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General Decay of Solutions of a Thermoelastic Bresse System with Viscoelastic Boundary Conditions

Ammar Khemmoudj

ABSTRACT: In this paper we consider a multidimensional thermoviscoelastic system of Bresse type where the heat conduction is given by Green and Naghdi theories. For a wider class of relaxation functions, We show that the dissipation produced by the memory effect is strong enough to produce a general decay results. We establish a general decay results, from which the usual exponential and polynomial decay rates are only special cases.

Key Words: Bresse system, Thermoelasticity, Relaxation function, General decay, Viscoelastic.

Contents

1. Introduction

In [21], Khemmoudj and Hamadouche taking into account the longitudinal deplacement w, considered the generalization of multidimential Timoshenko problems studed in [10], [22] and [34], that is they studied the stabilization for the following multi-dimensional Bresse system

$$\begin{cases} \rho_1 u_{tt} - \Delta u - \alpha_1 \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} - (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} + \beta_1 u + a(x) f_1(u,v,w) = 0, \\ \qquad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \rho_2 v_{tt} - \Delta v + \alpha_1 \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + \beta_2 v + \beta_2 w + a(x) f_2(u,v,w) = 0, \\ \qquad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \rho_1 w_{tt} - \Delta w + (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + \beta_2 v + \beta_2 w + a(x) f_3(u,v,w) = 0, \\ \qquad \text{in} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

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subject to the following boundary conditions

$$\begin{aligned} &u(x,t) = v(x,t) = w(x,t) = 0, && \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ &u(x,t) = -\int_0^t h_1(t-s)\left\{\tfrac{\partial u}{\partial \nu} + b_1(x)(v+w)\right\} ds, && \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ &v(x,t) = -\int_0^t h_2(t-s)\tfrac{\partial v}{\partial \nu} ds, && \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ &w(x,t) = -\int_0^t h_3(t-s)(\tfrac{\partial w}{\partial \nu} - b_2(x)u) ds, && \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \end{aligned} \tag{1.2}$$

and initial conditions

$$\begin{cases} (u(0), v(0), w(0)) = (u^0, v^0, w^0), \\ (\sqrt{\rho_1}u_t(0), \sqrt{\rho_2}v_t(0), \sqrt{\rho_1}w_t(0)) = (\sqrt{\rho_1}u^1, \sqrt{\rho_2}v^1, \sqrt{\rho_1}w^1), \end{cases} \tag{1.3}$$

where Ω is a bounded open set of $\mathbb{R}^n (n \geq 2)$ with a C^2-boundary $\Gamma = \partial\Omega$. Let Γ_0 and Γ_1 be closed nonempty disjoint subsets of Γ with $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \phi$ and $meas(\Gamma_0) > 0, meas(\Gamma_1) > 0$. $\nu(x)$ represent the exterior unit normal vector at $x \in \Gamma_1$. The authors have assumed that α_1, α_2, β_1, β_2 are a sufficiently small positive numbers, such that $\beta_1 > n\alpha_2$, $\beta_2 > n\alpha_1$, and

$$a \in C^1(\overline{\Omega}), \quad a(x) \geq a_0 > 0 \quad a.e. \quad in \quad \overline{\Omega},$$

where a_0 is a positive constant.

To state results of existence and stability, the following assumptions are made.

(i) **Assumptions on the relaxation functions.** *The relaxation functions $h_i, i = 1, 2, 3$ are considered positive, non-increasing and belonging to $W^{1,2}(0, +\infty)$.*

(ii) **Assumptions on the nonlinear functions.** *For the coupling terms $f_i, i = 1, 2, 3$, the authors supposed that*

1. $f_i \in C^1(\mathbb{R}^3), \quad i = 1, 2, 3.$
2. *Additionally, they assumed that there exists a nonnegative function $F(u, v, w) \in C^2(\mathbb{R}^3)$ such that*

$$f_1(u,v,w) = \frac{\partial F}{\partial u}, \quad f_2(u,v,w) = \frac{\partial F}{\partial v}, \quad f_3(u,v,w) = \frac{\partial F}{\partial w}. \tag{1.4}$$

3. *Further, they assumed that F is homogeneous of order $p+1$:*

$$F(\lambda u, \lambda v, \lambda w) = \lambda^{p+1}F(u,v,w), \text{ for all } \lambda > 0, \quad (u,v,w) \in \mathbb{R}^3. \tag{1.5}$$

Since F is homogeneous, the Euler Homogeneous Function theorem yields the following useful identity:

$$uf_1(u,v,w) + vf_2(u,v,w) + wf_3(u,v,w) = (p+1)F(u,v,w).$$

The homogeneity of F implies that there exists a constant $M > 0$ such that

$$|F(u,v,w)| \leq M\left(|u|^{p+1} + |v|^{p+1} + |w|^{p+1}\right). \tag{1.6}$$

Remark 1.1. *There is a large class of functions satisfying the assumptions (1.4)-(1.6). For instance functions of the form*

$$F(u,v,w) = a \mid u \mid^{p+1} + b \mid v \mid^{p+1} + c \mid w \mid^{p+1},$$

where a , b , c are positive constants, satisfy assumptions (1.4)-(1.6) with $p \geq 3$. Indeed, a quick calculation shows that there exists $c_0 > 0$ such that

$$F(u,v,w) = c_0 \left\{ \mid u \mid^{p+1} + \mid v \mid^{p+1} + \mid w \mid^{p+1} \right\}.$$

Moreover, it is easy to compute and find that

$$uf_1(u,v,w) + vf_2(u,v,w) + wf_3(u,v,w) = (p+1)F(u,v,w).$$

The authors established a general decay result, from which the usual exponential and polynomial decay rates are only special cases.

In this paper, the main purpose is to study the asymptotic behavior of the solutions to the following thermoviscoelastic multi-dimentional Bresse system

$$\begin{aligned}
u_{tt} - \Delta u - \alpha_1 \sum_{i=1}^{n} \partial_{x_i} v - (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \partial_{x_i} w + \beta_1 u &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\
v_{tt} - \Delta v + \alpha_1 \sum_{i=1}^{n} \partial_{x_i} u + \beta_2 (v + w) + \alpha_3 \sum_{i=1}^{n} \partial_{x_i} \vartheta &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\
w_{tt} - \Delta w + (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \partial_{x_i} u + \beta_2 (v + w) &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\
\vartheta_{tt} - k \Delta \vartheta - \delta \Delta \vartheta_t + \alpha_3 \sum_{i=1}^{n} \partial_{x_i} v_{tt} &= 0, \quad in \quad \Omega \times \mathbb{R}^+,
\end{aligned} \tag{1.7}$$

subject to the following boundary conditions

$$\begin{cases}
u(x,t) = v(x,t) = w(x,t) = 0, & on \quad \Gamma_0 \times \mathbb{R}^+, \\
u(x,t) = -\int_0^t h_1(t-s)(\frac{\partial u}{\partial \nu} + \sigma_1(x)(v+w))ds, & on \quad \Gamma_1 \times \mathbb{R}^+, \\
v(x,t) = -\int_0^t h_2(t-s)\frac{\partial v}{\partial \nu}ds, & on \quad \Gamma_1 \times \mathbb{R}^+, \\
w(x,t) = -\int_0^t h_3(t-s)(\frac{\partial w}{\partial \nu} - \sigma_2(x)u)ds, & on \quad \Gamma_1 \times \mathbb{R}^+, \\
\vartheta(x,t) = 0, & on \quad \Gamma \times \mathbb{R}^+,
\end{cases} \tag{1.8}$$

and initial conditions

$$\begin{cases}
(u(0), v(0), w(0), \vartheta(0)) = (u^0, v^0, w^0, \vartheta^0), \\
(u_t(0), v_t(0), w_t(0), \vartheta_t(0)) = (u^1, v^1, w^1, \vartheta^1),
\end{cases} \tag{1.9}$$

where the functions $\vartheta = \vartheta(x,t)$ is the difference temperature. Here the relaxation functions $h_i \in C^1(0,\infty)$, $\quad i = 1,2,3$, are positive and non-increasing and the parameters δ, k, α_3 are postive constants and α_1, α_2, β_1 and β_2 are sufficiently small positive numbers satisfying $\beta_1 > n\alpha_2, \beta_2 > n\alpha_1$.

