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Chaotic Optimization Algorithm Based on the Modified Probability Density Function of Lozi Map

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Abstract: Chaos optimization algorithms (COAs) usually utilize different chaotic maps(logistic, tent, Hénon, Lozi,...) to generate the pseudo-random numbers mapped as the design variables for global optimization. In this paper we are going to propose new technique to improve the chaotic optimization algorithm by using some transformations to modify the density of the map instead of changing it.

Key Words: Chaos optimization algorithm, Non linear test functions, Probability density function, Lozi map.

Contents

1. Introduction

Chaos theory has achieved great success since its early years through wide applications in other sciences such as physics, mechanics, electronics, biology, economy, astronomy, meteorology, optimisation,...ect [1,2,3,4,5].

Generally speaking, chaos has several important dynamical characteristics, namely, the sensitive dependence on initial conditions, ergodicity, pseudo-randomness, and strange attractor with self-similar fractal pattern. As far as optimization problems of some usual functions that are continuously differentiable are concerned, some traditional optimization algorithms, such as the Newton method, the gradient method and the Hessians methods [6,7,8] can get their global optimal points with the advantage of speed convergence and high precision.

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However, these traditional optimization algorithms will easily trap into local optimization when solving optimization problems of some multi-modal functions. Many authors use the discrete mappings that have chaotic behaviour in the optimisation algorithm in order to get out of this trap [9,10,11]. Recently, researchers have focused on developing hybrid algorithms by combining heuristic algorithms with chaos searching technique to solve non linear system of equations and optimization problems such as chaotic Monte Carlo optimization, chaotic BFGS, chaotic particle swarm optimization, chaotic genetic algorithms, chaotic harmony search algorithm, chaotic simulated annealing, gradient-based methods and so on [12,13,14]. Due to the non-repetition of chaos, the chaotic optimization algorithm can carry out overall searches at higher speeds than stochastic ergodic searches that depend on probabilities.

Different types of chaotic systems have been considered in literature for applications in optimization methods. The logistic equation and other equations, such as tent map, Gauss map, Lozi map, Hénon map, sinusoidal iterator, Chua's oscillator, Ikeda map, and others, have been adopted instead of random ones with very interesting results [15,16,17,18,19,20,21].

The disadvantage of this method is that if the density of chaotic sequences generated by the map selected is not high in the vicinity of the global optimum point, it is very likely that we won't find this point. Motivated by this reason we are going to suggest our main idea.

2. The Main Idea

As mentioned above the probability density function of chaotic sequences generated by chaotic maps obviously influences the efficiency of hybrid chaos optimization algorithms. If many sample points lie in the vicinity of global optimum, then the success ratio to find the global optimum in design space is high. In other words, if the probability density function of the chaotic sequences is high in the neighbourhood of global optimum point, then the success ratio to find the global optimum point is high. This gave birth to the idea we are working on; which states that instead of switching the map used in optimization algorithm, we choose a map that has a good chaotic behaviour and then we use some transformations to change the density of this map. After that, for any test function we use the transformation that gives the best results.

It is important to note that in this paper we use the Lozi map [22] defined as follows:

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1\begin{pmatrix} x \\ y \end{pmatrix} \\ L_2\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 - a \mid x \mid + by \\ x \end{pmatrix}. \tag{2.1}$$

It is a 2-d invertible iterated map that gives a chaotic attractor called the Lozi attractor which is obtained for $a = 1.4$ and $b = 0.3$ as shown in figure 1.

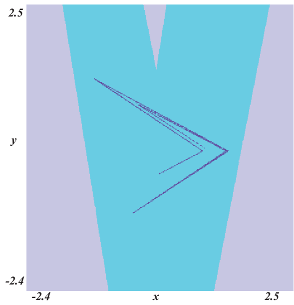

Figure 1: Lozi attractor obtained for $a = 1.7$ and $b = 0.3$.

