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On δ- Lorentzian Trans Sasakian Manifold With A Semi-Symmetric Metric Connection

Mohd Danish Siddiqi

ABSTRACT: δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric-metric connection have been studied. Expressions for curvature tensors, Ricci curvature tensors and scalar curvature of the δ-Lorentzian trans-Sasakian manifold with a semi-symmetric-metric connection have been obtained. Also, some results on quasi-projectively flat and φ-projectively flat manifolds endowed with a semi-symmetric-metric connection have been discussed. It is shown that the manifold satisfying $\bar{R}.\bar{S} = 0$, $\bar{P}.\bar{S} = 0$ is an η-Einstein manifold. Lastly, we obtain the conditions for the δ-Lorentzian trans-Sasakian manifold with a semi-symmetric-metric connection to be conformally flat and ξ-conformally flat.

Key Words: δ-Lorentzian trans-Sasakian manifold, Semi-symmetric metric connection, Quasi conformal curvature tensor, Ricci-curvature tensor, Weyl conformal curvature tensor, Einstein manifold.

Contents

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1. Introduction

The study of differentiable manifolds with Lorentzian metric is a natural and interesting topic in differential geometry. In [15], Ikawa and Erdogan studied the Sasakian manifold with Lorentzian metric. Notion of Lorentzian para-contact manifolds were introduced by Matsumoto [18]. Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [12]. In [32], Yildiz et al. studied Lorentzian α-Sasakian manifold and Lorentzian β-Kenmotsu manifold has been studied by Funda et. al. in [31]. After that in 2011, S. S. Pujar and V. J. Khairnar [21] have initiated the study of Lorentzian trans-Sasakian manifolds and studied the some basic results with some of its properties. Earlier to this , S. S. Pujar [22] studied the δ-Lorentzian α-Sasakian manifolds and δ-Lorentzian β-Kenmotsu manifolds.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. In 1969, Takahashi [28] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost conatct metric manifolds and indefinite Sasakian manifolds are known as (ϵ)-almost contact metric manifolds. The concept of (ϵ)-Sasakian manifolds was initiated by Bejancu and Duggal [6] and further investigation was taken up by X. Xufeng and C. Xiaoli [30]. U. C. De and A. Sarkar [9] studied the notion of (ϵ)-Kenmotsu manifolds. S. S. Shukla and D. D. Singh [24] extended the study to (ϵ)-trans-Sasakian manifolds with indefnite metric. M. D. Siddiqi et al. [25] also studied some properties of an (ε, δ)- trans-Sasakian manifolds which is closely related to this topic. The semi-Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This motivated Thripathi and others [27] to introduced (ϵ)-almost para-contact structure, where the vector filed ξ is space like or time like according as $(\epsilon) = 1$ or $(\epsilon) = -1$.

When M has a Lorentzian metric g, that is a symmetric non-degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case gives interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results in 2014, S. M Bhati [4] introduced the notion of δ-Lorentzian trans-Sasakian manifolds.

On other hand in 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [10]. In 1930, Bartolotti [5] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [13] defined and studied semi-symmetric metric connection. In 1970, K. Yano [33], started a systematic study of the semi-symmetric metric connection in a Riemannian manifold and this was further studied by various authors such as Sharfuddin Ahmad and S. I. Hussain [23], M. M. Tripathi [26], I. E. Hirică [14], Pathak. G. and U. C. De [20].

U. C. De [8] and C. S. Bagewadi et al. (cf. [1], [2], [3]) studied results on the existence of Projective, Pseudo projective, Conformal, Concircular, Quasi conformal curvature tensors on K-contact, Kenmotsu and trans-Sasakian manifolds.

Motivated by the above studies, in this paper, we study some curvature properties of δ-Lorentzian trans-Sasakian manifolds with respect to a semi-symmetric metric connection. Also, we have proved some results on curvature tensor, Ricci curvature tensor, scalar curvatures, quasi projectively flat, ϕ-projectively flat, $\bar{R}.\bar{S} = 0$, $\bar{P}.\bar{S} = 0$, Weyl conformally flat, Weyl ξ-conformally flat respectively in n-dimensional δ-Lorentzian trans-Sasakian manifolds with a semi-symmetric metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is said to be symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is said to be metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [10].

2. Preliminaries

Let M be a δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\varphi, \xi, \eta, g, \delta)$ consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\varphi^2 = X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi, \xi) = -\delta, \tag{2.3}$$

$$\eta(X) = \delta g(X, \xi), \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \delta\eta(X)\eta(Y) \tag{2.5}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\varphi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [4] on M, the vector field ξ is the time like [29], that is M contains a time like vector field.

In [29], Tanno classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tanno [29].

In the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds [11]. The class $C_6 \oplus C_5$ coincides with the class of trans-Sasakian structure of type (α, β) [17]. In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely. An almost contact metric structure [7] on M is called a trans-Sasakian [19] if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X\varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{2.6}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature, we define the δ-Lorentzian trans-Sasakian manifolds [4] as follows.

Definition 2.1. *A δ-Lorentzian manifold with structure $(\varphi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_X\varphi)Y = \alpha(g(X, Y)\xi - \delta\eta(Y)X) + \beta(g(\varphi X, Y)\xi - \delta\eta(Y)\varphi X) \tag{2.7}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [19]. δ-Lorentzian trans Sasakian manifold of type $(0, 0)$, $(0, \beta)$ $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and

$\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively.

Form (2.4), we have

$$\nabla_X \xi = \delta \{-\alpha\varphi(X) - \beta(X + \eta(X)\xi\}, \tag{2.8}$$

and

$$(\nabla_X \eta)Y = \alpha g(\varphi X, Y) + \beta[g(X,Y) + \delta\eta(X)\eta(Y)]. \tag{2.9}$$