Let us mention some known results on the decay rate for the Bresse system. In [16] a simple one dimensional Bresse model is usually considered in studying elastic structures of the arcs type whose motion is governed by the following system of three wave equations:

$$\begin{cases} \rho_1 \dfrac{\partial^2 u}{\partial t^2} - \kappa \dfrac{\partial^2 u}{\partial x^2} - \kappa \dfrac{\partial v}{\partial x} - \kappa l \dfrac{\partial w}{\partial x} - \kappa_0 l \left(\dfrac{\partial w}{\partial x} - lu \right) = 0, \\ \rho_2 \dfrac{\partial^2 v}{\partial t^2} - EI \dfrac{\partial^2 v}{\partial x^2} + \kappa \left(\dfrac{\partial u}{\partial x} + v + lw \right) = 0, \\ \rho_1 \dfrac{\partial^2 w}{\partial t^2} - \kappa_0 \dfrac{\partial^2 w}{\partial x^2} + \kappa_0 l \dfrac{\partial u}{\partial x} + \kappa l \left(\dfrac{\partial u}{\partial x} + v + lw \right) = 0, \end{cases} \tag{1.10}$$

where $(x, t) \in (0, L) \times (0, \infty)$ and the coefficients ρ_1, ρ_2, E and I denote respectively the mass per unit length, the mass moment of inertia of a cross-section of the beam, Young's modulus and the moment of inertia of a cross-section of the beam. The coefficient κ_0, κ and l are equal to EA, $\kappa' GA$ and R^{-1} respectively where G is the modulus of elasticity in shear, A is the cross sectional area, κ' is the shear factor and R for the radius of the curvature. We note that when $R \to \infty$, then $l \to 0$ and then this model reduces to the well-known Timoshenko beam equations.

The case of one frictional damping has been considered by Alabau Boussouira et al. [1], where the authors proved that the Bresse system is exponentially stable if and only if the velocities of waves propagations are the same. i.e. $\frac{\rho_1}{\rho_2} = \frac{\kappa}{EI}$ and $\kappa = \kappa_0$. Also, Noun and Wehbe [29] extended the results in [1] by considering only one locally distributed damping.

In [8], the authors considered the Bresse system with indefinite damping mechanism acting on the equation about the shear angle displacement. Under the equal speeds condition and only with Dirichlet-Neumann-Neumann boundary condition type, they proved the exponential stability of the system.

Concerning the asymptotic behavior of the Bresse system with past memory acting in the three equations we cite the work of Guesmia et al [12]. In that paper the authors showed under suitable conditions on the initial data and the memories, that the Bresse system converges to zero when time goes to infinity, and they provided a connection between the decay rate of energy and the growth of memories at infinity. In [2] Santos et al. considered the Bresse system with past history acting only in the shear angle displacement. They showed the exponential decay of the solution if and only if the wave speeds are the same. If not, they showed that the Bresse system is polynomial stable with optimal decay rate.

For Bresse system in classical thermoelasticity, Liu and Rao [17] considered the Bresse system with two different dissipative mechanism, given by two temperatures coupled to the system. The authors considered the problem

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) + l\kappa_1\theta^1 & = & 0, & in &]]0, L[\times\mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) + \kappa_1\theta_x^2 & = & 0, & in &]0, L[\times\mathbb{R}^+, \\ \rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) + \kappa_1\theta_x^1 & = & 0, & in &]0, L[\times\mathbb{R}^+ \\ \rho_3\theta_t^1 - \alpha\theta_{xx}^1 + \kappa_1\left(w_x - l\varphi\right) & = & 0, & in &]0, L[\times\mathbb{R}^+, \\ \rho_3\theta_t^2 - \alpha\theta_{xx}^2 + \kappa_1\psi_{tx} & = & 0, & in &]0, L[\times\mathbb{R}^+, \end{cases} \tag{1.11}$$

They proved that the exponential decay exists only when the velocities of the wave propagations are the same. If the wave speeds are different they showed that the energy of the system decays polynomially to zero with the rate $t^{-1/2}$ or $t^{-1/4}$, provided that the boundary conditions is of Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet type, respectively.

If $\theta^1 = 0$ in (1.11) Fatori and Munoz Rivera [18] analyzed the exponential stability of the obtained Bresse-Fourier system they showed that, in general, the system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. Recently, Najdi and Wehbe in [28] extended and improved the results of [18] when the thermal dissipation is locally distributed.

In the above system, the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation. That is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. To overcome this physical paradox, different models, have been introduced such as Cattaneo's law [5], Green and Naghdi's type-III theory, [13], [14] and others. The type-III Green and Naghdi's model of thermoelasticity includes temperature gradient and thermal displacement gradient among the constitutive variables and proposed a heat conduction law as

$$q(x,t) = -\left[\kappa\nabla\theta(x,t) + \kappa^*\nabla v(x,t)\right], \tag{1.12}$$

where $v_t = \theta$ and v is the thermal displacement gradient, κ and $\kappa*$ are two positive constants. Equation (1.12) together with the energy balance law

$$\rho_3\theta_t + \varrho div q = 0, \tag{1.13}$$

lead to the equation

$$\rho\theta_{tt} - \varrho\kappa\Delta\theta_t - \varrho\kappa^*\Delta\theta = 0 \tag{1.14}$$

which permits propagation of thermal waves at finite speed.

The coupling of equation (1.14) with some equations of elasticity has been an active area of research in the last two decades. See in this connection [30], [37] and the coupling in one-dimensional space with Bresse system, we refer the reader to Said-Houari and Hamadouche [32].

Recently, in thermoelasticity of type III, several authors (see [24], [26], [27] and the references therein) have studied the asymptotic behavior of solutions associated to the dynamic problem with memory boundary conditions. See [3], [36] for the study of Timoshenko system and Kirchhoff plates equations in classical thermoelasticity.

Models of viscoelastic problems without thermal effect where the memory term acting in the boundary were proposed in [6], [23], [33] for the study of wave propagation, in [22], [25], [34] for Timoshenko system, in [31], [35] for the von Karman plate system and in [4], [19] in the context of Kirchhoff equations.

The boundary condition of memory type for Timoshenko system, has been studied by Santos [34]. By considering k_i to be the resolvent kernels of $(-h_i'/h_i(0))$ for $i = 1, 2$, he showed that the energy of the solution decays exponentially (polynomially) when k_i and $-k_i'$, $i = 1, 2$, decay exponentially (polynomially). The same result has been established by Messaoudi and Soufyane [22] without assuming the exponential (polynomial) decay of k_1 and k_2 but only that their norms are small enough. In [25] the general decay for the same system has been proved.

Models with boundary conditions including a memory term which produces damping were proposed in [7], [6], [23] and [33] for the study of wave propagation, in [31] and [35] for the von Karman plate system and in [15], [11] and [36] in the context of Kirchhoff equations.

Motivated by their results, we investigated the asymptotic behavior of the system (1.7)- (1.9) for resolvent kernels of general-type decay and obtain a more general and explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases. The proof is mainly based on the use of a multiplier method coupled with some technical lemmas and some technical ideas and the introduction of a suitable Lyapounov functional.

The paper is organized as follows. In section 2 we establish the existence and uniqueness for regular and weak solutions of system (1.7)-(1.9). In section 3 we state and prove the general decay of the solutions of our studied system.

2. Preliminaries

In this section, we present some materials needed in the proof of our main result and we prove the existence and regularity of solutions for problem (1.7)-(1.9).

