



Fixed Point Theorems for Generalized $\beta - \phi$ -contractive Pair of Mappings Using Simulation Functions

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ABSTRACT: In this paper, our aim is to present a new class of generalized $\beta - \phi - \mathcal{Z}$ -contractive pair of mappings and we prove certain fixed point theorems for a pair of mappings using this concept. Our results generalizes some fixed point theorems in the literature. As an application some fixed point theorems endowed with a partial order in metric spaces are also proved.

Key Words: Common fixed points, Contractive type mapping, \mathcal{Z} -contractive pair of mappings, Partial order.

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1. Introduction

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic and this is a very active field of research at present. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical tools. It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach-Caccioppoli theorem which was published in 1922 in [1] and it also appears in [4]. Later in 1968, Kannan [6] studied a new type of contractive mapping. Since then, there have been many results related to mappings satisfying various types of contractive inequality, we refer to ([2], [3], [8], [9], [10] etc) and references therein.

Recently, Samet et al. [11] introduced a new category of contractive type mappings known as $\alpha - \phi$ -contractive type mappings. Further, Karapinar and Samet [7] generalized the $\alpha - \phi$ -contractive type mappings and obtained various fixed

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point theorems for this generalized class of contractive mappings. Our results unify and generalize the results derived by Karapinar and Samet [7], Samet et al. [7], Ćirić et al. [5] and various other related results in the literature. Very recently, Khojasteh, Shukla and Radenović [1] introduced a new class of mappings called simulation functions. Later, Argoubi, Samet and Vetro [14] slightly modified the definition of simulation functions by withdrawing a condition. Let Z^* be the set of simulation functions in the sense of Argoubi et al. [14].

Definition 1.1 ([14]) A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n = l \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

Note that the classes of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ denote by \mathbb{Z} .

Definition 1.2 [11] Let Φ be the family of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

(i) ϕ is nondecreasing.

(ii) $\sum_{n=1}^{+\infty} \phi^n(t) < \infty$ for all $t > 0$, where ϕ^n is the n^{th} iterate of ϕ .

Definition 1.3 [11] Let (X, d) be a metric space and $T : X \rightarrow X$ be a given self mapping, T is said to be an $\beta - \phi$ -contractive mapping if there exists two functions $\beta : X \times X \rightarrow [0, +\infty)$ and $\phi \in \Phi$ such that

$$\beta(x, y)d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x, y \in X$.

Definition 1.4 [11] Let $T : X \rightarrow X$ and $\beta : X \times X \rightarrow [0, +\infty)$. T is β -admissible if

$$x, y \in X, \beta(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$$

Theorem 1.5 [11] (i) T is β -admissible;

(ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Priya Shahi et al. [12] introduce the concept of α -admissible w.r.t. g mapping and generalized $\alpha - \psi$ -contractive pair of mappings as follows:

Definition 1.6 Let $f, g : X \times X \rightarrow [0, \infty)$. We say that f is α -admissible w.r.t. g if for all $x, y \in X$, we have

$$\alpha(gx, gy) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1.$$

Definition 1.7 Let (X, d) be a metric space and $f, g : X \rightarrow X$ be given mappings. We say that the pair (f, g) is a generalized $\alpha - \psi$ - contractive pair of mappings if there exists two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\alpha(gx, gy)d(fx, fy) \leq \psi(M(gx, gy)),$$

where

$$M(gx, gy) = \max \left\{ d(gx, gy), \frac{d(gx, fx) + d(gy, fy)}{2}, \frac{d(gx, fy) + d(gy, fx)}{2} \right\}.$$

Note: Throughout this paper $C(T, S)$ denotes the set of coincidence points of T and S are self maps on X , that is, $C(T, S) = \{u \in X, Tu = Su\}$.

2. Main Results

In the following theorem, we show the existence of common fixed point for four self-maps.