In a δ-Lorentzian trans Sasakian manifold M, we have the following relations:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2+\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] \\ &\quad +\delta[(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\phi^2 X - (X\beta)\varphi^2 Y], \end{aligned} \tag{2.10}$$

$$\begin{aligned} R(\xi,Y)X &= (\alpha^2+\beta^2)[\delta g(X,Y)\xi - \eta(X)Y] \\ &\quad +\delta(X\alpha)\varphi Y + \delta g(\varphi X, Y)(grad\alpha) \\ &\quad +\delta(X\beta)(Y + \eta(Y)\xi) - \delta g(\varphi Y, \varphi X))(grad\beta) \\ &\quad +2\alpha\beta[\delta g(\varphi X, Y)\xi + \eta(X)\varphi Y], \end{aligned} \tag{2.11}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= \delta(\alpha^2+\beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z) \\ &\quad +2\delta\alpha\beta[-\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi X, Z)] \\ &\quad -[(Y\alpha)g(\varphi X, Z) + (X\alpha)g(Y, \varphi Z)] \\ &\quad -(Y\beta)g(\varphi^2 X, Z) + (X\beta)g(\varphi^2 Y, Z)], \end{aligned} \tag{2.12}$$

$$S(X,\xi) = [((n-1)(\alpha^2+\beta^2) - (\xi\beta)]\eta(X) + \delta((\varphi X)\alpha) + (n-2)\delta(X\beta), \tag{2.13}$$

$$S(\xi,\xi) = (n-1)(\alpha^2+\beta^2) - \delta(n-1)(\xi\beta), \tag{2.14}$$

$$Q\xi = (\delta(n-1)(\alpha^2+\beta^2) - (\xi\beta))\xi + \delta\varphi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.15}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Further in an δ-Lorentzian trans Sasakian manifold, we have

$$\delta\varphi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.16}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.17}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.11), we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2+\beta^2) - \delta(\xi\beta). \tag{2.18}$$

It follows from (2.17) that ξ-sectional curvature does not depend on X. From (2.11)

$$\begin{aligned} g(R(\xi,Y)Z,\xi) &= [(\alpha^2+\beta^2) - \delta(\xi\beta)]g(Y,Z) \\ &\quad +[(\xi\beta) - \delta(\alpha^2+\beta^2)]\eta(Y)\eta(Z) + [2\alpha\beta + \delta(\delta\alpha)]g(\varphi Y, Z), \end{aligned} \tag{2.19}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.20}$$

An affine connection $\bar{\nabla}$ in M is called semi-symmetric connection [10], it is torsion tensor satisfies the following relations

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y], \tag{2.21}$$

and

$$\bar{T}(X,Y) = \eta(X)Y - \eta(Y)X. \tag{2.22}$$

Moreover, a semi-symmetric connection is called semi-symmetric metric connection if

$$\bar{g}(X,Y) = 0. \tag{2.23}$$

If ∇ is metric connection and $\bar{\nabla}$ is the semi-symmetric metric connection with non-vanishing torsion tensor T in M, then we have

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.24}$$

$$\bar{\nabla}_X Y - \nabla_X Y = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(X,Y)], \tag{2.25}$$

where

$$g(T(Z,X),Y) = g(T^{'}(X,Y),Z). \tag{2.26}$$

By using (2.4), (2.23) and (2.25), we get

$$g(T^{'}(X,Y),Z) = g(\eta(X)Z - \eta(Z)X, Y),$$

$$g(T^{'}(X,Y),Z) = \eta(X)g(Z,Y) - \delta g(X,Y)g(\xi,Z),$$

$$T^{'}(X,Y) = \eta(X)Y - \delta g(X,Y)\xi, \tag{2.27}$$

$$T^{'}(Y,X) = \eta(Y)X - \delta g(X,Y)\xi. \tag{2.28}$$

From (2.23), (2.24), (2.26) and (2.27), we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi.$$

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold and ∇ be the metric connection on M. The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the metric connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi. \tag{2.29}$$

3. Curvature tensor on δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. The curvature tensor $\bar{R}$ of M with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

By using (2.4), (2.28) and (3.1), we get

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + (\delta)[g(X,Z)Y - g(Y,Z)X] \\ &+(\beta+\delta)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &-(\beta\delta - 1)[\eta(Y)X - \eta(X)Y]\eta(Z), \\ &+\alpha[g(\varphi X,Z)Y - g(\varphi Y,Z)\varphi X] \\ &+\alpha[-g(X,Z)\varphi Y + g(Y,Z)\varphi X],\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of connection ∇.

Lemma 3.1. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$(\bar{\nabla}_X\varphi)(Y) = \alpha g(\varphi X,Y)\xi - \delta\eta(Y)X + \beta(g(\varphi X,Y)\xi - (\delta\beta+\delta)\eta(Y)\varphi X, l \tag{3.3}$$

$$\bar{\nabla}_X\xi = -(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\varphi X, \tag{3.4}$$

$$(\bar{\nabla}_X\eta)Y = \alpha g(\varphi X,Y)\xi + (\beta+\delta)g(X,Y) - (1+\beta\delta)\eta(X)\eta(Y). \tag{3.5}$$

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X\varphi Y = (\bar{\nabla}_X\varphi) + \varphi(\bar{\nabla}_X Y).$$

By using (2.1) and (2.28), we have

$$(\bar{\nabla}_X\varphi)Y = (\nabla_X\varphi)Y - \eta(Y)\varphi X.$$

In view of (2.7), the last equation gives

$$(\bar{\nabla}_X\varphi)(Y) = \alpha(g(\varphi X,Y)\xi - \delta\eta(Y)X + \beta(g(\varphi X,Y)\xi - (\delta\beta+\delta)\eta(Y)\varphi X.$$

To prove (3.5), we replace $Y=\xi$ in (2.28) and we have

$$\bar{\nabla}_X\xi = \nabla_X\xi + \eta(\xi)X - \delta g(X,\xi)\xi.$$

By using (2.2), (2.4) and (2.8), the above equation gives

$$\bar{\nabla}_X\xi = -(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\phi X.$$

In order to prove (3.5), we differentiate $\eta(Y)$ covariantly with respect to X and using (2.28), we have

$$\bar{\nabla}_X \eta(Y) = (\nabla_X \eta)Y + g(X,Y) - \eta(X)\eta(Y).$$

Using (2.9) in above equation, we get

$$(\bar{\nabla}_X \eta)Y = \alpha g(\varphi X, Y)\xi + (\beta+\delta)g(X,Y) - (1+\beta\delta)\eta(X)\eta(Y).$$

□

Lemma 3.2. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$\begin{aligned} \bar{R}(X,Y)\xi &= (\alpha^2+\beta^2-\delta\beta)[\eta(X)Y-\eta(Y)X] \\ &\quad +(2\alpha\beta+\delta\alpha)[\eta(Y)\varphi X-\eta(X)\varphi Y] \\ &\quad +\delta[(Y\alpha)\varphi X-(X\alpha)\varphi Y-(X\beta)\Phi^2 Y+(Y\beta)\varphi^2 X]. \end{aligned} \tag{3.6}$$