We first consider the following hypothesis

(**A1**) There exists a fixed point $x^0 \in \mathbb{R}^n$ such that, for $m(x) = x - x_0$

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu \leq 0\}, \quad \Gamma_1 = \{x \in \Gamma : m(x).\nu > 0\}.$$

Also we consider two functions $\sigma_i(x) \in W^{1,\infty}(\Gamma_1)$, given by

$$\sigma_i(x) = \alpha_i \left(\sum_{k=1}^{n} \nu_k(x) \right), \quad i = 1, 2. \tag{2.1}$$

In order to exhibit the dissipative nature of system (1.7) we introduce, as in [37], the new variable

$$\theta(x,t) = \int_0^t \vartheta(x,s)ds + \chi(x) \tag{2.2}$$

where $\chi \in H_0^1(\Omega)$ solves

$$k\Delta\chi = \vartheta_1 - \delta\Delta\vartheta_0 + \alpha_3 \sum_{i=1}^{n} \partial_{x_i} v_1. \tag{2.3}$$

Then, it is easy to see that problem (1.7) becomes

$$\begin{cases} u_{tt} - \Delta u - \alpha_1 \sum\limits_{i=1}^{n} \partial_{x_i} v - (\alpha_1 + \alpha_2) \sum\limits_{i=1}^{n} \partial_{x_i} w + \beta_1 u &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha_1 \sum\limits_{i=1}^{n} \partial_{x_i} u + \beta_2 (v + w) + \alpha_3 \sum\limits_{i=1}^{n} \partial_{x_i} \theta_t &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\ w_{tt} - \Delta w + (\alpha_1 + \alpha_2) \sum\limits_{i=1}^{n} \partial_{x_i} u + \beta_2 (v + w) &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \\ \theta_{tt} - k\Delta\theta - \delta\Delta\theta_t + \alpha_3 \sum\limits_{i=1}^{n} \partial_{x_i} v_t &= 0, \quad in \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{2.4}$$

with the initial conditions

$$(u, v, w, \theta, u_t, v_t, w_t, \theta_t)(x, 0) = (u^0, v^0, w^0, \theta^0, u^1, v^1, w^1, \theta^1) \tag{2.5}$$

Next, we will use the second, the third and the fourth equations in (1.8) to estimate the terms $\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}$ and $\frac{\partial w}{\partial \nu}$.

Defining the convolution product operator by

$$(h * \varphi)(t) = \int_0^t h(t-s)\varphi(s)ds, \tag{2.6}$$

and differentiating the second, the third and the fourth equations in (1.8), we arrive to the following Volterra equations:

$$\begin{cases} \frac{\partial u}{\partial \nu} + \sigma_1(x)(v + w) + \frac{1}{h_1(0)} h_1' * \left\{ \frac{\partial u}{\partial \nu} + \sigma_1(x)(v + w) \right\} &= -\frac{1}{h_1(0)} u_t, \\ \frac{\partial v}{\partial \nu} + \frac{1}{h_2(0)} h_2' * \frac{\partial v}{\partial \nu} &= -\frac{1}{h_2(0)} v_t, \\ \frac{\partial w}{\partial \nu} - \sigma_2(x) u + \frac{1}{h_3(0)} h_3' * \left\{ \frac{\partial w}{\partial \nu} - \sigma_2(x) u \right\} &= -\frac{1}{h_3(0)} w_t. \end{cases} \tag{2.7}$$

Applying the Volterra's inverse operator, we get

$$\begin{cases} \frac{\partial u}{\partial \nu} + \sigma_1(x)(v + w) &= -\frac{1}{h_1(0)} \{u_t + k_1 * u_t\}, \\ \frac{\partial v}{\partial \nu} &= -\frac{1}{h_2(0)} \{v_t + k_2 * v_t\}, \\ \frac{\partial w}{\partial \nu} - \sigma_2(x) u &= -\frac{1}{h_3(0)} \{w_t + k_3 * w_t\}. \end{cases} \tag{2.8}$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{h_i(0)} h_i' * k_i = -\frac{1}{h_i(0)} h_i' \quad for \quad i = 1, 2, 3. \tag{2.9}$$

Denoting by $\tau_1 = \frac{1}{h_1(0)}$, $\tau_2 = \frac{1}{h_2(0)}$ and $\tau_3 = \frac{1}{h_3(0)}$, the normal derivatives of u, v and w can be written as

$$\frac{\partial u}{\partial \nu} = -\tau_1\{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\} - \sigma_1(x)(v+w), \tag{2.10}$$

$$\frac{\partial v}{\partial \nu} = -\tau_2\{v_t + k_2(0)v - k_2(t)v_0 + k_2' * v\}, \tag{2.11}$$

$$\frac{\partial w}{\partial \nu} = -\tau_3\{w_t + k_3(0)w - k_3(t)w_0 + k_3' * w\} + \sigma_2(x)u. \tag{2.12}$$

Reciprocally, taking initial data such that $u^0 = v^0 = w^0 = 0$ on Γ_1, the identities (2.10), (2.11) and (2.12) imply the second, the third and the fourth equations in (1.8) respectively.

Since we are interested in relaxation functions of more general decay and the boundary conditions (2.10), (2.11) and (2.12) involve the resolvent kernels k_i, $i = 1, 2, 3$, we want to know if k_i has the same decay properties. The following lemma answers this question.

Let h be a relaxation function and k its resolvent kernel, that is,

$$k(t) - (k * h)(t) = h(t). \tag{2.13}$$

Lemma 2.1. *(See Lemma 2.1, [6]) If h is a positive continuous function, then k is also positive and continuous. Suppose that*

$$h(t) \le c_0 e^{-\int_0^t \gamma(\zeta)d\zeta}$$

where $\gamma : [0, +\infty) \to \mathbb{R}^+$, is a nonincreasing function satisfying, for some positive constant $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} < \frac{1}{c_0}.$$

Then k satisfies

$$k(t) \le \frac{c_0}{1 - c_0 c_1} e^{-\varepsilon \int_0^t \gamma(\zeta)d\zeta}.$$

According to the Lemma 2.1, in what follows, we are going to use the boundary relation (2.10), (2.11) and (2.12) instead of the second, the third and the fourth equations in (1.8) respectively.

Let us define

$$(h \circ \varphi)(t) = \int_0^t h(t-s)|(\varphi(t) - \varphi(s))|^2 ds \tag{2.14}$$

and

$$(h \odot \varphi)(t) = \int_0^t h(t-s)(\varphi(t) - \varphi(s))ds. \tag{2.15}$$

Using Hölder's inequality, we have

$$|(h \odot \varphi)(t)|^2 \leq \left(\int_0^t |g(s)|ds \right) (|g| \circ \varphi)(t). \tag{2.16}$$

The following lemma gives an important property for the convolution product.

Lemma 2.2. *(See Lemma 2.2, [6]) For real functions h, $\varphi \in C^1(\mathbb{R}^+)$, we have*

$$(h * \varphi)\varphi_t = -\frac{1}{2}|\varphi(t)|^2 + \frac{1}{2}h' \circ \varphi - \frac{1}{2}\frac{d}{dt}\left(h \circ \varphi - \left(\int_0^t h(s)ds \right) |\varphi(t)|^2 \right). \tag{2.17}$$

We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Define the following space:

$$V = \left\{ v \in H^1(\Omega); v = 0 \quad \text{on} \quad \Gamma_0 \right\}. \tag{2.18}$$

The well-posedness of system (1.7) - (1.9) is given by the following theorem.