Definition 2.1 Let (X, D) be a metric space and S, T be self maos on X . The pair (S, T) is called a generalized $\beta - \phi - \mathcal{Z}$ -contractive pair of mappings with respect to ζ if

$$\zeta(\beta(Tx, Ty)d(Sx, Sy), \phi(M(Tx, Ty))) \geq 0 \tag{2.1}$$

for all $x, y \in X$, where $\beta : X \times X \rightarrow [0, \infty]$ and $\phi \in \Phi$ and

$$M(Tx, Ty) = \max \left\{ d(Tx, Ty), \frac{d(Tx, Sx) + d(Ty, Sy)}{2}, \frac{d(Tx, Sy) + d(Ty, Sx)}{2} \right\}$$

Theorem 2.2 Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be such that $S(X) \subseteq T(X)$. Assume that the pair (S, T) is a generalized $\beta - \phi - \mathcal{Z}$ -contractive pair of mappings and the following conditions hold :

- (i) S is β -admissible w.r.t. T ;
- (ii) there exists $x_0 \in X$ such that $\beta(Tx_0, Sx_0) \geq 1$;
- (iii) If $\{Tx_n\}$ is a sequence in X such that $\beta(Tx_n, Tx_{n+1}) \geq 1$ for all n and $Tx_n \rightarrow Tz \in T(X)$ as $n \rightarrow \infty$, then there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\beta(Tx_{n(k)}, Tz) \geq 1$ for all k .

Proof. In view of condition (ii), let $x_0 \in X$ be such that $\beta(Tx_0, Sx_0) \geq 1$. Since $S(X) \subseteq T(X)$, we can choose a point $x_1 \in X$ such that $Sx_0 = Tx_1$. Continuing this process having chosen x_1, x_2, \dots, x_n we choose x_{n+1} in X such that

$$Sx_n = Tx_{n+1}, \quad n = 0, 1, 2, \dots \tag{2.2}$$

Since S is β -admissible w.r.t. T , we have

$$\beta(Tx_0, Sx_0) = \beta(Tx_0, Tx_1) \geq 1 \Rightarrow \beta(Sx_0, Sx_1) = \beta(Tx_0, Tx_2) \geq 1$$

Using mathematical induction, we get

$$\beta(Tx_n, Tx_{n+1}) \geq 1 \text{ for all } n = 0, 1, 2, \dots \quad (2.3)$$

If $Sx_{n+1} = Sx_n$ for some n , then by (2)

$$Sx_n = Tx_{n+1}, \quad n = 0, 1, 2, \dots$$

that is, S and T have a coincidence point at $x = x_{n+1}$ and so we have finished the proof. For this, we suppose that $d(Sx_n, Sx_{n+1}) > 0$ for all n .

Now, putting $x = x_n$, $y = x_{n+1}$ in (1), we get

$$\begin{aligned} 0 &\leq \zeta(\beta(Tx_n, Tx_{n+1})d(Sx_n, Sx_{n+1}), \phi(M(Tx_n, Tx_{n+1})) \\ &< \phi(M(Tx_n, Tx_{n+1}) - \beta(Tx_n, Tx_{n+1})d(Sx_n, Sx_{n+1})) \end{aligned}$$

or

$$\begin{aligned} \beta(Tx_n, Tx_{n+1})d(Sx_n, Sx_{n+1}) &< \phi(M(Tx_n, Tx_{n+1})), \\ d(Sx_n, Sx_{n+1}) &\leq \beta(Tx_n, Tx_{n+1})d(Sx_n, Sx_{n+1}) \\ &< \phi(M(Tx_n, Tx_{n+1})), \end{aligned}$$

where

$$\begin{aligned} M(Tx_n, Tx_{n+1}) &= \max \left\{ d(Tx_n, Tx_{n+1}), \frac{d(Tx_n, Sx_n) + d(Tx_{n+1}, Sx_{n+1})}{2}, \right. \\ &\quad \left. \frac{d(Tx_n, Sx_{n+1}) + d(Tx_{n+1}, Sx_n)}{2} \right\} \\ &\leq \max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}. \end{aligned} \quad (2.4)$$

Owing to monotonicity of the function ϕ and using the inequality (2) and (4), we have for all $n \geq 1$

$$d(Sx_n, Sx_{n+1}) = \phi(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}). \quad (2.5)$$

If for some $n \geq 1$, we have $d(Sx_{n-1}, Sx_n) \leq d(Sx_n, Sx_{n+1})$, from (5), we obtain that

$$d(Sx_n, Sx_{n+1}) \leq \phi(d(Sx_n, Sx_{n+1}) < d(Sx_n, Sx_{n+1})),$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\} = d(Sx_{n-1}, Sx_n). \quad (2.6)$$