Proof. By replacing $Z = \xi$ in (3.3), we have

$$\begin{aligned} \bar{R}(X,Y)\xi &= R(X,Y)\xi + (\delta)[g(X,\xi)Y - g(Y,\xi)X] \\ &\quad +(\beta+\delta)[g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y)]\xi \\ &\quad -(\beta\delta-1)[\eta(Y)X-\eta(X)Y]\eta(\xi) \\ &\quad +\alpha[g(\phi X,\xi)Y - g(\varphi Y,\xi)\varphi X - g(X,\xi)\varphi Y + g(Y,\xi)\varphi X] \end{aligned}$$

In view of (2.2), (2.4) and (2.10), the above equation reduces to

$$\begin{aligned} \bar{R}(X,Y)\xi &= (\alpha^2+\beta^2-\delta\beta)[\eta(X)Y-\eta(Y)X] \\ &\quad +(2\alpha\beta+\delta\alpha)[\eta(Y)\varphi X-\eta(X)\varphi Y] \\ &\quad +\delta[(Y\alpha)\varphi X-(X\alpha)\varphi Y-(X\beta)\varphi^2 Y+(Y\beta)\varphi^2 X]. \end{aligned}$$

□

Remark 3.3. *Replace $Y = \xi$ and using (3.3), (2.11), (2.2) and (2.4), we obtain*

$$\begin{aligned} \bar{R}(X,\xi)\xi &= (\alpha^2+\beta^2-\delta\beta)[-X-\eta(X)Y] \\ &\quad +(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\varphi X+\delta(\xi\beta)\varphi^2 X]. \end{aligned} \tag{3.7}$$

Remark 3.4. *Now, again replace $X = \xi$ in (3.7), using (2.1), (2.2) and (2.4), we obtain*

$$\begin{aligned} \bar{R}(\xi,Y)\xi &= (\alpha^2+\beta^2-\delta\beta)[-\eta(Y)\xi-Y] \\ &\quad -(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\varphi Y-\delta(\xi\beta)\varphi^2 Y]. \end{aligned} \tag{3.8}$$

Remark 3.5. *Replace $Y = X$ in (3.8), we get*

$$\bar{R}(\xi,X)\xi = -(\alpha^2+\beta^2-\delta\beta)[-X-\eta(X)\xi] -(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\varphi X-\delta(\xi\beta)\varphi^2X]. \quad (3.9)$$

From (3.8) and (3.9), we obtain

$$\bar{R}(X,\xi)\xi = -\bar{R}(\xi,X)\xi. \quad (3.10)$$

Now, contracting X in (3.3), we get

$$\bar{S}(Y,Z) = S(Y,Z)-[(\delta)(n-2)+\beta]g(Y,Z) -(\beta\delta-1)(n-2)\eta(Z)\eta(Y)-\alpha(n-2)g(\varphi Y,Z), \quad (3.11)$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M.
This gives

$$\bar{Q}Y = QY-[(\delta)(n-2)+\beta]Y -(\beta\delta-1)(n-2)\eta(Y)\xi-\alpha(n-2)\varphi Y, \quad (3.12)$$

where $\bar{Q}$ and Q are Ricci operator with respect to the semi-symmetric metric connection and metric connection respectively and define as $g(\bar{Q}Y,Z)=\bar{S}(Y,Z)$ and $g(QY,Z)=S(Y,Z)$ respectively. Replace $Y=\xi$ in (3.12) and using (2.15), we get

$$\bar{Q}\xi = \delta(n-1)(\alpha^2+\beta^2)\xi-(\xi\beta)\xi-2\delta(n-2)\xi +\delta\varphi(grad\alpha)-\delta(n-2)(grad\beta)-\beta(n-1)\xi. \quad (3.13)$$

Putting $Y=Z=e_i$ and taking summation over i, $1\le i\le n-1$ in (3.11), using (2.14) and also the relations $r=S(e_i,e_i)=\sum_{i=1}^{n}\delta_i R(e_i,e_i,e_i,e_i)$, we get

$$\bar{r} = r-(n-1)[(\delta)(n-2)+2\beta], \quad (3.14)$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.

Now, we have the following lemmas.

Lemma 3.6. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with the semi-symmetric metric connection, then*

$$\bar{S}(\varphi Y,Z) = -\delta(\varphi^2Y)\alpha-\delta(n-2)(\varphi Y)\beta-\alpha(n-2)g(\varphi Y,\varphi Z), \quad (3.15)$$

$$\bar{S}(Y,\xi) = [(n-1)(\alpha^2+\beta^2-\delta(\xi\beta)-\delta\beta(n-1)]\eta(Y) +\delta(n-2)(Y\beta)+\delta(\varphi Y)\beta, \quad (3.16)$$

$$\bar{S}(\xi,\xi) = [(n-1)(\alpha^2+\beta^2-\delta(\xi\beta)-\delta\beta(n-1)]\eta(Y). \quad (3.17)$$

Proof. By replacing $Y = \varphi Y$ in equation (3.11) and using (2.13) and (2.5), we have (3.17). Taking $Y = \xi$ in (3.11) and using (2.13) we get (3.16). (3.17) follows from considering $Y = \xi$ in (3.16) we get (3.17). □

Lemma 3.7. *Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold with the semi-symmetric metric connection, then*

$$\begin{aligned}\bar{S}(grad\alpha,\xi) &= \delta(n-1)(\alpha^2+\beta^2(\xi\beta)-\beta(n-1)(\xi\alpha)-(\xi\alpha)(\xi\beta) \\ &\quad +\delta(\varphi grad\alpha)\alpha+\delta(n-2)g(grad\alpha,grad\beta),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\bar{S}(grad\beta,\xi) &= \delta(n-1)(\alpha^2+\beta^2(\xi\beta)-\beta(n-1)(\xi\beta)-(\xi\beta)^2 \\ &\quad +\delta(\varphi grad\beta)\alpha+\delta(n-2)g(grad\beta)^2.\end{aligned} \tag{3.19}$$

Proof. From equation (3.11) and (3.16) and using $Y = grad\alpha$ we have (3.18) . Similarly taking $\xi = grad\beta$ in (3.11) and using (3.16), we get (3.19).
□

4. Quasi-projectively flat δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

Let M be n-dimensional δ-Lorentzian trans-Sasakian manifold. If there exists a one to one correspondence between each co-ordinate neighborhood of M and a domain in Euclidean space such that any geodesic of δ-Lorentzian trans-Sasakian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. The projective curvature tensor $\bar{P}$ with respect to semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{4.1}$$

Definition 4.1. *A δ-Lorentzian trans-Sasakian manifold M is said to be quasi-projectively flat with respect to semi-symmetric metric connection, if*

$$g(\bar{P}(\varphi X,Y)Z,\varphi U) = 0, \tag{4.2}$$

where $\bar{P}$ is the projective curvature tensor with respect to semi-symmetric metric connection.