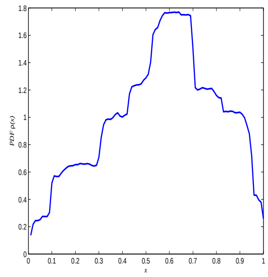

Figure 2: Density of iterated values of $x(k)$ of map (1) over the interval $[0, 1]$ splitted in 100 boxes for $10, 000, 000$ iterated values.

Numerical computation of the density $\rho(s)$ of iterated values $x(k)$ is displayed in figure 2. In this figure, the iterated values $x(k)$ are normalized in the range $[0, 1]$ i.e. $\int_0^1 \rho(x)\,\mathrm{d}x = 1$ where we observe that the highest value of $\rho(s)$ is approximately 1.8 when s is in the neighbourhood of 0.6.

For further explanation we suggest the following example:

Example 2.1. *Let the transformation S (Figure 3) defined on the interval [0,1] in itself such as*

$$S(x) = \begin{cases} \frac{1}{3}x \; if \; 0 \leq x \leq 0.75 \\ 3x - 2 \; if \; 0.75 \leq x \leq 1 \end{cases} \tag{2.2}$$

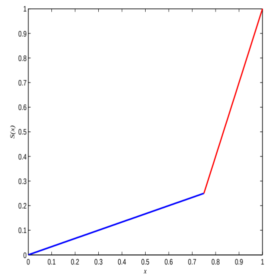

Figure 3: The transformation S.

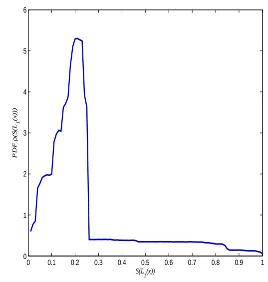

Figure 4: Probability density function of $S(L_1)$.

then the density of iterated values of S is as shown in Figure 4.

As shown in Figure 4 the majority of the values $S(L_1)$ are in the neighbourhood of 0.2; so if the test function has its optimal point in this region, it is quite possible to find it. But if the global optimum point is in the vicinity of 1, it is unlikely that we will find it because the density function of the chaotic sequence is low.

3. Chaotic Optimization Method (COM)

Many unconstrained optimization problems with continuous variables can be formulated as the following functional optimization problem.
Find x to minimize $f(x)$, $x = (x_1, x_2, ..., x_n)$. Subject to $x_i \in [L_i, U_i]$, $i = 1, 2, ..., n$, where f is the objective function, and x is the decision solution vector consisting of n variables $x_i \in \mathbb{R}$ bounded by lower and upper limits L_i and U_i respectively.

In order to test the effectiveness of this idea we will combine it with chaotic optimization method proposed by Coelho in [23] and improved by Hamaizia and Lozi in [24] and we use it to find the optimal solution of some test functions. Therefore the COM becomes as follows:

Firstly, the Lozi map is adopted to have a chaotic behaviour used to generate several sequences of points by using different initial conditions (the number of sequences is equal to the dimension of the objective function).

Secondly, we use a transformation in order to modify the density function of the Lozi map as in the above example.

Thirdly, every sequence $\{y(i), i = 1, 2, ...n\}$ is normalized in the range $[0, 1]$ as follows:

$$z(i) = \frac{y(i) - \alpha}{\beta - \alpha}, \tag{3.1}$$

for all $i = 1, 2, ...n$, where $\alpha = min\{(y(i), i \geq 1\}$, $\beta = max\{(y(i), i \geq 1\}$.

The rest are

Algorithm 1.