Theorem 2.3. *Let $k_i \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$. Assume that $(u^0, v^0, w^0) \in (H^2 \cap V)^3$, $(u^1, v^1, w^1) \in V^3$ and $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\theta_1 \in H_0^1(\Omega)$ with*

$$\begin{cases} \frac{\partial u^0}{\partial \nu} + \tau_1 u^0 + \sigma_1(v^0 + w^0) = 0, & on \quad \Gamma_1, \\ \frac{\partial v^0}{\partial \nu} + \tau_2 v^0 = 0 & on \quad \Gamma_1, \\ \frac{\partial w^0}{\partial \nu} + \tau_3 w^0 - \sigma_2 u^0 = 0 & on \quad \Gamma_1, \end{cases} \tag{2.19}$$

then there exists only one strong solution (u, v, w, θ) of the system (1.7)-(1.9) satisfying

$$\begin{aligned} u, v, w \in L^\infty\left([0,\infty); H^2(\Omega) \cap V\right), \quad \theta \in L^\infty([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)), \\ u_t, v_t, w_t, \in L^\infty\left([0,\infty), L^2(\Omega)\right), \\ \theta_t \in L^\infty([0,\infty); H_0^1(\Omega)), \\ u_{tt}, v_{tt}, w_{tt}, \theta_{tt} \in L^\infty([0,\infty), L^2(\Omega)). \end{aligned}$$

The Theorem 2.3 can be proved, using the Galerkin method and following exactly the procedure of [9,33,34].

3. Asymptotic behavior of solutions

In this section we study the asymptotic behavior of the solutions of system (1.7)-(1.9) when the resolvent kernels k_i, $i = 1, 2, 3$, satisfy the following

(A2) $k_i(0) > 0, \quad k_i(t) \geq 0, \quad k_i'(t) \leq 0, \quad k_i''(t) \geq \gamma_i(t)(-k_i'(t)), \quad i = 1, 2, 3$

. where $\gamma_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma_i(t) > 0, \quad \gamma_i'(t) \le 0, \quad \textit{and} \quad \int\limits_0^{+\infty} \gamma_i(t)dt = +\infty, \quad i = 1, 2, 3.$$

Multiplying the first equation in (2.4) by u_t, the second by v_t, the third by w_t and the fourth by θ_t, integrating over Ω, using a integration by parts, the boundary conditions, and (2.10)-(2.12), one can easily find that the first order energy of system (2.4) is given by

$$\begin{aligned}
E(t) \;=\; & \frac{1}{2}\int\limits_\Omega \left(\rho_1|u_t|^2 + \rho_2|v_t|^2 + \rho_1|w_t|^2 + \rho_3|\theta_t|^2\right) dx \\
&+\frac{1}{2}\int\limits_\Omega \left((\beta_1 - n\alpha_2)|u|^2 + (\beta_2 - n\alpha_1)(|v| + |w|)^2\right) dx \\
&+\frac{1}{2}\int\limits_\Omega |\nabla v|^2 dx + \frac{(1-\alpha_1)}{2}\int\limits_\Omega |\nabla u|^2 dx + \frac{(1-\alpha_2)}{2}\int\limits_\Omega |\nabla w|^2 dx \\
&+\frac{k}{2}\int\limits_\Omega |\nabla\theta|^2 dx + \frac{\alpha_1}{2}\sum_{i=1}^n \int\limits_\Omega \left|\frac{\partial u}{\partial x_i} + v + w\right|^2 dx \\
&+\frac{\alpha_2}{2}\sum_{i=1}^n \int\limits_\Omega \left|\frac{\partial w}{\partial x_i} - u\right|^2 dx \\
&+\frac{\tau_1}{2}\int\limits_{\Gamma_1} (k_1(t)|u|^2 - k_1' \circ u) d\Gamma_1 \\
&+\frac{\tau_2}{2}\int\limits_{\Gamma_1} (k_2(t)|v|^2 - k_2' \circ v) d\Gamma_1 \\
&+\frac{\tau_3}{2}\int\limits_{\Gamma_1} (k_3(t)|w|^2 - k_3' \circ w) d\Gamma_1.
\end{aligned}$$

Theorem 3.1. *Given*

$$((u^0, u^1), (v^0, v^1), (w^0, w^1), (\theta_0, \theta_1)) \in (V \times L^2(\Omega))^3 \times (H_0^1(\Omega) \times L^2(\Omega)).$$

*Assume that (**A1**) and (**A2**) hold with*

$$\lim_{t\to\infty} k_i(t) = 0, \quad i = 1, 2, 3. \tag{3.1}$$

Then for some t_0 large enough, we have, $\forall t \ge t_0$.
If $(u_0, v_0, w_0) = (0, 0, 0)$ on Γ_1, then

$$E(t) \le cE(0)e^{-\omega \int\limits_0^t \gamma(s)ds}. \tag{3.2}$$

Otherwise, If $(u_0, v_0, w_0) \neq (0,0,0)$ *on* Γ_1 *then*

$$E(t) \leq c\left\{E(0) + \int_0^t k_0(s)[1 + e^{\omega\int_{t_0}^{s}\gamma(\zeta)d\zeta}]ds\right\} e^{-\omega\int_0^t \gamma(s)ds} \tag{3.3}$$

where

$$\gamma(t) = \min\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\}$$

and

$$k_0(t) = k_1^2(t)\int_{\Gamma_1} |u^0|^2 d\Gamma_1 + k_2^2(t)\int_{\Gamma_1} |v^0|^2 d\Gamma_1 + k_3^2(t)\int_{\Gamma_1} |w^0|^2 d\Gamma_1$$

ω *is a fixed positive constant and* c *is a generic positive constant.*

Lemma 3.2. *Under the assumptions of Theorem 3.1, the energy* $E(t)$ *given by* (3.1)*, satisfies*

$$\begin{aligned}\frac{dE}{dt} \leq & -\delta\int_\Omega |\nabla\theta_t|^2 dx \\ & +\frac{\tau_1}{2}\int_{\Gamma_1}\left\{-|u_t|^2 + k_1^2(t)|u^0|^2 - k_1'' \circ u\right\} d\Gamma_1 \\ & +\frac{\tau_2}{2}\int_{\Gamma_1}\left\{-|v_t|^2 + k_2^2(t)|v^0|^2 - k_2'' \circ v\right\} d\Gamma_1 \\ & +\frac{\tau_3}{2}\int_{\Gamma_1}\left\{-|w_t|^2 + k_3^2(t)|w^0|^2 - k_3'' \circ w\right\} d\Gamma_1. \end{aligned} \tag{3.4}$$

Proof: Multiplying the first equation in (2.4) by u_t and integrating the result by parts over Ω, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_1|u_t|^2 + |\nabla u|^2 + \beta_1|u|^2)dx \\ & -\alpha_1\int_\Omega \sum_{i=1}^n \frac{\partial v}{\partial x_i}u_t dx - (\alpha_1+\alpha_2)\int_\Omega \sum_{i=1}^n \frac{\partial w}{\partial x_i}u_t dx \\ = & \int_{\Gamma_1} \frac{\partial u}{\partial \nu}u_t d\Gamma_1. \end{aligned} \tag{3.5}$$

Using Gauss's Theorem, we have

$$\alpha_1\int_\Omega \sum_{i=1}^n \frac{\partial v}{\partial x_i}u_t dx = \alpha_1\int_{\Gamma_1} vu_t(\sum_{i=1}^n \nu_i)d\Gamma_1 - \alpha_1\int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i}v dx \tag{3.6}$$

and

$$\alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial w}{\partial x_i} u_t dx = \alpha_1 \int_{\Gamma_1} w u_t (\sum_{i=1}^n \nu_i) d\Gamma_1 - \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} w dx. \tag{3.7}$$

Plugging the estimates (3.6) and (3.7) into (3.5), we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_1|u_t|^2 + |\nabla u|^2 + \beta_1|u|^2)dx \\ &+\alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} v dx + \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} w dx - \alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial w}{\partial x_i} u_t dx \\ = &\int_{\Gamma_1} \frac{\partial u}{\partial \nu} u_t d\Gamma_1 + \alpha_1 \int_{\Gamma_1} v u_t (\sum_{i=1}^n \nu_i) d\Gamma_1 + \alpha_1 \int_{\Gamma_1} w u_t (\sum_{i=1}^n \nu_i) d\Gamma_1. \end{aligned} \tag{3.8}$$