Now, from (4.1) taking inner product with U, we get

$$\begin{aligned}g(\bar{P}(X,Y)Z,U) &= g(\bar{R}(X,Y)Z,U) \\ &\quad -\frac{1}{(n-1)}[\bar{S}(Y,Z)g(X,U)-\bar{S}(X,Z)g(Y,U)].\end{aligned} \tag{4.3}$$

Replace $X = \varphi X$ and $U = \varphi U$ in (4.3), we get

$$g(\bar{P}(\varphi X, Y)Z, \varphi U) = g(\bar{R}(\varphi X, Y)Z, \varphi U) - \frac{1}{(n-1)}[\bar{S}(Y, Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, Z)g(Y, \varphi U)]. \tag{4.4}$$

From (4.2) and (4.4), we have

$$g(\bar{R}(\varphi X, Y)Z, \varphi U) = \frac{1}{(n-1)}[\bar{S}(Y, Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, Z)g(Y, \varphi U)]. \tag{4.5}$$

Now, using equations (2.1), (2.4), (3.11) and (3.15) in equation (4.5), we have

$$\begin{aligned} g(\bar{R}(\varphi X, Y)Z, \varphi U) &= \frac{1}{(n-1)}[\bar{S}(Y, Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, Z)g(Y, \varphi U)] \\ &- \frac{(\delta+\beta)}{(n-1)}g(\varphi X, Z)g(Y, \varphi U) + \frac{(\delta+\beta)}{(n-1)}g(Y, Z)g(\varphi X, \varphi U) \\ &- \frac{(\delta\beta-1)}{(n-1)}\eta(Y)\eta(Z)g(\varphi X, \varphi U) \\ &+ \frac{(\delta\alpha)}{(n-1)}\eta(X)\eta(Z)g(\varphi X, \varphi U) \\ &- \frac{\alpha}{(n-1)}g(X, Z)g(Y, \varphi U) - \frac{\alpha}{(n-1)}g(\varphi Y, Z)g(\varphi X, \varphi U) \\ &+ \alpha g(Y, Z)g(X, \varphi U) + \alpha g(\varphi X, Z)g(\varphi X, \varphi U). \end{aligned} \tag{4.6}$$

Let $\{e_1, e_2 ....e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M, then $\{\varphi e_1, \varphi e_2........\varphi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now, putting $X = U = e_i$ in equation (4.6) and using (2.2), (2.3),(2.19), (3.11) and (3.16), we have

$$\begin{aligned} S(Y, Z) &= [(n-2)(\beta+\delta) + \delta(n-1)(\alpha^2+\beta^2) - (n-1)(\xi\beta)]g(Y, Z) \\ &+ [\delta(n-2)(\xi\beta) + (n-2)(\beta\delta-1)]\eta(Y)\eta(Z) \\ &- [2\delta(n-1)\alpha\beta + (n-1)(\xi\alpha) - \alpha]g(\varphi Y, Z) \\ &- \delta\eta(Y)(\phi Z)\alpha - \delta(n-2)(\xi\beta)\eta(Y). \end{aligned} \tag{4.7}$$

If $\alpha = 0$ and $\beta = constant$ in (4.7), we get

$$S(Y, Z) = [(n-2)(\beta+\delta) + (n-1)\delta\beta^2]g(Y, Z) + (\beta\delta-1)(2-n)\eta(Y)\eta(Z). \tag{4.8}$$

Therefore, we have

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where $a = (n-2)(\beta+\delta) + (n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
These results shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 4.2. *An n-dimensional quasi projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

5. φ-Projectively flat δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

An n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection is said to be ϕ-projectively flat if

$$\varphi^2(\bar{P}(\varphi X, \varphi Y)\varphi Z) = 0, \tag{5.1}$$

where $\bar{P}$ is the projective curvature tensor of M n-dimensional δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. Suppose M to be φ-projectively flat δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. It is know that $\varphi^2(\bar{P}(\varphi X, \varphi Y)\varphi Z) = 0$ holds if and only if

$$g(\bar{P}(\varphi X, \varphi Y)\varphi Z, \varphi U) = 0, \tag{5.2}$$

for any $X, Y, Z, U \in TM$. Replace $Y = \varphi Y$ and $U = \varphi U$ in (4.4), we have

$$\begin{aligned} g(\bar{P}(\varphi X, \varphi Y)\varphi Z, \varphi U) &= g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi U) \\ &\quad -\frac{1}{(n-1)}[-\bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U) \\ &\quad -\frac{1}{(n-1)}[-\bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U)] \end{aligned} \tag{5.3}$$

From (5.2) and (5.3), we have

$$\begin{aligned} g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi U) &= \frac{1}{(n-1)}[\bar{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi U) \\ &\quad -\bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U)]. \end{aligned} \tag{5.4}$$

Now, using (2.1),(2.2),(2.4),(2.5), (3.3) and (3.11) in equation (5.4), we have

$$\begin{aligned} g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi U) &= \frac{1}{(n-1)}[\bar{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U)] \\ &\quad -\frac{(\delta+\beta)}{(n-1)}g(\varphi Y, \varphi Z)g(\varphi X, \varphi U) \\ &\quad +\frac{(\delta+\beta)}{(n-1)}g(\varphi X, \varphi Z)g(\varphi Y, \varphi U) \\ &\quad -\frac{\alpha}{(n-1)}g(Y, \varphi Z)g(\varphi X, \varphi U) \\ &\quad -\frac{\alpha}{(n-1)}g(X, \varphi Z)g(\varphi X, \varphi U) \\ &\quad +\alpha g(\varphi Y, \varphi Z)g(X, \varphi U) - \alpha g(\varphi X, \varphi Z)g(Y, \varphi U). \end{aligned}$$

