2}$$

As in (2.1), we have that (3.1) implies:

$$\| [x, y; F] - [u, v; F] \| \leq \omega(\| x - u \| + \| y - v \|) \quad \text{for all} \quad x, y, u, v \in \Omega. \tag{3.3}$$

Definition 3.2. *Let $x_{-1}, x_0 \in \Omega$ be fixed. The dd$[x, y; F]$ is said to be ω_0- center regularly continous on Ω_0 if there exists $\omega_0 \in \mathfrak{T}$, such that for all $x, y \in \Omega_0$:*

$$\begin{aligned} &\omega_0^{-1}\Bigg(\min\{\| [x, y; F] \|, \| [x_0, x_{-1}; F] \|\} - \underline{h_0} + \| [x, y; F] - [x_0, x_{-1}; F] \| \Bigg) \\ &-\omega_0^{-1}\Bigg(\min\{\| [x, y; F] \|, \| [x_0, x_{-1}; F] \|\} - \underline{h_0} \Bigg) \leq \| x - x_0 \| + \| y - x_{-1} \|, \end{aligned} \tag{3.4}$$

condition (3.4) implies

$$\| [x, y; F] - [u, v; F] \| \leq \omega_0(\| x - x_0 \| + \| y - x_{-1} \|). \tag{3.5}$$

Definition 3.3. *The* $dd[x,y;F]$ *is said to be* $\omega-$ *regularly continuous on* Ω_0*, if there exist* $\omega \in \mathcal{T}$*, and a constant* $\underline{h}_0 \in [0, \underline{h}_0([x,y;F])]$ *such that for all* $x, y, u, v \in \Omega_0$

$$\begin{aligned} &\omega^{-1}\Bigg(\min\{\| [x,y;F] \|, \| [u,v;F] \|\} - \underline{h_0} + \| [x,y;F] - [u,v;F] \| \Bigg) \\ &-\omega^{-1}\Bigg(\min\{\| [x,y;F] \|, \| [u,v;F] \|\} - \underline{h_0}\Bigg) \leq \| x-u \| + \| y-v \|, \end{aligned} \tag{3.6}$$

where

$$\underline{h}_0([x,y;F]) = \inf_{x,y \in \Omega_0^2} \{[x,y;F]\} \tag{3.7}$$

Next, comments as the ones given after the Definitions 2.1–2.3 can follow in an analogous way. We shall now define a more precise majorant generator than the one in [6].

Let:

$$\overline{t} := \| x - x_0 \|, \ \overline{\alpha} := \omega_0^{-1}(\| [x, x_-; F] \| - \underline{h_0}), \ \overline{\beta} := \| x - x_- \|,$$

$$\overline{\delta} := \| x_+ - x \|, \ [x_0, x_{-1}; F] = \mathcal{I},$$

$$\alpha_0 := \omega_0^{-1}(1 - \underline{h_0}), \quad \overline{\beta}_0 := \| x_0 - x_{-1} \|, \quad \overline{a} := \alpha_0 - \overline{\beta}_0,$$

$$\psi(s,t) = \omega(s+t) - \omega(s), \quad \psi_0(s,t) = \omega_0(s+t) - \omega_0(s),$$

$$\overline{d} = \psi_0((\overline{a} - 2\,\overline{t} - \overline{\delta})_+, 2\,\overline{t} + \overline{\delta} + \overline{\beta_0}), \quad a = \alpha_0 - \beta_0 = \omega_0^{-1}(1 - \underline{h}_0) - \beta_0,$$

$$\Delta_n = \| x_n - x_\infty \| .$$

Our majorant generator is defined as follows: $g : Q \subseteq \mathbb{R}^4 \longrightarrow \mathbb{R}^4$: $q = (t, \alpha, \beta, \delta)$, for

$$Q = \{q \ : \ g_0((\alpha_0 - 2\,t - \delta - \beta_0)_+, \beta_0 + 2\,t + \delta) < 1 \text{ for } t \geq 0, \alpha \geq 0, \beta \geq 0, \text{ and } \delta \geq 0\}$$

into $q_+ = (t_+, \alpha_+, \beta_+, \delta_+)$ as follows:

$$t_+ := t + \delta, \quad \alpha_+ := a - 2\,t - \delta, \quad \beta_+ := \delta,$$

$$\delta_+ := \delta \left(\frac{\omega(a - 2\,t + \beta) - \omega(a - 2\,t - \delta)}{\omega_0(a - 2\,t - \delta)} \right), \tag{3.8}$$