Next, by multiplying the second equation in (2.4) by v_t and integrating by parts over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_2|v_t|^2 + |\nabla v|^2 + \beta_2|v|^2)dx \\ &+\alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} v_t dx + \beta_2 \int_\Omega w v_t dx + \alpha_3 \int_\Omega \frac{\partial \theta_t}{\partial x_i} v_t dx \\ = &\int_{\Gamma_1} \frac{\partial v}{\partial \nu} v_t d\Gamma_1. \end{aligned} \tag{3.9}$$

Similarly, multiplying the third equation in (2.4) by w_t and integrating by parts over Ω we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_3|w_t|^2 + |\nabla w|^2 + \beta_2|w|^2)dx \\ &+(\alpha_1+\alpha_2) \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} w_t dx + \beta_2 \int_\Omega v w_t dx \\ = &\int_{\Gamma_1} \frac{\partial w}{\partial \nu} w_t d\Gamma_1. \end{aligned} \tag{3.10}$$

Again, using Gauss's Theorem, we get

$$\alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} w_t dx = \alpha_2 \int_{\Gamma_1} u w_t (\sum_{i=1}^n \nu_i) d\Gamma_1 - \alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial w_t}{\partial x_i} u dx. \tag{3.11}$$

Substituting the equation (3.11) in (3.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}(\rho_3|w_t|^2+|\nabla w|^2+\beta_2|w|^2)dx \\
&+\alpha_1\int_{\Omega}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}w_t dx \\
&-\alpha_2\int_{\Omega}\sum_{i=1}^{n}\frac{\partial w_t}{\partial x_i}u dx+\beta_2\int_{\Omega}v w_t dx \\
&=\int_{\Gamma_1}\frac{\partial w}{\partial \nu}w_t d\Gamma_1-\alpha_2\int_{\Gamma_1}u w_t(\sum_{i=1}^{n}\nu_i)d\Gamma_1
\end{aligned} \tag{3.12}$$

Finally, multiplying the fourth equation in (2.4) by θ_t and integrating by parts over Ω, we obtain

$$\rho_3\int_{\Omega}\theta_{tt}\theta_t dx-k\int_{\Omega}\Delta\theta\theta_t dx-\delta\int_{\Omega}\Delta\theta_t\theta_t dx+\alpha_3\int_{\Omega}\frac{\partial v_t}{\partial x_i}\theta_t dx=0. \tag{3.13}$$

Summing up the equations (3.8), (3.9), (3.12) and (3.13), using (2.10), (2.11), (2.12) and Lemma 2.2, we deduce the desired result. □

Let us define the following functionals:

$$\begin{aligned}
F_1(t) &= \int_{\Omega}(2m\cdot\nabla u+(n-\varepsilon_0)u)\rho_1 u_t dx, \\
F_2(t) &= \int_{\Omega}(2m\cdot\nabla v+(n-\varepsilon_0)v)\rho_2 v_t dx, \\
F_3(t) &= \int_{\Omega}(2m\cdot\nabla w+(n-\varepsilon_0)w)\rho_1 w_t dx, \\
F_4(t) &= \int_{\Omega}(n-\varepsilon_0)\theta(\rho_3\theta_t+\alpha_3\sum_{i=1}^{n}\frac{\partial v}{\partial x_i})dx,
\end{aligned}$$

where ε is a small positive constant, and then, we consider the following functional

$$F(t)=F_1(t)+F_2(t)+F_3(t)+F_4(t). \tag{3.14}$$

The following lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.3. *Under the assumptions of Theorem 3.1, the time derivative of the functional $F(t)$, satisfies*

$$\begin{aligned}\frac{d}{dt}F(t) \leq\ & \int_{\Gamma_1} m.\nu(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)d\Gamma_1 - \int_{\Gamma_1} m.\nu|\nabla u|^2 d\Gamma_1 \\ & -c_1\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}+v+w\right|^2 dx + \int_{\Gamma_1}\frac{\partial u}{\partial \nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1 \\ & -(1-\varepsilon_0)\int_{\Omega}(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2)dx - c_2\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial w}{\partial x_i}-u\right|^2 dx \\ & +\int_{\Gamma_1}\frac{\partial v}{\partial \nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1 + \int_{\Gamma_1}\frac{\partial w}{\partial \nu}(2m.\nabla w+(n-\varepsilon_0)w)d\Gamma_1 \\ & -\int_{\Gamma_1} m.\nu|\nabla v|^2 d\Gamma_1 - \int_{\Gamma_1} m.\nu|\nabla w|^2 d\Gamma_1 \\ & -\varepsilon_0\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)dx \\ & +(c_3+\rho_3(n-\varepsilon_0)c_4+C(\varepsilon_1)\delta)\int_{\Omega}|\nabla\theta_t|^2 dx \\ & -(n-\varepsilon_0)(k-\delta\varepsilon_1)\int_{\Omega}|\nabla\theta|^2 dx. \end{aligned} \tag{3.15}$$

Proof: A multiplication of the first equation in (2.4) by $2m.\nabla u+(n-\varepsilon_0)u$ gives

$$\begin{aligned} & \frac{d}{dt}\int_{\Omega}(2m\cdot\nabla u+(n-\varepsilon_0)u)\rho_1 u_t dx \\ =\ & \int_{\Omega} 2\rho_1 m.\nabla u_t u_t dx + (n-\varepsilon_0)\int_{\Omega}\rho_1|u_t|^2 dx \\ & +\int_{\Omega} 2m.\nabla u\Delta u dx + (n-\varepsilon_0)\int_{\Omega} u\Delta u dx \\ & +\alpha_1\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial v}{\partial x_i}dx \\ & +(\alpha_1+\alpha_2)\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial w}{\partial x_i}dx \\ & -\int_{\Omega}\beta_1 u[2m.\nabla u+(n-\varepsilon_0)u]dx. \end{aligned}$$

Integrating by parts and using the relation $divm = n$, we get

$$
\begin{aligned}
& \frac{d}{dt}\int_{\Omega}(2m\cdot\nabla u+(n-\varepsilon_0)u)\rho_1 u_t dx \\
\leq\ & \int_{\Gamma_1} m.\nu\rho_1|u_t|^2 d\Gamma_1-\varepsilon_0\int_{\Omega}\rho_1|u_t|^2 dx \\
& +\int_{\Gamma_1}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial u}{\partial\nu}d\Gamma_1-\int_{\Gamma_1} m.\nu|\nabla u|^2 d\Gamma_1 \\
& -(1-\varepsilon_0)\int_{\Omega}|\nabla u|^2 dx+\alpha_1\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial v}{\partial x_i}dx \\
& +(\alpha_1+\alpha_2)\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial w}{\partial x_i}dx \\
& +\beta_1\varepsilon_0\int_{\Omega}|u|^2 dx-\beta_1\int_{\Gamma_1} m.\nu|u|^2 d\Gamma_1.
\end{aligned}
$$