Let $\{e_1, e_2 ....e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M, then $\{\varphi e_1, \varphi e_2 ....\varphi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now putting

$X = U = e_i$ in equation (5.5) and using (2.1)–(2.5), (2.19), (3.11) and (3.16), we have

$$\begin{aligned} S(Y,Z) &= [(n-2)(\beta+\delta)+\delta(n-1)(\alpha^2+\beta^2)-(n-1)(\xi\beta)]g(Y,Z) \\ &+[2\delta(n-2)(\xi\beta)+(n-2)(\beta\delta-1)]\eta(Y)\eta(Z) \\ &+[\alpha-2\delta\alpha\beta(n-1)-(n-1)(\xi\alpha)]g(\varphi Y,Z) \\ &-[\delta(\varphi Z)\alpha+\delta(n-2)(Z\beta)]\eta(Y)-[\delta(\varphi Y)\alpha+\delta(n-2)(Y\beta)]\eta(Z). \end{aligned} \tag{5.5}$$

If $\alpha = 0$ and $\beta = constant$ in (5.5), we get

$$S(Y,Z) = [(n-2)(\beta+\delta)+(n-1)\delta\beta^2]g(Y,Z)+(\beta\delta-1)(2-n)\eta(Y)\eta(Z). \tag{5.6}$$

Therefore,

$$S(Y,Z) = ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a = (n-2)(\beta+\delta)+(n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
This result shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 5.1. *An n-dimensional ϕ-projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

6. δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying $\bar{R}.\bar{S} = 0$

Now, suppose that M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying the condition:

$$\bar{R}(X,Y).\bar{S} = 0. \tag{6.1}$$

Then, we have

$$\bar{S}(\bar{R}(X,Y)Z,U)+\bar{S}(Z,\bar{R}(X,Y)U) = 0. \tag{6.2}$$

Now, we replace $X = \xi$ in equation (6.2), using equations (2.11) and (6.2), we have

$$\begin{gathered} \delta(\alpha^2+\beta^2)g(Y,Z)\bar{S}(\xi,U)-(\alpha^2+\beta^2)\eta(Z)\bar{S}(Y,U)-2\delta\alpha\beta g(\varphi Y,Z)\bar{S}(\xi,U) \\ +2\alpha\beta\eta(Z)\bar{S}(\varphi Y,U)+\delta(Z\alpha)\bar{S}(\varphi Y,U)-\delta g(\varphi Y,Z)\bar{S}(grad\alpha,U) \\ -\delta g(\phi Y,\varphi Z)\bar{S}(grad\beta,U)+\delta(Z\beta)\bar{S}(Y,U)-\delta(Z\beta)\eta(Y)\bar{S}(\xi,U) \\ -\delta g(Y,Z)\bar{S}(\xi,U)+\delta\eta(Z)\bar{S}(Y,U)+\alpha g(\varphi Y,Z)\bar{S}(\xi,U)-\delta\alpha\eta(Z)\bar{S}(\varphi Y,U) \\ +\delta(\alpha^2+\beta^2)g(Y,U)\bar{S}(\xi,Z)-(\alpha^2+\beta^2)\eta(U)\bar{S}(Y,Z)-2\delta\alpha\beta g(\varphi Y,U)\bar{S}(\xi,Z) \\ +2\alpha\beta\eta(U)\bar{S}(\varphi Y,Z)+\delta(U\alpha)\bar{S}(\varphi Y,Z)-\delta g(\varphi Y,U)\bar{S}(grad\alpha,Z) \\ -\delta g(\varphi Y,\varphi U)\bar{S}(grad\beta,Z)+\delta(U\beta)\bar{S}(Y,Z)-\delta(U\beta)\eta(Y)\bar{S}(\xi,Z) \\ -\delta g(Y,U)\bar{S}(\xi,Z)+\delta\eta(U)\bar{S}(Y,Z)+\alpha g(\varphi Y,U)\bar{S}(\xi,Z)-\delta\alpha\eta(U)\bar{S}(\varphi Y,Z) = 0. \end{gathered} \tag{6.3}$$

Using equations (2.1)–(2.5), (2.13), (2.14), (3.11) and (3.16)–(3.18) in equation (6.3)

$$
\begin{aligned}
&[(\alpha^2+\beta^2)-\delta(\xi\beta)-\delta\beta]S(Y,Z)\\
&=[\delta(n-1)(\alpha^2+\beta^2)-2\beta(n-1)(\alpha^2+\beta^2)-2(n-1)(\alpha^2+\beta^2)(\xi\beta)\\
&+2\delta\beta(n-1)(\xi\beta)-\delta(\xi\beta)^2+(\phi grad\beta)\alpha+(n-2)(grad\beta)^2\\
&+\delta\beta^2(n-2)+\delta(n-2)(\alpha^2+\beta^2)+\beta(\alpha^2+\beta^2)\\
&-2\alpha^2\beta(n-2)-\delta\alpha(\xi\alpha)-(n-2)(\xi\beta)-\delta\beta(\xi\beta)\\
&-\beta(n-2)+\delta\alpha^2(n-2)]g(Y,Z)+[-\delta(\phi grad\beta)\alpha\\
&-\delta(n-2)(grad\beta)^2+(n-2)(\beta\delta-1)(\alpha^2+\beta^2)\\
&+2\delta\alpha^2\beta(n-2)+\alpha(n-2)(\xi\alpha)+(\beta+\delta)(n-2)(\xi\beta)\\
&+\beta(\beta+\delta)(n-2)-\alpha^2(n-2)]\eta(Y)\eta(Z)+[-2\delta\alpha\beta(n-1)(\alpha^2+\beta^2)\\
&+2(n-2)\alpha\beta^2+2\alpha\beta(n-2)(\xi\beta)-(n-1)(\alpha^2+\beta^2)(\xi\alpha)\\
&+\delta\beta(n-2)(\xi\alpha)+\delta(\xi\alpha)(\xi\beta)+(\varphi grad\alpha)\alpha+(n-2)(g(grad\alpha,grad\beta)\\
&+\alpha(\alpha^2+\beta^2)-\delta\alpha(\xi beta)-2\alpha\beta(n-2)(\delta)-(n-2)(\delta\alpha)+\alpha(n-2)]g(\varphi Y,Z)\\
&+[\delta(\xi\alpha)+2\alpha\beta-\delta\alpha]S(\varphi Y,Z)+[(n-2)(\xi\beta)(Z\beta)\\
&+[\delta(\alpha^2+\beta^2)(\varphi Z)\alpha-\delta(n-2)(\alpha^2+\beta^2)(Z\beta)+(\xi\beta)(\varphi Z)\alpha\\
&\beta(\phi Z)\alpha+\beta(n-2)(Z\beta)]\eta(Y)+[\delta(\alpha^2+\beta^2)(\varphi Y)\alpha+\delta(n-2)(\alpha^2+\beta^2)(Y\beta)\\
&-2\delta\alpha\beta(\varphi^2Y)\alpha-2\delta\alpha\beta(n-2)(\varphi Y\beta)-\beta(\varphi Y)\alpha\\
&-\beta(n-2)(Y\beta)+\alpha(\varphi^2Y)\alpha+\alpha(n-2)(\varphi Y\beta)]\eta(Z)\\
&-(n-2)(Y\beta)(Z\beta)+(n-2)(Z\beta)(\xi\beta).
\end{aligned}
$$