Remark 3.4. *Note that if* $\omega_0 = \omega = \omega_1$*, majorant generator (3.8) coincides with the corresponding one in [8]. Otherwise (i.e. if* $\omega_0 < \omega_1$*) it is more precise.*

We say that the quadruple $q' = (t', \alpha', \beta', \delta')$ *is majorizing* $q = (t, \alpha, \beta, \delta)$ $(q \prec q')$*, if* $t \leq t'$*,* $\alpha \geq \alpha'$*,* $\beta \leq \beta'$*,* $\delta \leq \delta'$*.*

Let $q_0 = (t_0, \alpha_0, \beta_0, \delta_0) \in Q$, then we have the iteration:

$$q_{n+1} = g(q_n) \qquad (n \geq 0). \tag{3.9}$$

If $d_n = \psi_0(\alpha_{n+1}, \beta_0 + 2\, t_n + \delta_n) < 1$, iteration $\{q_n\}$ is well defined.

In particular, if

$$\overline{d} = \psi_0((\overline{a} - 2\,\overline{t} - \overline{\delta})_+, 2\,\overline{t} + \overline{\delta} + \overline{\beta_0}) < 1, \tag{3.10}$$

then, dd $[x_+, x; F]$ is boundedly invertible, and

$$\| \, [x_+, x; F]^{-1} \, \| \leq (1 - \overline{d})^{-1}. \tag{3.11}$$

Indeed, we have in view of (3.4):

$$\begin{aligned}
\| \, \mathcal{I} - [x_+, x; F] \, \| &= \| \, [x_0, x_{-1}; F] - [x_+, x; F] \, \| \\
&\leq \omega_0 \Big(\omega_0^{-1} (\min\{1, \| \, [x_+, x; F] \, \|\} - \underline{h}_0) \\
&\quad + \| \, x_+ - x_0 \, \| + \| \, x - x_{-1} \, \| \Big) \\
&\quad - \min\{1, \| \, [x_+, x; F] \, \|\} + \underline{h}_0 \\
&\leq \omega_0(\min\{\alpha_0, \overline{\alpha_+}\} + \overline{t_+} + \overline{t} + \overline{\beta_0}) - \omega_0(\min\{\alpha_0, \overline{\alpha_+}\}) \\
&\leq \psi_0(\min\{\alpha_0, \overline{\alpha_+}\}, \, 2\,\overline{t} + \overline{\delta} + \overline{\beta_0}) = \overline{d} < 1,
\end{aligned}$$

which together with the Banach lemma on invertible operators implies (3.11).

Estimate (3.11) is tighter than the corresponding one in [8] using ω_1 instead of ω_0.

This substitution in the proofs of the results in [8] produces the advantages as already stated in the introduction of this study.

Hence, we arrived at:

Theorem 3.5. *We have:*

For all $n \in \mathbb{N}$, we have: $d_n < 1 \Longleftrightarrow 2\, t_n + \delta_n < a$;

If

$$d_n < 1, \quad (n \geq 0) \tag{3.12}$$

then, the sequence q_n generated by (3.9) is well defined, and converges to $(t_\infty, a - 2\, t_\infty, 0, 0)$, where $t_\infty \leq 0.5\, a$;

$$\overline{q_0} \prec q_0 \Longrightarrow \overline{q_n} \prec q_n,$$

sequence $\{x_n\}$ generated by the Secant method (1.3) is well defined, remains in $U(x_0, t_\infty - t_0)$, and converges to a solution x_∞ of the equation $F(x) = 0$;

Moreover, the following estimates hold for all $n \geq 0$:

$$\Delta_{n+1} \leq \Delta_n \, \frac{\omega(a - t_{n-1} - t_n) - \omega(a - t_{n-1} - t_n - \Delta_{n-1})}{\omega_0(a - t_n - t_{n+1})} < \frac{\Delta_n \, \omega(\Delta_{n-1})}{\omega_0(a - 2\, t_\infty)}$$

and

$$\| \, x_n - x_\infty \, \| \leq t_\infty - t_n.$$

Furthermore, x_∞ is the only solution in $U(x_0, a - t_\infty)$.