Notice, that in view of (5) and (6), we get for all $n \geq 1$, that

$$d(Sx_n, Sx_{n+1}) \leq \phi(d(Sx_{n-1}, Sx_n)). \quad (2.7)$$

Continuing this process inductively, we obtain

$$d(Sx_n, Sx_{n+1}) \leq \phi^n(d(Sx_0, Sx_1)), \text{ for all } n \geq 1 \quad (2.8)$$

From (8) and using the triangular inequality, for all $k \geq 1$, we have

$$\begin{aligned} d(Sx_n, Sx_{n+k}) &\leq d(Sx_n, Sx_{n+1}) + \dots + d(Sx_{n+k-1}, Sx_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \phi^p(d(Sx_1, Sx_0)) \\ &\leq \sum_{p=n}^{+\infty} \phi^p(d(Sx_1, Sx_0)) \end{aligned} \tag{2.9}$$

Letting, $p \rightarrow \infty$ in (9), we obtain that $\{Sx_n\}$ is a Cauchy sequence in (X, d) . Since by (2), we have $\{Sx_n\} = \{Tx_{n+1}\} \subseteq T(X)$ and $T(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} Tx_n = Tz. \tag{2.10}$$

Now, we show that z is a coincidence point of S and T . On contrary, assume that $d(Sz, Tz) > 0$. Since, by condition (iii) and (10), we have

$$\beta(Tx_{n(k)}, Tz) \geq 1 \text{ for all } k.$$

Using $x = x_{n(k)}, y = z$ in (i), we get

$$\begin{aligned} 0 &\leq \zeta(\beta(Tx_{n(k)}, Tz)d(Sx_{n(k)}, Sz), \phi(M(Tx_{n(k)}, Tz))) \\ &< \phi(M(Tx_{n(k)}, Tz) - \beta(Tx_{n(k)}, Tz))d(Sx_{n(k)}, Sz) \end{aligned}$$

or

$$\beta(Tx_{n(k)}, Tz)d(Sx_{n(k)}, Sz) < \phi M(Tx_{n(k)}, Tz)$$

But $\beta(Tx_{n(k)}, Tz) \geq 1$

$$\begin{aligned} d(Sx_{n(k)}, Sz) &\leq \beta(Tx_{n(k)}, Tz)d(Sx_{n(k)}, Sz) \\ &< \phi(M(Tx_{n(k)}, Tz)), \end{aligned} \tag{2.11}$$

$$\begin{aligned} M(Tx_{n(k)}, Tz) &= \max \left\{ d(Tx_{n(k)}, Tz), \frac{d(Tx_{n(k)}, Sx_{n(k)}) + d(Tz, Sz)}{2}, \right. \\ &\quad \left. \frac{d(Tx_{n(k)}, Sz) + d(Tz, Sx_{n(k)+1})}{2} \right\} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M(Tx_{n(k)}, Tz) &= \max \left\{ d(Tx_{n(k)}, Tz), \frac{d(Tx_{n(k)}, Sx_{n(k)}) + d(Tz, Sz)}{2}, \right. \\ &\quad \left. \frac{d(Tx_{n(k)}, Sz) + d(Tz, Sx_{n(k)})}{2} \right\} \end{aligned}$$

Making $k \rightarrow \infty$ in (11), we obtain

$$\begin{aligned} d(Tz, Sz) &\leq \phi \lim_{k \rightarrow \infty} (M(Tx_{(k)}, Tz)) \\ &\leq \phi \left(\max \left\{ d(Tx_{n(k)}, Tz), \frac{d(Tx_{n(k)}, Sx_{n(k)}) + d(Tz, Sz)}{2}, \right. \right. \\ &\quad \left. \left. \frac{d(Tx_{n(k)}, Sz) + d(Tz, Sx_{n(k)})}{2} \right\} \right) \end{aligned}$$

Letting $k \rightarrow \infty$ in the above inequality yields $d(Tz, Sz) \leq \phi\left(\frac{d(Sz, Tz)}{2}\right) < \frac{d(Sz, Tz)}{2}$, which is a contradiction. Hence, our supposition is wrong and $\phi(Sz, Tz) = 0$, that is, $Sz = Tz$.

This shows that S and T have a coincidence point.