If $\alpha=0$ and $\beta=constant$ in (5.5), we get

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a=-[\frac{(n-1)\delta\beta^4+(n-2)(grad\beta)^2+(n-2)\delta\beta^2+(n-2)\delta\beta^2-(n-2)\beta+(2n-3)\beta^3}{(\beta+\delta)\beta}]$ and $b=-[\frac{(n-2)(\beta\delta-1)\beta^2+(n-2)(\beta+\delta)\beta-(n-2)\delta(grad\beta^2}{(\beta+\delta)\beta}]$. This show that M is an η-Einstein manifold. Thus,we can state the following theorem:

Theorem 6.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfying $\bar{R}.\bar{S}=0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha=0$ and $\beta=constant$.*

7. δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$

Now, we consider δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying

$$(\bar{P}(X,Y).\bar{S})(Z,U) = 0, \tag{7.1}$$

where $\bar{P}$ is the projective curvature tensor and $\bar{S}$ is the Ricci tensor with semi-symmetric metric connection.Then, we have

$$\bar{S}(\bar{P}(X,Y)Z,U) + \bar{S}(Z,\bar{P}(X,Y)U) = 0. \tag{7.2}$$

Replace $X = \xi$ in the equation (7.2), we get

$$\bar{S}(\bar{P}(\xi,Y)Z,U) + \bar{S}(Z,\bar{P}(\xi,Y)U) = 0. \tag{7.3}$$

Putting $X = \xi$ in (4.1), we get

$$\bar{P}(\xi,Y)Z = \bar{R}(\xi,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)\xi - \bar{S}(\xi,Z)Y]. \tag{7.4}$$

Using (2.1), (2.2), (2.4), (2.11), (3.3), (3.11), (3.17) and (7.4) in (7.3), we get

$$\begin{aligned}
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,Z)\bar{S}(\xi,U) - \frac{1}{(n-1)}S(Y,Z)\bar{S}(\xi,U) \quad (7.5)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(Z)\bar{S}(\xi,U) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\varphi Y,Z)\bar{S}(\xi,U)\\
&-\delta g(\varphi Y,Z)\bar{S}(grad\alpha,U) - \delta g(\varphi Y,\varphi Z)\bar{S}(grad\beta,U) + 2\alpha\beta\eta(Z)\bar{S}(\varphi Y,U)\\
&+\delta(Z\alpha)\bar{S}(\varphi Y,U) + \delta(Z\beta)\bar{S}(Y,U) - \delta(Z\beta)\eta(Y)\bar{S}(\xi,U) - \delta\alpha\eta(Z)\bar{S}(\varphi Y,U)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,U)\frac{(n-2)}{(n-1)}\delta(Z\beta)\bar{S}(Y,U) - \frac{1}{(n-1)}\delta(\varphi Z)\alpha\bar{S}(Y,U)\\
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,U)\bar{S}(\xi,Z) - \frac{1}{(n-1)}S(Y,U)\bar{S}(\xi,Z)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(U)\bar{S}(\xi,Z) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\varphi Y,U)\bar{S}(\xi,Z)\\
&-\delta g(\varphi Y,U)\bar{S}(grad\alpha,Z) - \delta g(\varphi Y,\phi U)\bar{S}(grad\beta,Z) + 2\alpha\beta\eta(U)\bar{S}(\varphi Y,Z)\\
&+\delta(U\alpha)\bar{S}(\varphi Y,Z) + \delta(Z\beta)\bar{S}(Y,Z) - \delta(U\beta)\eta(Y)\bar{S}(\xi,Z) - \delta\alpha\eta(U)\bar{S}(\varphi Y,Z)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,Z)\frac{(n-2)}{(n-1)}\delta(U\beta)\bar{S}(Y,Z) - \frac{1}{(n-1)}\delta(\varphi U)\alpha\bar{S}(Y,Z) = 0
\end{aligned}$$

Putting $U = \xi$ and Using (2.1)–(2.5), (3.11) and (3.15)–(3.19) in (7.4), we get