We would like to know how small $q_0 \in Q$ should be to force the series $\sum \delta_n$ or equivalently sequence $\{t_n\}$ to converge. That is we need to find the convergence domain Q_c of generator (3.8).

Let

$$Q(t) = \{(t_0, \beta_0, \delta_0) \,:\, t_\infty = t_\infty(t_0, \beta_0, \delta_0) = t\}$$

denotes the attraction bassin of the fixed point $(t, 0, 0)$ of generator (3.8).

Clearly, we have:

$$Q_c = \bigcup_{0 \le t \le .5\ a} Q(t). \tag{3.13}$$

For all triples (t', t, β) with $0 \le t \le t' \le .5\ a$, and $\beta \ge 0$, the equation

$$t_\infty(t, \beta, \delta) = t' \tag{3.14}$$

is uniquely solvable for δ: $\delta = X_{t'}(t, \beta)$.

The function $X_{t'}(t, \beta)$ uniquely solves the system

$$X(t, \beta)\ \frac{\omega(a - 2\ t + \beta) - \omega(a - 2\ t - X(t, \beta))}{\omega_0(a - 2\ t - X(t, \beta))} = X(t + X(t, \beta), X(t, \beta)) \tag{3.15}$$

and

$$X(t', \beta) = 0.$$

Application 3.6. *If $\omega_1(s) = c\ s$, and $\omega_0(s) = c_0\ s$, the generator (3.8) reduces to*

$$t_+ = t + \delta, \quad \beta_+ = \delta, \quad \delta_+ = \lambda\ \delta\ \frac{\beta_+\ \delta}{c_0^{-1} - \beta_0 - 2\ t - \delta}, \quad \lambda = \frac{c}{c_0}. \tag{3.16}$$

The change of variables $(u, v, w) = \dfrac{1}{c_0^{-1} - \beta_0}\ (t, \beta, \delta)$ *leads to*

$$u_+ = u + w, \quad v_+ = w, \quad w_+ = \lambda\ w\ \frac{v + w}{1 - 2\ u - w}. \tag{3.17}$$

In view of (3.17), system (3.15) reduces to:

$$\begin{aligned} &\lambda\ X(u, v)\ \frac{v + X(u, v)}{1 - 2\ u - X(u, v)} = X(u + X(u, v), X(u, v)) \\ &X(u', v) = 0, \qquad u' \in [0, 0.5] \end{aligned} \tag{3.18}$$

1. *Case $\lambda = 1$. The solution of the system is:*

$$X_{u'}(u, v) = \frac{(.5 - u)^2 - (.5 - u')^2}{2\ (.5 - u) + v}. \tag{3.19}$$

In view of (3.19), we have:

$$\delta_n = X_{t'}(t_n, \beta_n) = \frac{r_n^2 - (.5\ a - t')^2}{2\ r_n + \beta_n}, \tag{3.20}$$

where, $r_n = .5\ (c^{-1} - \beta_0) - t_n$.

Hence, we get:

$$X_{.5\ a}(0, \beta_0) = \frac{(1 - c\ \beta_0)^2}{4\ c}. \tag{3.21}$$

Condition (3.12) in Theorem 3.5 for starters x_{-1}, x_0 *can be replaced by:*

$$\| x_{-1} - x_0 \| \leq \beta_0 \leq c^{-1}, \quad \| F(x_0) \| \leq \frac{(1 - c\ \beta_0)^2}{4\ c}, \tag{3.22}$$

whereas in the general case:

$$\| x_{-1} - x_0 \| \leq \beta_0 \leq \omega^{-1}(1 - \underline{h}_0), \quad \| F(x_0) \| \leq \delta_0 \leq X_{.5\ a}(0, \beta_0), \tag{3.23}$$

where $X_{t'}(t, \beta)$ *is the only solution of the system (3.18), and* $a = \omega_0^{-1}(1 - \underline{h}_0) - \beta_0$.