Inputs:

M_g***:*** *max number of iterations of chaotic Global search.*
Mgl_1***:*** *max number of iterations of first chaotic Local search in Global search.*
Mgl_2***:*** *max number of iterations of second chaotic Local search in Global search.*
M_l***:*** *max number of iterations of chaotic Local search.*
$Mt = M_g(Mgl_1 + Mgl_2) + M_l$***:*** *stopping criterion of chaotic optimization method in iterations.*
λ_{gl1}***:*** *step size in first Global-Local search.*
λ_{gl2}***:*** *step size in second Global-Local search.*
λ *step size in chaotic local search.*

Outputs:

$\bar{x}$***:*** *best solution from current run of chaotic search.*
$\bar{f}$***:*** *best objective function (minimization problem).*

Step 1: *Initialization of the numbers* M_g*,* Mgl_1*,* Mgl_2*,* M_l *of steps of chaotic search and initialization of parameters* λ_{gl1}*,* λ_{gl1}*,* λ *and initial conditions. Set* $k = 1$*,* $y_1(1)$*,* $y_2(1)$*,* $a = 1.1$ *and* $b = 0.3$*. Set the initial best objective function* $\bar{f} = +\infty$*.*

Step 2: Algorithm of chaotic global search:

while $k \leq M_g$ ***do***
$x_i(k) = L_i + z_i(k)(U_i - L_i)$, $i = 1, 2, ..., n$
if $f(x(k)) < \bar{f}$, ***then***
$\bar{x} = x(k)$, $\bar{f} = f(x(k))$
end if

Step 2-1: sub algorithm of first chaotic global-local search:

while $j \leq M_{gl1}$ ***do***
for $i = 1$ *to* n ***do***
if $r \leq 0.5$ ***then (where*** r ***is a uniformly distributed random variable with range*** $[0, 1]$***)***
$x_i(j) = \bar{x_i} + \lambda_{gl1} z_i(j)(U_i - \bar{x_i})$
else
$x_i(j) = \bar{x_i} - \lambda_{gl1} z_i(j)(\bar{x_i} - L_i)$
end if
end for
if $f(x(j)) < \bar{f}$, ***then***

$\bar{x} = x(j),\ \bar{f} = f(x(j))$
end if
$j = j + 1$
end while

Step 2-2: sub algorithm of second chaotic global-local search:
while $s \leq M_{gl2}$ ***do***
for $i = 1$ *to* n ***do***
if $r \leq 0.5$ ***then***
$x_i(s) = \bar{x_i} + \lambda_{gl2} z_i(s)(U_i - \bar{x_i})$
else
$x_i(s) = \bar{x_i} - \lambda_{gl2} z_i(s)(\bar{x_i} - L_i)$
end if
end for
if $f(x(s)) < \bar{f}$, ***then***
$\bar{x} = x(s),\ \bar{f} = f(x(s))$
end if
$s = s + 1$
end while
$k = k + 1$
end while

Step 3: algorithm of chaotic local search:
while $k \leq M_l$ ***do***
for $i = 1$ *to* n ***do***
if $r \leq 0.5$ ***then***
$x_i(k) = \bar{x_i} + \lambda z_i(k)(U_i - \bar{x_i})$
else
$x_i(k) = \bar{x_i} - \lambda z_i(k)(\bar{x_i} - L_i)$
end if
end for
if $f(x(k)) < \bar{f}$, ***then***
$\bar{x} = x(k),\ \bar{f} = f(x(k))$
end if
$k = k + 1$
end while

In order to enrich our study, we are going to use different values of steps' size λ, λ_{gl1} and λ_{gl2} and different values of the number of iterations M_g, Mgl_1, Mgl_2 and M_l as in table 1. During the chaotic local search, the step size λ (resp $\lambda_{gl1}, \lambda_{gl2}$) is an important parameter in convergence behaviour of optimization method which adjusts small ergodic ranges around x^*. The steps size λ and λ_{gl} are employed to control the impact of the current best solution on generating a new trial solution. Small λ and λ_{gl} tend to perform exploitation to refine results by local search, while large ones tend to facilitate a global exploration of search space.