Similarly, multiplying the second equation in (2.4) by $(2m.\nabla v+(n-\varepsilon_0)v)$, integrating over Ω and using integration by parts, we arrive at

$$
\begin{aligned}
& \frac{d}{dt}\int_{\Omega}(2m\cdot\nabla v+(n-\varepsilon_0)v)\rho_2 v_t dx \\
\leq\ & \int_{\Gamma_1} m.\nu\rho_2|v_t|^2 d\Gamma_1-\varepsilon_0\int_{\Omega}\rho_2|v_t|^2 \\
& +\int_{\Gamma_1}(2m.\nabla v+(n-\varepsilon_0)v)\frac{\partial v}{\partial\nu}d\Gamma_1-\int_{\Gamma_1} m.\nu|\nabla v|^2 d\Gamma_1 \\
& -(1-\varepsilon_0)\int_{\Omega}|\nabla v|^2 dx+\beta_2\varepsilon_0\int_{\Omega}|v|^2 dx+\beta_2\varepsilon_0\int_{\Omega} vw dx \\
& -\alpha_1\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla v+(n-\varepsilon_0)v)\frac{\partial u}{\partial x_i}dx \\
& -\beta_2\int_{\Gamma_1} m.\nu|v|^2 d\Gamma_1-\beta_2\int_{\Gamma_1} m.\nu vw d\Gamma_1 \\
& +\alpha_3\sum_{i=1}^{n}\int_{\Omega}(2m\cdot\nabla v+(n-\varepsilon_0)v)\frac{\partial\theta_t}{\partial x_i}dx.
\end{aligned}
$$

Also, we multiply the third equation in (2.4) by $(2m.\nabla w+(n-\varepsilon_0)w)$ and integrate over Ω, using integration by parts, to arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega}(2m\cdot\nabla w+(n-\varepsilon_0)w)\rho_1 w_t dx\\
\leq\;& \int_{\Gamma_1} m.\nu\rho_1|w_t|^2 d\Gamma_1-\varepsilon_0\int_{\Omega}\rho_1|w_t|^2dx\\
&-\int_{\Gamma_1} m.\nu|\nabla w|^2 d\Gamma_1+\int_{\Gamma_1}(2m.\nabla w+(n-\varepsilon_0)w)\frac{\partial w}{\partial\nu}d\Gamma_1\\
&+\beta_2\varepsilon_0\int_{\Omega} vwdx-(1-\varepsilon_0)\int_{\Omega}|\nabla w|^2dx+\beta_2\varepsilon_0\int_{\Omega}|w|^2dx\\
&-(\alpha_1+\alpha_2)\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla w+(n-\varepsilon_0)w)\frac{\partial u}{\partial x_i}dx\\
&-\beta_2\int_{\Gamma_1} m.\nu|w|^2d\Gamma_1-\beta_2\int_{\Gamma_1} m.\nu vwd\Gamma_1.
\end{aligned}$$

Finally, we multiply the fourth equation in (2.4) by $(n-\varepsilon_0)\theta$ and integrate over Ω, using integration by parts, to arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega}(n-\varepsilon_0)\theta(\rho_3\theta_t+\alpha_3\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}dx\\
=\;& \rho_3(n-\varepsilon_0)\int_{\Omega}|\theta_t|^2dx-k(n-\varepsilon_0)\int_{\Omega}|\nabla\theta|^2dx\\
&-\delta(n-\varepsilon_0)\int_{\Omega}\nabla\theta\nabla\theta_t dx+\alpha_3(n-\varepsilon_0)\int_{\Omega}\theta_t\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}dx.
\end{aligned}$$

Summing the above four inequalities, we easily deduce that

$$\begin{aligned}
\frac{d}{dt}F(t)\leq&\int_{\Gamma_1} m.\nu(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)d\Gamma_1-\beta_2\int_{\Gamma_1} m.\nu(|v|^2+|w|^2)d\Gamma_1\\
&-\int_{\Gamma_1} m.\nu(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2d\Gamma_1+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1\\
&-\varepsilon_0\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)dx\\
&-(1-\varepsilon_0)\int_{\Omega}(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2)dx+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_1\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial v}{\partial x_i}dx\\
&+\int_{\Gamma_1}\frac{\partial w}{\partial\nu}(2m.\nabla w+(n-\varepsilon_0)w)d\Gamma_1-\beta_1\int_{\Gamma_1}m.\nu|u|^2d\Gamma_1\\
&+(\alpha_1+\alpha_2)\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla u+(n-\varepsilon_0)u)\frac{\partial w}{\partial x_i}dx-2\beta_2\int_{\Gamma_1}m.\nu vwd\Gamma_1\\
&-\alpha_1\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla v+(n-\varepsilon_0)v)\frac{\partial u}{\partial x_i}dx+\beta_1\varepsilon_0\int_{\Omega}|u|^2dx\\
&+\beta_2\varepsilon_0\int_{\Omega}(|v|^2+|w|^2)dx-(\alpha_1+\alpha_2)\sum_{i=1}^{n}\int_{\Omega}(2m.\nabla w+(n-\varepsilon_0)w)\frac{\partial u}{\partial x_i}dx\\
&+2\beta_2\varepsilon_0\int_{\Omega}vwdx+\alpha_3\sum_{i=1}^{n}\int_{\Omega}2m\cdot\nabla v\frac{\partial\theta_t}{\partial x_i}dx+\rho_3(n-\varepsilon_0)\int_{\Omega}|\theta_t|^2dx\\
&-k(n-\varepsilon_0)\int_{\Omega}|\nabla\theta|^2dx-\delta(n-\varepsilon_0)\int_{\Omega}\nabla\theta\nabla\theta_tdx.
\end{aligned}
$$

We use the fact that there exists a positive constant c_3 such that

$$
\begin{aligned}
&\alpha_1\sum_{i=1}^{n}\int_{\Omega}\left(\frac{\partial v}{\partial x_i}m.\nabla u+\frac{\partial u}{\partial x_i}m.\nabla v+\frac{\partial w}{\partial x_i}m.\nabla u+\frac{\partial u}{\partial x_i}m.\nabla w\right)dx\\
&+\alpha_2\sum_{i=1}^{n}\int_{\Omega}\left(\frac{\partial w}{\partial x_i}m.\nabla u+\frac{\partial u}{\partial x_i}m.\nabla w\right)dx+\alpha_3\sum_{i=1}^{n}\int_{\Omega}2m\cdot\nabla v\frac{\partial\theta_t}{\partial x_i}dx\\
\leq\;& c_3\max\{\alpha_1,\alpha_2,\alpha_3\}\int_{\Omega}\left(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2+|\nabla\theta_t|^2\right)dx
\end{aligned}
$$

and using Poincaré inequality, taking ε_0 small enough, we get

$$
\begin{aligned}
\frac{d}{dt}F(t)\leq&\int_{\Gamma_1}m.\nu(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)d\Gamma_1\\
&-\varepsilon_0\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2+\rho_1|w_t|^2)dx\\
&+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1-\int_{\Gamma_1}m.\nu|\nabla u|^2d\Gamma_1\\
&-(1-\varepsilon_0)\int_{\Omega}(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2)dx
\end{aligned}
$$

$$
\begin{aligned}
&- c_1 \sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx + \int\limits_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m.\nabla v + (n - \varepsilon_0) v) d\Gamma_1 \\
&- \int\limits_{\Gamma_1} m.\nu |\nabla v|^2 d\Gamma_1 - c_2 \sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx \\
&+ \int\limits_{\Gamma_1} \frac{\partial w}{\partial \nu} (2m.\nabla w + (n - \varepsilon_0) w) d\Gamma_1 - \int\limits_{\Gamma_1} m.\nu |\nabla w|^2 d\Gamma_1 \\
&+ (c_3 + \rho_3 (n - \varepsilon_0) c_4 + C(\varepsilon)\delta) \int\limits_{\Omega} |\nabla \theta_t|^2 dx \\
&- (n - \varepsilon_0)(k - \delta\varepsilon) \int\limits_{\Omega} |\nabla \theta|^2 dx.
\end{aligned}
$$