Theorem 2.3 In addition to the hypothesis of Theorem 2.2, suppose that for all $u, v \in C(T, S)$, there exists $w \in X$ such that $\beta(Tu, Tw) \geq 1$ and $\beta(Tv, Tw) \geq 1$ and S, T commute at their coincidence points. Then, S and T have a unique common fixed point.

Proof. We prove this theorem in three steps.

First of all we claim that if $u, v \in C(T, S)$, then $Tu = Tv$. By hypothesis, there exists $w \in X$ such that

$$\beta(Tu, Tw) \geq 1, \beta(Tv, Tw) \geq 1 \quad (2.12)$$

From this fact $S(X) \subseteq T(X)$, let us define the sequence $\{w_n\}$ in X by $Tw_{n+1} = Sw_n$ for all $n \geq 0$ and $w_0 = w$. Since S is β -admissible w.r.t. T , we obtain it from (12) that

$$\beta(Tu, Tw_n) \geq 1, \beta(Tv, Tw_n) \geq 1 \quad (2.13)$$

for all $n \geq 0$.

Thus, putting $x = u, y = w_{n+1}$ in (1), we get

$$\begin{aligned} 0 &\leq \zeta(\beta(Tu, Tw_{n+1})d(Su, Sw_{n+1}), \phi(M(Tu, Tw_{n+1})) \\ &< \phi(M(Tu, Tw_{n+1}) - \beta(Tu, Tw_{n+1})d(Su, Sw_{n+1})) \end{aligned}$$

or

$$\beta(Tu, Tw_{n+1})d(Su, Sw_{n+1}) < \phi(M(Tu, Tw_{n+1}))$$

But $\beta(Tu, Tw_{n+1}) \geq 1$

$$\begin{aligned} d(Su, Sw_{n+1}) &\leq \beta(Tu, Tw_{n+1})d(Su, Sw_{n+1}) \\ &< \phi(M(Tu, Tw_{n+1})) = \phi(M(Su, Sw_n)) \end{aligned}$$

$$\begin{aligned} M(Su, Sw_n) &= \max \left\{ d(Su, Sw_n), \frac{d(Su, Tu) + d(Sw_n, Tw_n)}{2}, \right. \\ &\quad \left. \frac{d(Su, Tw_n) + d(Sw_n, Tu)}{2} \right\} \\ &\leq \max\{d(Tu, Tw_n), d(Tu, Tw_{n+1})\} \\ &\leq \max\{d(Tu, Tw_n), d(Tu, Tw_{n+1})\} \end{aligned} \quad (2.14)$$

Using the above inequality, (14) and owing to the monotone property of ϕ , we get

$$d(Tu, Tw_{n+1}) \leq \phi(\max\{d(Tu, Tw_n), d(Tu, Tw_{n+1})\}) \tag{2.15}$$

for all n . Without restriction to the generality, we can suppose that $d(Tu, Tw_n) \geq 0$ for all n . If $\max\{d(Tu, Tw_n), d(Tu, Tw_{n+1})\} = d(Tu, Tw_{n+1})$, we get it from (16), that

$$d(Tu, Tw_{n+1}) \leq \phi(d(Tu, Tw_{n+1})) < d(Tu, Tw_{n+1}), \tag{2.16}$$

which is a contradiction. Thus, we have

$$\max\{d(Tu, Tw_n), d(Tu, Tw_{n+1})\} = d(Tu, Tw_n),$$

$$d(Tu, Tw_{n+1}) \leq \phi(d(Tu, Tw_n)),$$

for all n .

$$d(Tu, Tw_n) \leq \phi^n(d(Tu, Tw_0)), \forall n \geq 1 \tag{2.17}$$

Letting, $n \rightarrow \infty$ in the above inequality, we have

$$\lim_{n \rightarrow \infty} d(Tu, Tw_n) = 0 \tag{2.18}$$

Similarly, we can prove that

$$\lim_{n \rightarrow \infty} d(Tv, Tw_n) = 0 \tag{2.19}$$

It follows from (19) and (20) that $Tu = Tv$.

Now in second step we will show the existence of a common fixed point. Let $u \in C(T, S)$, that is, $Tu = Su$. Owing to the commutativity of S and T at their coincidence points, we get

$$T^2u = TSu = STu \tag{2.20}$$

Let us denote $Tu = z$, then from (21), $Tz = Sz$. Thus, z is a coincidence points of S and T . Now, from step 1, we have $Tu = Tz = z = Sz$. Then, z is a common fixed point of S and T .