Table 1: The set of parameters values for every run of the COM algorithm

	λ	λ_{gl1}	λ_{gl2}	M_g	M_l	Mgl_1	Mgl_2	Mt
C1	0.01	0.04	0.01	50	10	5	5	510
C2	0.01	0.4	0.01	50	10	5	5	510
C3	0.01	0.04	0.01	100	50	5	5	1050
C4	0.001	0.04	0.01	200	100	5	5	2100

4. Numerical Examples and Discussion

4.1. Some Transformations

In this section, we are going to suggest some transformations in order to use them for changing the density functions of the Lozi map.

1. $$S_1(x) = x^2 + 3x. \tag{4.1}$$

2. Let m be the minimum of the sequence generated by Lozi map and M be its maximum and let α and β be two real numbers. We define the transformation S_2 as follows:
$$S_2(x) = \frac{-2\alpha}{m - M}x + \beta - \alpha - \frac{-2\alpha}{m - M}m. \tag{4.2}$$
We note that if we have doubt that the optimal point is x^* we can choose α and β so that the majority of the points $S_2(x)$ are in the neighbourhood of x^*. In this paper we set $\alpha = 0.001$ and $\beta = \frac{1}{2}$.

3. $$S_3(x) = \tan(x). \tag{4.3}$$

4. $$S_4(x) = \log(|x|)\sin(x). \tag{4.4}$$

Figure 5 shows the probability density function ρ_1 of $S_1(L_1)$ through which we see that the function ρ_1 is decreasing on the interval $[0.5, 1]$ which contains less points of $S_1(L_1)$ than the interval $[0, 0.5]$. The density ρ_2 of $S_2(L_1)$ is shown in Figure 6 where we notice that the majority of the points are in the neighbourhood of 0.5 so if the optimal point is near 0.5, it is very likely to find it but if the global minimum is far from 0.5, then this transformation is not suitable. The density ρ_3 of $S_3(L_1)$ shown in Figure 7 resembles the normal density. Finally, the density function of $S_4(L_1)$ is like that of the logistic map, see Figure 8.

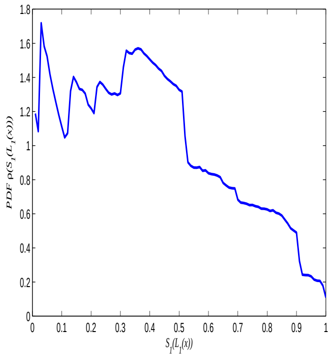

Figure 5: Probability density function of $S_1(L_1)$.

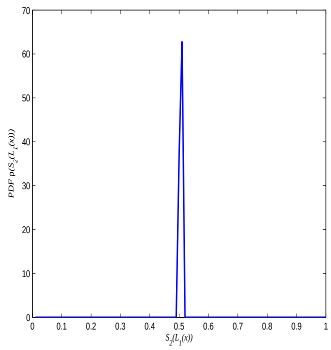

Figure 6: Probability density function of $S_2(L_1)$.

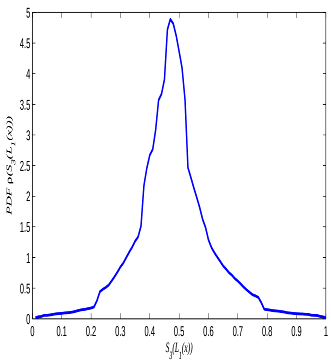

Figure 7: Probability density function of $S_3(L_1)$.

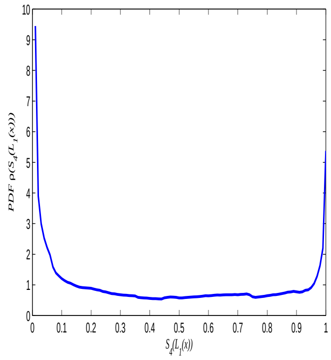

Figure 8: Probability density function of $S_4(L_1)$.

4.2. Some Test Functions

In this subsection we are going to give some test functions that are used to examine the effectiveness of this new method.

1.
$$f_1(x_1, x_2, ..., x_n) = \frac{\sum_{i=1}^{n}(x_i^4 - 16x_i^2 + 5x_i)}{2}, \tag{4.5}$$
where $-5 \leq x_i \leq 5$ for $1 \leq i \leq n$.