Note that for $\beta_0 = 0$, *(3.22), (3.23) can replace condition (2.24) for Newton's method, provided that the Lipschitz constant* $\ell = 2\ c$ $(\ell_0 = 2\ c_0)$. *That is we have for Newton's method:*

$$\| F_0(x_0) \| \leq \delta_0 \leq \frac{1}{2\ \ell}, \tag{3.24}$$

or

$$\| F_0(x_0) \| \leq \delta_0 \leq X_{.5\ a}(0, 0), \tag{3.25}$$

where $X_{t'}(t, \beta)$ *is the only solution of the system (3.18), and* $a = \omega^{-1}(1 - \underline{h})$.

Condition (3.24) is the famous for its simplicity and clarity Newton–Kantorovich hypothesis for solving nonlinear equations in the Lipschitz case [3], [9].

2. *Case* $\lambda \neq 1$. *The solution* $X_{t'}(t, \beta)$ *is given implicitly, and is of only theoretical use. A more direct approach however leads to a weaker sufficient convergence for Newton's method than (3.24):*

$$\| F_0(x_0) \| \leq \delta_0 \leq \frac{1}{2\ L}, \tag{3.26}$$

where,

$$L = \frac{1}{8}\ (\sqrt{\ell_0 \ell} + 4\ \ell_0 + \sqrt{8\ell_0^2 + \ell_0\ \ell}) \tag{3.27}$$

[2]– [4] (see, also (2.20)).

Note that

$$L < \ell, \tag{3.28}$$

which improves (3.24).

Applications for this case can be found in Section 4.

In the case $\omega_0(t) = c_0\, t^p$, $\omega_1(t) = c\, t^p$, $p \in (0,1)$, *the analogous to condition* (3.26) *is given by:*

$$\| F_0(x_0) \| \leq \delta_0 \leq h(c_1)^{\frac{1}{p}}, \tag{3.29}$$

where,

$$c_1 = \frac{\ell + \sqrt{\ell^2 + 4\, \ell_0\, \ell\, (1+p)^p\, p^{1-p}}}{2\, \ell} \tag{3.30}$$

and

$$h(c_1) = \left(1 - \frac{1}{t}\right)^p \frac{1+p}{\left((\ell_0\, (1+p))^{\frac{1}{1-p}} + (\ell\, t\, (t-1))^{\frac{1}{1-p}}\right)^{1-p}} \tag{3.31}$$

[5]–[7].

Extension 3.7. *The results obtained here can be weakened even further, if* Ω_0 *in condition* (2.9) *and after is replaced by* $\Omega_0^* = \Omega \cap U(x_1, r_0 - a)$, *since* $\Omega_0^* \subseteq \Omega_0$ *so the corresponding function* ω^* *will be at least as tight as* ω. *We are still using the initial data, since* $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$ *for Newton's method and* $x_1 = x_0 - [x_0, x_{-1}, ; F]^{-1}F(x_0)$ *for the Secant method. In particular, condition* (3.26) *shall become*

$$\|F_0(x_0)\| \leq \delta_0 \leq \frac{1}{2L_1}, \tag{3.32}$$

where $L_1 = \frac{1}{8}\,(\sqrt{\ell_0 \ell_1} + 4\, \ell_0 + \sqrt{8\ell_0^2 + \ell_0\, \ell_1})$, *and* ℓ_1 *is the Lipschitz-constant on* Ω_0^*.

Remark 3.8. *If* $\omega(t) \leq \omega_0(t)$ *for* $t \in [0, r_0)$ *holds instead of* (2.12) *then, clearly the preceding results hold with function* ω_0 *replacing* ω *in all estimates. Moreover, the previously stated advantages also hold in this setting.*

4. Numerical Examples

We present two numerical examples.