The proof of Lemma 3.3 is completed. □

Now, we introduce the Lyapunov functional. So, for $N > 0$ large enough, let

$$
L(t) = NE(t) + F(t). \tag{3.16}
$$

Applying Young's inequality and Poincaré inequality to the boundary integrals we have, for $\varepsilon > 0$,

$$
\begin{aligned}
\int\limits_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m.\nabla u + (n - \varepsilon_0) u) d\Gamma_1 &\leq \varepsilon \left(\int\limits_{\Omega} |\nabla u|^2 dx + \int\limits_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 \right) \\
&\quad + C_\varepsilon \int\limits_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma_1,
\end{aligned}
$$

$$
\begin{aligned}
\int\limits_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m.\nabla v + (n - \varepsilon_0) v) d\Gamma_1 &\leq \varepsilon \left(\int\limits_{\Omega} |\nabla v|^2 dx + \int\limits_{\Gamma_1} m.\nu |\nabla v|^2 d\Gamma_1 \right) \\
&\quad + C_\varepsilon \int\limits_{\Gamma_1} \left| \frac{\partial v}{\partial \nu} \right|^2 d\Gamma_1
\end{aligned}
$$

and

$$
\begin{aligned}
\int\limits_{\Gamma_1} \frac{\partial w}{\partial \nu} (2m.\nabla w + (n - \varepsilon_0) w) d\Gamma_1 &\leq \varepsilon \left(\int\limits_{\Omega} |\nabla w|^2 dx + \int\limits_{\Gamma_1} m.\nu |\nabla w|^2 d\Gamma_1 \right) \\
&\quad + C_\varepsilon \int\limits_{\Gamma_1} \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma_1.
\end{aligned}
$$

By rewriting the boundary conditions (2.10), (2.11) and (2.12) as

$$
\begin{aligned}
\frac{\partial u}{\partial \nu} &= -\tau_1\{u_t + k_1(t)u - k_1(t)u_0 - k_1' \odot u\}, \quad on \quad \Gamma_1 \times \mathbb{R}^+, \\
\frac{\partial v}{\partial \nu} &= -\tau_2\{v_t + k_2(t)v - k_2(t)v_0 - k_2' \odot v\}, \quad on \quad \Gamma_1 \times \mathbb{R}^+, \\
\frac{\partial w}{\partial \nu} &= -\tau_3\{w_t + k_3(t)w - k_3(t)w_0 - k_3' \odot w\}, \quad on \quad \Gamma_1 \times \mathbb{R}^+,
\end{aligned}
$$

and combining all above relations, we obtain

$$
\begin{aligned}
\frac{dL}{dt}(t) \leq &- (\frac{N\tau_1}{2} - C_\varepsilon - m.\nu\rho_1) \int\limits_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{N\tau_1}{2} \int\limits_{\Gamma_1} k_1^2(t)|u^0|^2 d\Gamma_1 \\
&- (1-\varepsilon) \int\limits_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 - \frac{N\tau_1}{2} \int\limits_{\Gamma_1} k_1'' \circ u d\Gamma_1 \\
&- (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k_1^2(t)) \int\limits_{\Omega} |\nabla u|^2 dx \\
&- (\frac{N\tau_2}{2} - C_\varepsilon - m.\nu\rho_2) \int\limits_{\Gamma_1} |v_t|^2 d\Gamma_1 + \frac{N\tau_2}{2} \int\limits_{\Gamma_1} k_2^2(t)|v^0|^2 d\Gamma_1 \\
&- (1-\varepsilon) \int\limits_{\Gamma_1} m.\nu |\nabla v|^2 d\Gamma_1 - \frac{N\tau_2}{2} \int\limits_{\Gamma_1} k_2'' \circ v d\Gamma_1 \\
&- (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k_2^2(t)) \int\limits_{\Omega} |\nabla v|^2 dx \\
&- (\frac{N\tau_3}{2} - C_\varepsilon - m.\nu\rho_1) \int\limits_{\Gamma_1} |w_t|^2 d\Gamma_1 + \frac{N\tau_3}{2} \int\limits_{\Gamma_1} k_3^2(t)|w^0|^2 d\Gamma_1 \\
&- (1-\varepsilon) \int\limits_{\Gamma_1} m.\nu |\nabla w|^2 d\Gamma_1 - \frac{N\tau_3}{2} \int\limits_{\Gamma_1} k_3'' \circ w d\Gamma_1 \\
&- (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k_3^2(t)) \int\limits_{\Omega} |\nabla w|^2 dx \\
&- \varepsilon_0 \int\limits_{\Omega} \left(\rho_1 |u_t|^2 + \rho_2 |v_t|^2 + \rho_1 |w_t|^2\right) dx - c_1 \sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx \\
&- c_2 \sum_{i=1}^{n} \int\limits_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&-(N\delta - c_3 - \rho_3(n-\varepsilon_0)c_4 - C(\varepsilon_1)\delta)\int_\Omega |\nabla\theta_t|^2 dx \\
&-(n-\varepsilon_0)(k-\delta\varepsilon_1)\int_\Omega |\nabla\theta|^2 dx \\
&+C_\varepsilon \int_{\Gamma_1} |k_1' \odot u|^2 d\Gamma_1 + C_\varepsilon \int_{\Gamma_1} |k_2' \odot v|^2 d\Gamma_1 + C_\varepsilon \int_{\Gamma_1} |k_3' \odot w|^2 d\Gamma_1 \\
&+C_\varepsilon k_1^2(t)\int_{\Gamma_1} |u^0|^2 d\Gamma_1 + C_\varepsilon k_2^2(t)\int_{\Gamma_1} |v^0|^2 d\Gamma_1 + C_\varepsilon k_3^2(t)\int_{\Gamma_1} |w^0|^2 d\Gamma_1.
\end{aligned}
$$

Now, we take

$$
\varepsilon = \varepsilon_0 < \min\{\frac{1}{2}, \frac{k}{\delta}\}. \tag{3.17}
$$

Once ε is fixed, we pick N large enough so that

$$
\begin{aligned}
N\tau_1 &> 2(C_\varepsilon + \max|m.\nu\rho_1|), \\
N\tau_2 &> 2(C_\varepsilon + \max|m.\nu\rho_2|), \\
N\tau_3 &> 2(C_\varepsilon + \max|m.\nu\rho_1|), \\
N\delta &> c_3 + \rho_3(n-\varepsilon_0)c_4 + C(\varepsilon)\delta.
\end{aligned}
$$

Using the fact that $\lim_{t\to\infty} k_i(t) = 0$ for $i = 1, 2, 3$, and (2.16) , we arrive at

$$
\begin{aligned}
\frac{dL}{dt}(t) \leq\ & -C_1 E(t) \\
& +C_2 \int_{\Gamma_1} \left\{\gamma_1(t)k_1^2(t)|u^0|^2 + \gamma_2(t)k_2^2(t)|v^0|^2 + \gamma_3(t)k_3^2(t)|w^0|^2\right\} d\Gamma_1 \\
& -C_3 \int_{\Gamma_1} \{k_1' \circ u + k_2' \circ v + k_3' \circ w\} d\Gamma_1, \quad \forall t \geq t_0
\end{aligned} \tag{3.18}
$$

for some t_0 large enough and some positive constants C_1, C_2 and C_3.

If $\gamma(t) = \min\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\}$ for $t \geq t_0$, we multiply both sides of (3.18) by $\gamma_1(t)$ to get

$$
\begin{aligned}
\gamma(t)\frac{dL}{dt}(t) \leq & - C_1\gamma(t)E(t) \\
& + C_2 \int_{\Gamma_1} \left\{\gamma_1(t)k_1^2(t)|u^0|^2 + \gamma_2(t)k_2^2(t)|v^0|^2 + \gamma_3(t)k_3^2(t)|w^0|^2\right\} d\Gamma_1 \\
& - \gamma(t)C_3 \int_{\Gamma_1} \{k_1' \circ u + k_2' \circ v + k_3' \circ w\} d\Gamma_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq -C_1\gamma(t)E(t)\\
&\quad + C_2 \int_{\Gamma_1} \left\{\gamma_1(t)k_1^2(t)|u^0|^2 + \gamma_2(t)k_2^2(t)|v^0|^2 + \gamma_3(t)k_3^2(t)|w^0|^2\right\} d\Gamma_1\\
&\quad + C_3 \int_{\Gamma_1} \{\gamma_1(t)(-k_1)' \circ u + \gamma_2(t)(-k_2') \circ v + \gamma_3(t)(-k_3') \circ w\} d\Gamma_1,
\end{aligned}
$$

$\forall t \geq t_0$.