2.
$$\begin{aligned} f_2(x_1, x_2) = x_1^4 - 7x_1^2 + x_2^4 - 9x_2^2 - 5x_2 + 11x_1^2x_2^2 + \\ + 99\sin(71x_1) + 137\sin(97x_1x_2) + 131\sin(51x_2), \end{aligned} \tag{4.6}$$
where $-10 \leq x_1 \leq 10$ and $-10 \leq x_2 \leq 10$.

3.
$$\begin{aligned} f_3(x_1, x_2) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \times \\ [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)], \end{aligned} \tag{4.7}$$
where $-2 \leq x_1 \leq 2$ and $-2 \leq x_2 \leq 2$.

4.
$$f_4(x_1, x_2) = 100\sqrt{|x_2 - 0.01x_1^2|} + 0.01\,|x_1 + 10|\,, \tag{4.8}$$
where $-15 \leq x_1 \leq -5$ and $-3 \leq x_2 \leq 3$.

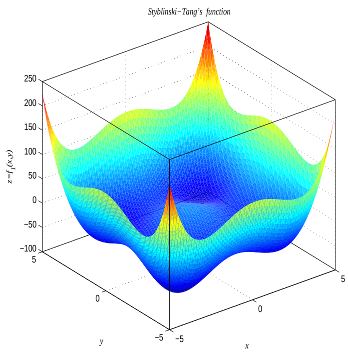

Figure 9: Styblinski-Tang's functionf_1.

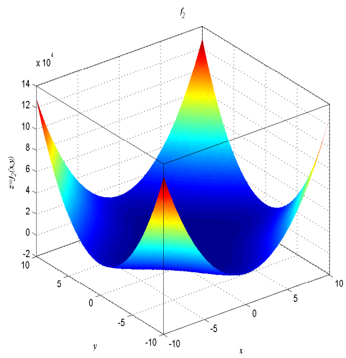

Figure 10: Function f_2.

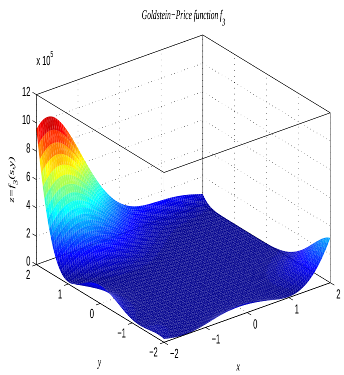

Figure 11: Goldstein-Price function f_3.

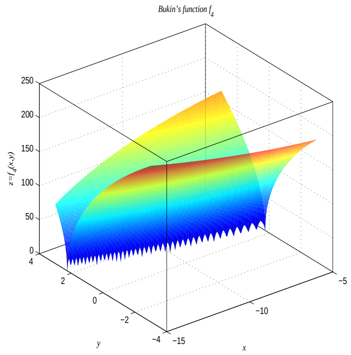

Figure 12: Bukin function f_4.

Figures 9 shows the 3D plots of the Styblinski-Tang function. f_1 is a d-dimensional function, and usually evaluated on the hypercube $x_i \in [-5, 5]$, for all $i = 1, ..., d$ where d is the dimension of f_1, it has a global minimum

$$-39.16617 \times d \leq f_4(-2.903534, ..., -2.903534) \leq -39.16616 \times d.$$

Concerning f_2 shown in Figure 10 used by Hamaizia and Lozi in [24] which possesses hundreds of local minima, its global minima is not yet theoretically known. Functionf_3 shown in Figure 11 is the Goldstein-Price function which is usually evaluated on the rectangle $x_1 \in [-2, 2]$ and $x_2 \in [-2, 2]$, has a lot of local minimum and one global minimum $f_3(0, -1) = 3$. Function f_4 shown in Figure 12 is the Bukin function, it is usually evaluated on the rectangle $x_1 \in [-15, -5]$ and $x_2 \in [-3, 3]$, has a lot of local minimum and one global minimum $f_4(-10, 1) = 0$.