Example 4.1. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $x_0 = 1$, $p \in [0, 0.5)$, $\Omega = \overline{U}\{x_0, 1-p\}$, *and define function* F *on* Ω *by*

$$F(x) = x^3 - p. \tag{4.1}$$

Then, we have $\delta_0 = \frac{1}{3}(1-p)$, $\ell_0 = 3 - p$, $\ell = 2(2-p)$. *Condition* (3.24) *is not satisfied, since* $\delta_0 > \frac{1}{2\ell}$, *for each* $p \in (0, 0.5)$. *Condition* (3.26) *is satisfied for* $p \in [0.4271907643, 0.5)$ *and* $\ell_1 = \frac{2}{3(3-p)}(-2p^2+5p+6)$. *Moreover, condition* (3.32) *is satisfied for* $p \in [0.408945626, 0.5)$. *That is, there exist several values of* p *for which the previous conditions cannot guarantee the convergence but our new ones can.*

Example 4.2. *Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{C}[0,1]$ be the space of continuous functions defined in $[0,1]$ equipped with max-norm. Let $S = \{x \in C[0,1]; \|x\| \le R\}$, such that $R > 0$ and F defined on S and given by*

$$[F(x)]\,(s) = x(s) - f(s) - \xi \int_0^1 G(s,t)\; x^3(t)\; dt, \quad x \in C[0,1], \quad s \in [0,1],$$

where $f \in C[0,1]$ is a given function, ξ is a real constant and the kernel G is the Green's function.

$$G(s,t) = \begin{cases} t\;(1-s), & t \le s \\ s\;(1-t), & s \le t. \end{cases}$$

In this case, for each $x \in S$, $F'(x)$ is a linear operator defined on S by the following expression:

$$[F'(x)v]\,(s) = v(s) - 3\xi \int_0^1 G(s,t)\;\; x^2(t)\; v(t)\; dt, \quad v \in C[0,1], \quad s \in [0,1].$$

If we choose $x_0(s) = f(s) = 1$, it follows $\|I - F'(x_0)\| \le 3|\xi|/8$. Thus, if $|\xi| < 8/3$, $F'(x_0)^{-1}$ is defined and

$$F'(x_0)^{-1} \le \frac{8}{8-3|\xi|}, \qquad \|F(x_0)\| \le |\xi|/8, \qquad \eta = \|F'(x_0)^{-1}F(x_0)\| \le \frac{|\xi|}{8-3|\xi|}.$$

On the other hand, for $x, y \in S$, we have

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \|x-y\| \frac{1+3|\xi|(\|x+y\|)}{8-3|\xi|} \le \|x-y\| \frac{1+6R|\xi|}{8-3|\xi|}$$

and

$$\|F'(x_0)^{-1}(F'(x) - F'(1))\| \le \|x-1\| \frac{1+3|\xi|(\|x+1\|)}{8-3|\xi|} \le \|x-1\| \frac{1+3(1+R)|\xi|}{8-3|\xi|}$$

Let $\xi = 1.075$ and $R = 2$, then we have

$$\delta_0 = 0.225131\cdots, \quad \ell = 2.91099\cdots, \quad \ell_0 = 2.2356\cdots \quad \text{and} \quad \ell_1 = 1.55622.$$

Using this values we obtain that conditions (3.24) and (3.26) are not satisfied, since $2\delta_0\ell = 1.31071\cdots > 1$ and $2\delta_0 L = 1.03064\cdots > 1$. But condition (3.32) is satisfied, since $2\delta_0 L_1 = 0.979333\cdots < 1$. That is, we can guarantee the convergence of Newton's method.

Conclusion

Using the concept of regular smothness and our new idea of restricted convergence domains, we provided under the same computational cost as in [5]-[7]:

1. Weaker sufficient convergence conditions for Newton's and the Secant methods;

2. Finer error sequences for Newton's and the Secant methods (under the same hypotheses (see Theorems 2.4 and 3.5)). In practice, this improvement leads to the computation of fewer iterates needed to achieve a certain error tolerance $\epsilon > 0$;

3. An at least as precise information on the location of the solution;

4. The technique of convergence domains can be used to study other methods involving inverses of linear operators.

Finally, special cases and numerical examples further validating the theoretical results are also given in this study.

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Ioannis K. Argyros,
Cameron University,
Department of Mathematics Sciences,
Lawton, OK 73505, USA.
E-mail address: iargyros@cameron.edu

and

Santhosh George,
Department of Mathematical and Computational Sciences
NIT Karnataka, Surathkal
India-575 025.
E-mail address: sgeorge@nitk.ac.in

and

Shobha M. Erappa,
Department of Mathematics
Manipal Institute of Technology
Manipal, Karnataka, India-576104.
E-mail address: shobha.me@gmail.com