Using (**A2**) and the fact that $\gamma_i(t)$ is nonincreasing, we obtain

$$
\begin{aligned}
\gamma(t)\frac{dL}{dt}(t) \quad \leq \quad & -C_1\gamma(t)E(t)\\
& + C_2 \int_{\Gamma_1} \left\{\gamma_1(t)k_1^2(t)|u^0|^2 + \gamma_2(t)k_2^2(t)|v^0|^2 + \gamma_3(t)k_3^2(t)|w^0|^2\right\} d\Gamma_1\\
& + C_3 \int_{\Gamma_1} \left\{\gamma_1(t)k_1^{''} \circ u + \gamma_2(t)k_2^{''} \circ vd + \gamma_3(t)k_3^{''} \circ w\right\} d\Gamma_1, \quad \forall t \geq t_0.
\end{aligned}
$$

Next, by using (3.4), we easily see that

$$
\begin{aligned}
\gamma(t)\frac{dL}{dt}(t) \quad \leq \quad & -C_1\gamma(t)E(t)\\
& + C_4 \int_{\Gamma_1} \left\{k_1^2(t)|u^0|^2 + k_2^2(t)|v^0|^2 + k_3^2(t)|w^0|^2\right\} d\Gamma_1\\
& - C\frac{dE}{dt}, \quad \forall t \geq t_0,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\gamma(t)\frac{dL}{dt}(t) + C\frac{dE}{dt} \quad \leq \quad & -C_1\gamma(t)E(t)\\
& + C_4 \int_{\Gamma_1} \left\{k_1^2(t)|u^0|^2 + k_2^2(t)|v^0|^2 + k_3^2(t)|w^0|^2\right\} d\Gamma_1,
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{d}{dt}(\gamma(t)L(t) + CE(t) - \gamma'(t)L(t)\\
&\leq -C_1\gamma(t)E(t) + C_4 \int_{\Gamma_1} \left[k_1^2(t)|u^0|^2 + k_2^2(t)|v^0|^2 + k_3^2(t)|w^0|^2\right] d\Gamma_1. \quad (3.19)
\end{aligned}
$$

Again using the fact that $\gamma_1(t)$ is noninceasing and setting

$$
L_1(t) = \gamma(t)L(t) + CE(t) \sim E(t), \tag{3.20}
$$

estimate (3.19) gives

$$\begin{aligned}\frac{d}{dt}L_1(t) \quad &\leq \quad -\omega\gamma(t)L_1(t)\\ &\quad +c\int_{\Gamma_1} k_1^2(t)|u^0|^2 d\Gamma_1\\ &\quad +c\int_{\Gamma_1} k_2^2(t)|v^0|^2 d\Gamma_1\\ &\quad +c\int_{\Gamma_1} k_3^2(t)|w^0|^2 d\Gamma_1, \quad \forall t \geq t_0 \end{aligned} \tag{3.21}$$

(i) if $u^0 = v^0 = w^0 = 0$ on Γ_1, then (3.21) reduces to

$$\frac{d}{dt}L_1(t) \leq -\omega\gamma(t)L_1(t), \quad \forall t \geq t_0. \tag{3.22}$$

A integration over (t_0, t) yields

$$L_1(t) \leq L_1(t_0)e^{-\omega\int_{t_0}^{t}\gamma(s)ds}, \quad \forall t \geq t_0. \tag{3.23}$$

By using (3.20), then we obtain for some positive constant c

$$E(t) \leq cE(t_0)e^{-\omega\int_{t_0}^{t}\gamma(s)ds}, \quad \forall t \geq t_0. \tag{3.24}$$

So we get

$$E(t) \leq cE(0)e^{-\omega\int_{0}^{t}\gamma(s)ds}, \quad \forall t \geq t_0 \tag{3.25}$$

(ii) if $(u^0, v^0, w^0) \neq (0,0,0)$ on Γ_1, then (3.21) gives

$$\frac{d}{dt}L_1(t) \leq -\omega\gamma(t)L_1(t) + C_1k_1^2(t) + C_2k_2^2(t) + C_3k_3^2(t) \quad \forall t \geq t_0, \tag{3.26}$$

where

$$C_1 = c\int_{\Gamma_1} |u^0|^2 d\Gamma_1, \quad C_2 = c\int_{\Gamma_1} |v^0|^2 d\Gamma_1, \quad C_3 = c\int_{\Gamma_1} |w^0|^2 d\Gamma_1.$$

In this case, we introduce

$$\begin{aligned} L_2(t) &= L_1(t) \\ &-C_1 e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \int_{t_0}^{t} k_1^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds \\ &-C_2 e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \int_{t_0}^{t} k_2^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds \\ &-C_3 e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \int_{t_0}^{t} k_3^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds. \end{aligned}$$

Differentiating $L_2(t)$ and using (3.21), we find that

$$\frac{d}{dt} L_2(t) \leq -\omega\gamma(t) L_2(t), \quad \forall t \geq t_0. \tag{3.27}$$

Again, integrating over (t_0, t), we obtain

$$L_2(t) \leq L_2(t_0) e^{-\omega \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0, \tag{3.28}$$

which implies

$$\begin{aligned} L_1(t) \leq\ & L_1(t_0) e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \\ &+ \left(C_1 \int_{t_0}^{t} k_1^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds \right) e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \\ &+ \left(C_2 \int_{t_0}^{t} k_2^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds \right) e^{-\omega \int_{t_0}^{t} \gamma(s)ds} \\ &+ C_3 \left(\int_{t_0}^{t} k_3^2(s) e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} ds \right) e^{-\omega \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0. \end{aligned}$$

Using (3.20) and (3.4), then we obtain for some positive constant C

$$\begin{aligned} E(t) \leq & C E(0) e^{\omega \int_{0}^{t_0} \gamma(s)ds} e^{-\omega \int_{0}^{t} \gamma(s)ds} \\ &+ C \left\{ C_1 \int_{0}^{t} k_1^2(s) \left[\frac{\tau_1}{2c} + e^{\omega \int_{t_0}^{s} \gamma(\zeta)d\zeta} \right] ds \right\} e^{\omega \int_{0}^{t_0} \gamma(s)ds} e^{-\omega \int_{0}^{t} \gamma(s)ds} \end{aligned}$$

$$+C\left\{C_2\int_0^t k_2^2(s)\left[\frac{\tau_2}{2c}+e^{\omega\int_{t_0}^{s}\gamma(\zeta)d\zeta}\right]ds\right\}e^{\omega\int_0^{t_0}\gamma_1(s)ds}e^{-\omega\int_0^{t}\gamma_1(s)ds}$$
$$+C\left\{C_3\int_0^t k_3^2(s)\left[\frac{\tau_3}{2c}+e^{\omega\int_{t_0}^{s}\gamma_1(\zeta)d\zeta}\right]ds\right\}e^{\omega\int_0^{t_0}\gamma_1(s)ds}e^{-\omega\int_0^{t}\gamma_1(s)ds}.$$

This completes the proof of Theorem 3.1

Remark 3.4. *As in [21], here our result gives more general decay rates for which the exponential and polynomial decay estimates are just particular cases of* (3.2) *and* (3.3). *In fact, we obtain exponential decay for* $\gamma_i(t) = c$ *and polynomial decay for* $\gamma_i(t) = c(1+t)^{-1}$, $i = 1, 2, 3$, *where* c *is a positive constant.*

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Ammar Khemmoudj
Faculty of Mathematics,
University of Science and Technology Houari Boumedienne,
PO Box 32, El-Alia 16111, Bab Ezzouar,Algiers,
Algeria.
E-mail address: akhemmoudj@yahoo.fr