5. Numerical Results

Each optimization method was implemented in Matlab (MathWorks). All the programs were run on a 2.53 GHz, i3 processor with 4 GB of random access memory. On the other hand, since the COM algorithm gives random results, so in each case study, 50 independent runs are made involving 50 different initial trial conditions.

Table 2 shows the numerical results of the global minima research of Styblinski-Tang function f_1 in which we notice that the most suitable transformation used to find the optimal solution of f_1 is S_4. The best results concerning f_2 are obtained through using the transformations S_3 and S_4 but the first is better since its standard divergence is less than that of S_4 (see table 3). Table 4 represents the numerical findings of the optimization problem of the Goldstein-Price function f_3 where we see that all the transformations in subsection 4.1 are suitable for finding the global minima of f_3 but the most appropriate of all is S_4. Finally, the numerical results of the optimization problem of function f_4 are shown in table 5 from which we see that the most appropriate transformation used to find the optimal solution of Bukin function f_4 is S_1 because the numerical global minima is in the neighbourhood of the theoretical one and the standard divergence is near to zero.

Table 2: Optimization results of f_1 for every transformation

T.F	Trans	Cases	Optimal solu	Optimal point	Mean value	Std.Dev
f_1	S_1	C1	-117.4985	(-2.9045,-2.9044,-2.9026)	-117.4980	0.0003
		C2	-117.4969	(-2.9108,-2.9044,-2.8972)	-117.4532	0.0294
		C3	-117.4985	(-2.9031,-2.9041,-2.9023)	-117.4982	0.0001
		C4	-117.4985	(-2.9033,-2.9035,-2.9043)	-117.4984	0.0001
	S_2	C1	-117.4970	(-2.9095,-2.9059,-2.9103)	-117.4888	0.0061
		C2	-117.4584	(-2.9356,-2.9020,-2.9390)	-116.5590	0.5693
		C3	-117.4980	(-2.9047,-2.9049,-2.9085)	-117.4939	0.0025
		C4	-117.4984	(-2.9050,-2.9016,-2.9031)	-117.4949	0.0029
	S_3	C1	-117.4947	(-2.9022,-2.8920,-2.8941)	-117.4773	0.0105
		C2	-116.5576	(-2.7716,-3.0596,-3.0117)	-114.5126	1.2206
		C3	-117.4979	(-2.9040,-2.8978,-2.9018)	-117.4805	0.0103
		C4	-117.4958	(-2.9132,-2.9113,-2.9055)	-117.4816	0.0082
	S_4	C1	-117.4985	(-2.9035,-2.9037,-2.9037)	-117.4985	0.0000
		C2	-117.4985	(-2.9039,-2.9039,-2.9030)	-117.4980	0.0002
		C3	-117.4985	(-2.9034,-2.9034,-2.9038)	-117.4985	0.0000
		C4	-117.4985	(-2.9035,-2.9036,-2.9035)	-117.4985	0.0000

Table 3: Optimization results of f_2 for every transformation

Test function	Transfor -mations	Cases	Optimal solution	Optimal point	Mean value	Std.Dev
f_2	S_1	C1	-395.6748	(0.1533,2.4332)	-389.4862	7.8471
		C2	-395.4022	(0.1535,2.4322)	-384.5009	4.5688
		C3	-395.8736	(0.2433,2.0636)	-389.2662	8.1911
		C4	-395.8498	(0.2433,2.0632)	-377.1366	7.8063
	S_2	C1	-395.8338	(0.2433,2.0641)	-378.3163	8.2958
		C2	-395.7457	(0.2431,2.0644)	-378.7449	6.2101
		C3	-395.7776	(0.2434,2.0640)	-381,5902	9.5200
		C4	-395.7475	(0.2435,2.0634)	-381.9671	8.9664
	S_3	C1	-395.8474	(0.2432,2.0636)	-393.0928	1.4983
		C2	-388.3178	(0.0661,1.6989)	-385.7685	0.3642
		C3	-395.7612	(0.2431,2.0635)	-393.6255	1.0753
		C4	-395.7959	(0.2432,2.0642)	-387.6166	7.0161
	S_4	C1	-385.1566	(1.9246,0.0925)	-373.0012	3.1790
		C2	-395.6758	(0.2432,2.0647)	-389.1207	5.6259
		C3	-395.8349	(0.2432,2.0641)	-392.9497	2.1407
		C4	-395.8751	(0.2433,2.0635)	-393.5276	1.2457