In the third step we will prove the Uniqueness. Assume that z^* is another common fixed point of S and T . Then, $z^* \in C(T, S)$. By step 1, we have $z^* = Tz^* = Tz = z$. This completes the proof.

3. Consequences

Following results can be obtained from our previous results:

Corollary 3.1 Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be such that $S(X) \subseteq T(X)$. Suppose that there exists a function $\phi \in \Phi$ such that

Proof. By taking $\beta(x, y) = 1$ for $x, y \in X$ and $\zeta(t, s) = \lambda s - t$, for all $t, s > 0$, $\lambda \in (0, 1)$, the result holds.

$$d(Sx, Sy) \leq \lambda(\phi(M(Tx, Ty))), \quad (3.1)$$

for all $x, y \in X$. Also suppose that $T(X)$ is closed. Then, S and T have a coincidence point. Further, if S, T commute at their coincidence points, then S and T have a common fixed point.

Corollary 3.2 Let (X, d) be a complete metric space $S : X \rightarrow X$. Suppose that there exists a function $\phi \in \Phi$ such that

$$d(Sx, Sy) \leq \lambda(\phi(M(x, y))), \quad (3.2)$$

for all $x, y \in X$. Also, S has a unique fixed point.

Corollary 3.3 Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ such that $S(X) \subseteq T(X)$. Suppose that there exists a function $\phi \in \Phi$ such that

$$d(Sx, Sy) \leq \phi(d(Tx, Ty)), \quad (3.3)$$

for all $x, y \in X$. Also, suppose, $T(X)$ is closed. Then, S and T have a coincidence point. Further, if S and T commute at their coincidence points, then S and T have a common fixed point.

Corollary 3.4 Let (X, d) be a complete metric space and $S : X \rightarrow X$. Suppose that there exists a function $\phi \in \Phi$ such that

$$d(Sx, Sy) \leq \phi(d(x, y)), \quad (3.4)$$

for all $x, y \in X$. Then, S has a unique fixed point.

Corollary 3.5 Let (X, d) be a complete metric space and $S : X \rightarrow X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, 1)$, such that

Proof. By putting $T = I$ in equation (22),

$$d(Sx, Sy) \leq \lambda(\phi(M(x, y))),$$

Now, put $\phi = I$

$$d(Sx, Sy) \leq \lambda(M(x, y))$$

where,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Sx) + d(y, Sy)}{2}, \frac{d(x, Sy) + d(y, Sx)}{2} \right\}.$$

for all $x, y \in X$. Then, S has a unique fixed point.

Corollary 3.6 Let (X, d) be a complete metric space and $S : X \rightarrow X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, 1)$ such that $d(Sx, Sy) \leq \lambda(M(x, y))$

Proof. By putting $M = d$

$$d(Sx, Sy) \leq (d(x, y))$$

for all $x, y \in X$. Then S has a unique fixed point.

Corollary 3.7 Let (X, d) be a complete metric space and $S : X \rightarrow X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, \frac{1}{2})$ such that

$$d(Sx, Sy) \leq \lambda \left[\frac{d(x, Sx) + d(y, Sy)}{2} \right] \times 2$$

$$d(Sx, Sy) \leq \lambda [d(x, Sx) + d(y, Sy)]$$

for all $x, y \in X$. Then, S has a unique fixed point.

Corollary 3.8 Let (X, d) be a complete metric space and $S : X \rightarrow X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, \frac{1}{2})$ such that

$$d(Sx, Sy) \leq \lambda \left[\frac{d(x, Sy) + d(y, Sx)}{2} \right] \times 2$$

$$d(Sx, Sy) \leq \lambda [d(x, Sy) + d(y, Sx)]$$

for all $x, y \in X$. Then, S has a unique fixed point.

4. Fixed point theorems on Metric spaces endowed with a partial order:

Definition 4.1[7] Let (X, \preceq) be a partially ordered set and $T : X \rightarrow X$ be a given mapping. We say that T is non decreasing with respect to \preceq if

$$x, y \in X, x \preceq y \Rightarrow Tx = Ty.$$

Definition 4.2[7] Let (X, \preceq) be a partially ordered set. A sequence $\{x_n\}$ is said to be nondecreasing with respect to \preceq if $x_n \preceq x_{n+1}$ for all n .

