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A New Proof of Classical Dixon's Summation Theorem for the Series ${}_3F_2(1)$ *

Sungtae Jun, Insuk Kim* and Arjun K. Rathie

ABSTRACT: The aim of this short note is to provide a new proof of classical Dixon's summation theorem for the series ${}_3F_2(1)$.

Key Words: Dixon's summation theorem, Hypergeometric series, Generalized Hypergeometric Function.

Contents

1. Introduction

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz play a key role. Applications of the above mentioned theorems are well known now. For very interesting applications of these theorems, we refer a paper by Bailey [1].

Here we shall mention the following summation theorems that will be required in our present investigation.

Gauss summation theorem: [2,3,4]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.1}$$

provided $\mathrm{Re}(c-a-b) > 0$.

A known result: [4]

$$ {}_2F_1\left[\begin{matrix} -k, & a+k \\ & 1+a-c \end{matrix}; 1\right] = \frac{(-1)^k (c)_k}{(1+a-c)_k}, \tag{1.2}$$

which can be obtained by (1.1).

Kummer summation theorem: [2,3,4]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.3}$$

* Corresponding author
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The aim of this note is to provide a new proof of the following classical Dixon's summation theorem [2] for the series ${}_3F_2$ viz.

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right] \tag{1.4} $$
$$ = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}, $$

provided $\mathrm{Re}(a-2b-2c) > -2$.

2. A new proof of Dixon's summation theorem (1.4)

Consider the following integral valid for $\mathrm{Re}(b) > 0$

$$ I = \int_0^\infty e^{-t}\, t^{b-1}\, {}_2F_2\left[\begin{matrix} a, & c \\ 1+a-b, & 1+a-c \end{matrix}; t\right] dt. $$

Expressing the generalized hypergeometric function ${}_2F_2$ in series, we have

$$ I = \int_0^\infty e^{-t}\, t^{b-1} \sum_{k=0}^{\infty} \frac{(a)_k\ (c)_k\ t^k}{(1+a-b)_k\ (1+a-c)_k\ k!} dt. $$

Changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series, we have

$$ I = \sum_{k=0}^{\infty} \frac{(a)_k\ (c)_k}{(1+a-b)_k\ (1+a-c)_k\ k!} \int_0^\infty e^{-t}\, t^{b+k-1} dt. $$

Evaluating the gamma integral and using the result

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, $$

we have

$$ I = \Gamma(b) \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k\ (c)_k}{(1+a-b)_k\ (1+a-c)_k\ k!}. \tag{2.1} $$

Finally, summing up the series, we get

$$ I = \Gamma(b)\ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right]. \tag{2.2} $$

On the other hand, writing (2.1) in the form

$$ I = \Gamma(b) \sum_{k=0}^{\infty} \frac{(-1)^k\ (a)_k\ (b)_k}{(1+a-b)_k\ k!} \left\{ \frac{(-1)^k\ (c)_k}{(1+a-c)_k} \right\}. $$

Using (1.2), this becomes

$$I = \Gamma(b) \sum_{k=0}^{\infty} \frac{(-1)^k \, (a)_k \, (b)_k}{(1+a-b)_k \, k!} \, {}_2F_1 \left[\begin{matrix} -k, & a+k \\ 1+a-c \end{matrix} ; \, 1 \right].$$

Expressing ${}_2F_1$ as a series, we have after some simplification

$$I = \Gamma(b) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k \, (a)_k \, (b)_k \, (-k)_m \, (a+k)_m}{(1+a-b)_k \, (1+a-c)_m \, k! \, m!}.$$

Using the identities

$$(a)_k (a+k)_m = (a)_{k+m} \quad \text{and} \quad (-k)_m = \frac{(-1)^m \, k!}{(k-m)!},$$

we have, after some calculation

$$I = \Gamma(b) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^{k+m} \, (a)_{k+m} \, (b)_k}{(1+a-b)_k \, (1+a-c)_m \, m! \, (k-m)!}.$$

Now, using a known result [4, p.57, Equ.(2)]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+k),$$

we have

$$I = \Gamma(b) \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^k \, (a)_{k+2m} \, (b)_{k+m}}{(1+a-b)_{k+m} \, (1+a-c)_m \, m! \, k!}.$$

Using the identities

$$(a)_{k+2m} = (a)_{2m}(a+2m)_k \quad \text{and} \quad (b)_{k+m} = (b)_m (b+m)_k,$$

we have, after some simplification

$$I = \Gamma(b) \sum_{m=0}^{\infty} \frac{(a)_{2m} \, (b)_m}{(1+a-b)_m \, (1+a-c)_m \, m!} \times \sum_{k=0}^{\infty} \frac{(-1)^k (a+2m)_k \, (b+m)_k}{(1+a-b+m)_k \, k!}.$$

Summing up the inner series, we have

$$I = \Gamma(b) \sum_{m=0}^{\infty} \frac{(a)_{2m} \, (b)_m}{(1+a-b)_m \, (1+a-c)_m m!} \times {}_2F_1 \left[\begin{matrix} a+2m, & b+m \\ 1+a-b+m \end{matrix} ; -1 \right].$$

Now using Kummer's summation theorem (1.3) and then applying the identity

$$(a)_{2m} = 2^{2m} \left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a + \frac{1}{2}\right)_m,$$

we get after some simplification

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m\ (b)_m}{(1+a-c)_m\ m!}.$$

Summing up the series, we get

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}a, & b \\ 1+a-c\end{matrix};\ 1\right].$$

Applying Gauss summation theorem (1.1), we finally have

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}. \tag{2.3}$$

Therefore, equating (2.2) and (2.3), we get the desired Dixon's summation theorem (1.4).

This completes our new proof of Dixon's summation theorem for the series ${}_3F_2(1)$.

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Sungtae Jun,
General Education Institute,
Konkuk University, Chungju 380-701,
Republic of Korea.
E-mail address: sjun@kku.ac.kr

and

Insuk Kim,*
Department of Mathematics Education,
Wonkwang University, Iksan 570-749,
Republic of Korea.
E-mail address: iki@wku.ac.kr

and

Arjun K. Rathie,
Department of Mathematics,
Vedant College of Engineering and Technology (Rajasthan Technical University),
Bundi 323021, Rajasthan,
India.
E-mail address: arjunkumarrathie@gmail.com