Table 4: Optimization results of f_3 for every transformation

Test function	Transfor -mations	Cases	Optimal solution	Optimal point	Mean value	Std.Dev
f_3	S_1	C1	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
		C2	3.0003	(-0.0011,-0.9998)	3.0028	0.0021
		C3	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
		C4	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
	S_2	C1	3.0000	(-0.0000,-1.0002)	3.0008	0.0012
		C2	3.0003	(-0.0002,-0.9992)	3.0639	0.0689
		C3	3.0000	(-0.0003,-1.0000)	3.0004	0.0003
		C4	3.0000	(-0.0003,-0.9997)	3.0005	0.0003
	S_3	C1	3.0000	(-0.0002,-0.9999)	3.0042	0.0038
		C2	3.0574	(-0.0100,-0.9933)	3.1461	0.0501
		C3	3.0002	(-0.0003,-0.9995)	3.0045	0.0041
		C4	3.0001	(0.0004,-0.9997)	3.0045	0.0034
	S_4	C1	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
		C2	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
		C3	3.0000	(-0.0000,-1.0000)	3.0000	0.0000
		C4	3.0000	(-0.0000,-1.0000)	3.0000	0.0000

Table 5: The optimal solution of f_4 for every transformation

Test function	Transfor -mations	Cases	Optimal solution	Optimal point	Mean value	Std.Dev
f_4	S_1	C1	0.0173	(-10.7026,1.1455)	0.1600	0.0720
		C2	0.0247	(-11.4549,1.3121)	0.4569	0.2221
		C3	0.0058	(-9.6433,0.9299)	0.0916	0.0517
		C4	0.0054	(-10.3195,1.0649)	0.0764	0.0423
	S_2	C1	0.0126	(-10.1153,1.0232)	0.2494	0.1133
		C2	0.0440	(-9.3116,0.8671)	0.6132	0.3021
		C3	0.0138	(-10.2609,1.0529)	0.1403	0.0776
		C4	0.0103	(-10.4391,1.0897)	0.1029	0.0532
	S_3	C1	0.0203	(-10.0553,1.0111)	0.2316	0.01306
		C2	0.0821	(-12.3331,1.5211)	0.8963	0.4186
		C3	0.0185	(-9.9977,0.9995)	0.1903	0.1097
		C4	0.0100	(-10.2287,1.0463)	0.1183	0.0615
	S_4	C1	0.0291	(-12.6313,1.5955)	0.0911	0.0338
		C2	0.0220	(-9.3076,0.8663)	0.2144	0.1155
		C3	0.0247	(-12.4120,1.5406)	0.0780	0.0212
		C4	0.0299	(-12.9314,1.6722)	0.0665	0.0200

6. Conclusion

In this paper, we have presented a new technique of chaotic optimization algorithm by using some transformations in order to modify the density of the Lozi map. In order to test the numerical performance of this new technique, the four non linear multi modal benchmark functions are employed.

As a result of this study, there is no map that gives the best solution of optimization problem for all test functions. So, to obtain global minimum of objective test function you should choose a map that has a good chaotic behaviour and select the transformation that fits the shape of the test function under study. More detailed analysis about this new technique by using other maps and other transformations and testing them on a large number of test functions in higher dimension will be provided in near future.

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