$$[(\alpha^2+\beta^2) - \delta(\xi\beta) - \delta\beta]S(Y,Z) \tag{7.6}$$

$$
\begin{aligned}
&= [\delta(n-1)(\alpha^2+\beta^2) + (n-2)(\beta\delta)(\alpha^2+\beta^2) - \beta(n-1)(\alpha^2+\beta^2) \\
&-\delta(n-2)(\beta\delta-1) - 2(n-1)(\xi\beta)(\alpha^2+\beta^2) - (n-2)(\beta\delta-1)(\xi\beta) \\
&2\alpha^2\beta(n-2)\delta\alpha(n-2)(\xi\alpha) + \delta\alpha^2(n-2) + \delta\beta(n-1) + \delta(\xi\beta)^2 \\
&+(\varphi grad\alpha)\alpha + (n-2)(grad\beta)^2]g(Y,Z) + [(n-2)\beta(\beta+\delta) - (n-2)(\alpha^2+\beta^2) \\
&+2(n-2)\delta\alpha^2\beta + \alpha(n-2)(\xi\alpha) + (n-2)(\beta+\delta)(\xi\beta) - \alpha^2(n-2) \\
&-\delta(n-2)(grad\beta)^2 - \delta(\varphi grad\beta)\alpha]\eta(Y)\eta(Z) + [\alpha(\alpha^2+\beta^2) \\
&-2\delta\alpha\beta(\alpha^2+\beta^2)(n-1) - 2\alpha\beta^2 n - \delta(\xi\beta) - \delta\beta(\xi\alpha) + 2\alpha\beta(\xi\beta) \\
&-2\delta\alpha\beta(n-2) - (n-1)(\xi\alpha) + \alpha(n-2) - (n-1)(\alpha^2+\beta^2)(\xi\alpha) + (n-1)\delta\beta(\xi\alpha) \\
&+\delta(\xi\alpha)(\xi\beta) + (\varphi grad\alpha)\alpha +)n-2)g(grad\alpha, grad\beta)]g(\varphi Y, z) \\
&+[\delta\alpha + \delta(\xi\alpha) - \delta\alpha]S(\varphi Y, Z) \\
&+[\delta(n+3)(\alpha^2+\beta^2)(Z\beta) + \beta(n-2)(Z\beta) - \delta(\alpha^2+\beta^2)(\varphi Z)\alpha \\
&+(n-1)\beta(\varphi Z)\alpha + (\xi\beta)(\varphi Z)\alpha)]\eta(Y) + [-2\delta\alpha\beta(\varphi^2 Y)\alpha - 2\delta\alpha\beta(n-2)(\varphi Y\beta) \\
&+\alpha(\varphi^2 Y)\alpha + \alpha(n-2)(\varphi Y\beta) + \delta(\alpha^2+\beta^2)(\varphi Y)\alpha + \delta(n-2)(\alpha^2+\beta^2)(Y\beta) \\
&-\beta(\varphi Y)\alpha - \beta(n-2)(Y\beta)]\eta(Z) - (Z\alpha)(\varphi^2 Y)\alpha \\
&-(n-2)(Z\beta)(\varphi Y\beta) - (Z\beta)(\varphi Y)\alpha - \beta(n-2)(Y\beta).
\end{aligned}
$$

If $\alpha = 0$ and $\beta = constant$ in (7.6), we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{7.7}$$

where $a = -[\frac{(n-1)\beta^4+(n-2)\beta^2(\beta\delta)+(n-1)\beta^3-(n-2)\beta(\beta\delta-1)+(n-1)\delta\beta+(n-2)(grad\beta)^2}{\beta(\beta\delta)}]$
and
$b = -[\frac{(n-2)\beta(\beta+\delta)+(n-2)\beta^2-(n-2)\delta(grad\beta)^2}{\beta(\beta+\delta)}]$.

This result show that the manifold under the consideration is an η-Einstein manifold. Thus we have the following theorem:

Theorem 7.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfying $\bar{P}.\bar{S} = 0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

8. Weyl conformal curvature tensor on δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

The Weyl conformal curvature tensor $\bar{C}$ of type $(1,3)$ of M an n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection $\bar{\nabla}$ is given by [16]

$$\begin{aligned}\bar{C}(X,Y)Z &= \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &+ g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{8.1}$$

where $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection $\bar{\nabla}$.

Now, taking inner product with U in (8.1), we get

$$\begin{aligned}g(\bar{C}(X,Y)Z,U) &= g(\bar{R}(X,Y)Z,U) \\ &- \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) \\ &+ g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{8.2}$$

Using (2.4), (3.3), (3.11), (3.12) and (3.14) in (8.2), we get

$$\begin{aligned}\bar{C}(X,Y,Z,U) &= g(\bar{R}(X,Y)Z,U) \\ &- \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],\end{aligned} \tag{8.3}$$

where $g(\bar{C}(X,Y)Z,U) = \bar{C}(X,Y,Z,U)$ and $R(X,Y)Z,U) = C(X,Y,Z,U)$ are Weyl curvature tensor with respect to semi-symmetric metric connection respectively, we have

$$\bar{C}(X,Y,Z,U) = C(X,Y,Z,U), \tag{8.4}$$

where

$$\begin{aligned}C(X,Y,Z,U) &= g(\bar{R}(X,Y)Z,U) \\ &- \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{8.5}$$

Theorem 8.1. *The Weyl conformal curvature tensor of a δ-Lorentzian trans-Sasakian manifold M with respect to a metric connection is equal to the Weyl curvature of δ-Lorentzian trans-Sasakian manifold with respect to the semi-symmetric metric connection.*

9. δ-Lorentzian trans-Sasakian manifold with Weyl conformal flat conditions with semi-symmetric metric connection

Let us consider that the δ-Lorentzian trans-Sasakian manifold M with respect to the semi-symmetric metric connection is Weyl conformally flat, that is $\bar{C} = 0$. Then from equation (8.1), we get

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &\quad + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &\quad + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{9.1}$$

Now, taking the inner product of equation (9.1) with U. then we get

$$\begin{aligned} g(\bar{R}(X,Y)Z,U) &= \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) \\ &\quad + g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] \\ &\quad - \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{9.2}$$

Using equations (2.4), (3.3), (3.11), (3.12) and (3.14) in equation (9.2), we get

$$\begin{aligned} g(R(X,Y)Z,U) &= \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &\quad + g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] \\ &\quad - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{9.3}$$

Putting $X = U = \xi$ in equation (9.3) and using (2.2), (2.3) and (2.4), we get

$$\begin{aligned} g(R(\xi,Y)Z,\xi) &= \frac{1}{(n-2)}[\delta S(Y,Z) - \delta\eta(Y)S(\xi,Z) \\ &\quad + g(Y,Z)S(\xi,\xi) - \delta\eta(Z)S(Y,\xi)] \\ &\quad - \frac{r}{(n-1)(n-2)}[\delta g(Y,Z) - \eta(Y)\eta(Z)], \end{aligned}$$

where $g(QY,Z) = S(Y,Z)$.