Definition 4.3[7] Let (X, \preceq) be a partially ordered set and d be a metric on X . We say that (X, \preceq, d) is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k .

Definition 4.4[5] Suppose (X, \preceq) is a partially ordered set and $S, T : X \rightarrow X$ are mappings of X into itself. One says S is T -non-decreasing if for $x, y \in X$

$$T(x) \preceq T(y) \Rightarrow S(x) \preceq S(y) \tag{4.1}$$

Corollary 4.5 Let (X, \preceq) be a partially ordered set and d be a metric on X such that (X, d) is complete. Assume that $S, T : X \rightarrow X$ be such that $S(X) \subseteq T(X)$ and S be a T -non-decreasing mapping w.r.t. \preceq . Suppose that there exists a function $\phi \in \Phi$ such that

$$d(Sx, Sy) \leq \phi(M(Tx, Ty)) \quad (4.2)$$

for all $x, y \in X$ with $Tx \preceq Ty$. Suppose also that the following conditions hold:

- (i) there exists $x_0 \in X$ such that $Tx_0 \leq Sx_0$;
- (ii) (X, \preceq, d) is T -regular.

Also suppose that $T(X)$ is closed. Then, S and T have a coincidence point. Moreover, if for every pair $(x, y) \in C(T, S) \times C(T, S)$ there exists $Z \in X$ such that $Tx \preceq Tz$ and $Ty \preceq Tz$, and if S and T commute at their coincidence points, then we obtain uniqueness of the common fixed point.

Proof. Define the mapping $\beta : X \times X \rightarrow [0, \infty)$ by

$$\beta(x, y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

Clearly, the pair (S, T) is a generalized $\beta - \phi$ contractive pair of mappings, that is,

$$\beta(Tx, Ty)d(Sx, Sy) \leq \phi(M(Tx, Ty))$$

for all $x, y \in X$. Notice that in view of condition (i), we have $\beta(Tx_0, Sx_0) \geq 1$. Moreover, for all $x, y \in X$, from the T -monotone property of S , we have

$$\begin{aligned} \beta(Tx, Ty) &\geq 1 \\ \Rightarrow Tx \preceq Ty \text{ or } Tx \succeq Ty \\ \Rightarrow Sx \preceq Sy \text{ or } Sx \succeq Sy \\ \Rightarrow \beta(Sx, Sy) &\geq 1 \end{aligned} \quad (4.4)$$

which amounts to say that S is β -admissible w.r.t. T . Now, let $\{Tx_n\}$ be a sequence in X such that $\beta(Tx_n, Tx_{n+1}) \geq 1$ for all n and $Tx_n \rightarrow Tz \in X$ as $n \rightarrow \infty$. From the T -regularity hypothesis, there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\{Tx_{n(k)}\} \leq Tz$ for all k . So, by the definition of β , we obtain that $\beta(\{Tx_{n(k)}, Tz\}) \geq 1$. Now, all the hypothesis 2.2 are satisfied. Hence, we deduce that S and T have a coincidence point z , that is, $Sz = Tz$. By hypothesis, there exists $z \in X$ such that $Tx \preceq Tz$ and $Ty \preceq Tz$, which implies from the definition of β and $\beta(Tx, Ty) \geq 1$ and $\beta(Ty, Tz) \geq 1$. Thus, we deduce the existence and uniqueness of the common fixed point by Theorem 2.3.

Corollary 4.6 Let (X, \preceq) be a partially ordered set and d be a metric on X such that (X, d) is complete. Assume that $S, T : X \rightarrow X$ be a non-decreasing mapping w.r.t. \preceq . Suppose that there exists a function $\phi \in \Phi$ such that

$$d(Sx, Sy) \leq \phi(d(Tx, Ty))$$

for all $x, y \in X$ with $Tx \preceq Ty$. Suppose also that the following conditions hold;

(i) there exists $x_0 \in X$ such that $Tx_0 \preceq Sx_0$;

(ii) (X, \preceq, d) is T -regular.

Also, suppose $T(X)$ is closed. Then, S and T have a coincidence point. Moreover, if for every pair $(x, y) \in C(T, S) \times C(T, S)$ there exists $z \in X$ such that $Tx \preceq Tz$ and $Ty \preceq Tz$ and if S and T commute at their coincidence points, then we obtain uniqueness of the common fixed point.

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