Now, using equations (2.13), (2.14) and (2.16), we get

$$\begin{aligned} S(Y,Z) &= [(\delta(\alpha^2+\beta^2)-(\xi\beta)]+\frac{r}{(n-1)}]g(Y,Z)+[\delta(n-4)(\xi\beta) \\ &\quad +n(\alpha^2+\beta^2)-\frac{\delta}{r}(n-1)]\eta(Y)\eta(Z)-[2\delta\alpha\beta(n-2)+(n-2)(\xi\alpha)] \\ &\quad g(\phi Y,Z)-[\delta(\varphi Z)\alpha+\delta(Z\beta)(n-2)]\eta(Y) \\ &\quad -[\delta(\varphi Y)\alpha+\delta(n-2)(Y\beta)]\eta(Z). \end{aligned} \tag{9.4}$$

If $\alpha = 0$ andd $\beta = constant$ in (7.6), we get

$$S(Y,Z) = [\delta\beta^2 + \frac{r}{(n-1)}]g(Y,Z) + [n\beta^2 - \frac{\delta r}{(n-1)}]\eta(Y)\eta(Z). \tag{9.5}$$

Therefore

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = [\delta\beta^2 + \frac{r}{(n-1)}]$ and $b = [n\beta^2 - \frac{\delta r}{(n-1)}]$. This shows that M is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 9.1. *An n-dimensional Weyl conformally flat δ-Lorentzian trans-Sasakian manifold with respect to semi-symmetric metric connection $\bar{\nabla}$ is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

Now, taking equation (8.1)

$$\begin{aligned} \bar{C}(X,Y)Z &= \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &\quad + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ &\quad + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{9.6}$$

Using (2.20), (3.3), (3.11), (3.12) and (3.14) in equation (9.6), we get

$$\begin{aligned} \bar{C}(X,Y)Z &= C(X,Y)Z + \delta[g(X,Z)Y - g(Y,Z)X] \\ &\quad +(\delta+\beta)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi \\ &\quad -(\beta\delta-1)\eta(Z)[\eta(Y)X - \eta(X)Y] + \alpha[g(\varphi X,Z)Y \\ &\quad -g(\varphi,Z)X - g(Y,Z)\varphi X + g(X,Z)\varphi Y] \\ &\quad +\frac{1}{(n-2)}(\beta\delta-1)(n-2)\eta(Y)\eta(Z) - ((\delta)(n-2)+\beta)g(Y,Z)X \\ &\quad +\alpha(n-2)g(\varphi Y,Z)X + ((\delta)(n-2)+\beta)g(X,Z)Y \\ &\quad +(\beta\delta-1)(n-2)\eta(X)\eta(Z)Y - \alpha(n-2)g(\varphi X,Z)Y \\ &\quad -((\delta)(n-2)+\beta)g(Y,Z)X + (\beta+\delta)(n-2)g(Y,Z)\eta(X)\xi \\ &\quad \alpha(n-2)g(Y,Z)\varphi X + ((\delta)(n-2)+\beta)g(X,Z)Y \\ &\quad -(\beta+\delta)(n-2)g(X,Z)\eta(Y)\xi - \alpha(n-2)g(X,Z)\varphi Y] \\ &\quad -\frac{\beta+\delta+(n-2)}{(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

Let X and Y are orthogonal basis to ξ. Putting $Z = \xi$ and using (2.1), (2.2) and (2.4) in (9.7), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi.$$

Thus, we have the following:

Theorem 9.2. *A n-dimensinal δ-Lorentzian trans-Sasakian manifold M is Weyl ξ-conformally flat with respect to the semi-symmetric metric connection if and only if the manifold is also Weyl ξ-conformally flat with respect to the metric connection provided that the vector fields are horizontal vector fields.*

10. Example of 3-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection

We consider the three dimensional manifold $M = [(x, y, z) \in R^3 \mid z \neq 0]$, where (x, y, z) are the cartesian coordinates in R^3. Choosing the vector fields

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = -z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric define by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_2, e_2) = 0,\ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \delta,$$

where $\delta = \pm 1$. Let η be the 1-form defined by $\eta(Z) = \delta g(Z, e_3)$ for any vector field Z on TM. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\varphi^2 Z = Z + \eta(Z)e_3,\ \eta(e_3) = 1 \text{ and } g(\varphi Z, \varphi W) = g(Z, W) - \delta\eta(Z)\eta(W)$$

for any vector fields Z, W on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = \delta e_1,\ [e_2, e_3] = \delta e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) \\ &\quad + g([Z, X], Y). \end{aligned}$$

From above equation which is known as Koszul's formula, we have

$$\nabla_{e_1} e_3 = \delta e_1,\ \nabla_{e_2} e_3 = \delta e_2,\ \nabla_{e_3} e_3 = 0, \tag{10.1}$$

$$\nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_2 = -\delta e_3,\ \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_1} e_1 = -\delta e_3,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_3} e_1 = 0.$$

Using the above relations, for any vector field X on M, we have

$$\nabla_X \xi = \delta(X + \eta(X)\xi)$$

for $\xi \in e_3$, $\alpha = 0$ and $\beta = 1$. Hence the manifold M under consideration is an δ-Lorentzian trans Sasakian of type $(0, 1)$ manifold of dimension three.

Now, we consider this example for semi-symmetric metric connection, from (2.29), we obtain:

$$\bar{\nabla}_{e_1} e_3 = (1+\delta)e_1,\ \bar{\nabla}_{e_2} e_3 = (1+\delta)e_2,\ \bar{\nabla}_{e_3} e_3 = 0, \tag{10.2}$$

$$\bar{\nabla}_{e_1} e_2 = 0,\ \bar{\nabla}_{e_2} e_2 = -(1+\delta)e_3,\ \bar{\nabla}_{e_3} e_2 = 0,$$

$$\bar{\nabla}_{e_1} e_1 = -(1+\delta)e_3,\ \bar{\nabla}_{e_2} e_1 = 0,\ \bar{\nabla}_{e_3} e_1 = 0.$$

Then the Riemannian and the Ricci curvature tensor fields with respect to semi-symmetric metric connection are given by:

$$\bar{R}(e_1, e_2)e_2 = -(1+\delta)^2 e_1,\ \bar{R}(e_1, e_3)e_3 = -\delta(1+\delta)e_2,\ \bar{R}(e_2, e_1)e_1 = -(1+\delta)^2 e_2,$$

$$\bar{R}(e_2, e_3)e_3 = -\delta(1+\delta)e_2,\ \bar{R}(e_3, e_1)e_1 = \delta(1+\delta)e_3,\ \bar{R}(e_3, e_2)e_2 = -\delta(1+\delta)e_3,$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -(1+\delta)(1+2\delta),\ \bar{S}(e_3, e_3,) = 2\delta(1+\delta).$$

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Mohd Danish Siddiqi,
Department of Mathematics,
Faculty of Science, Jazan Univesity
Kingdom of Saudi Arabia.
E-mail address: anallintegral@gmail.com, msiddiqi@jazanu.edu.sa