Bol. Soc. Paran. Mat. (3s.) **v. 39** 5 (2021): 25–41.
©SPM –ISSN-2175-1188 ON LINE
ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm
doi:10.5269/bspm.40370

Additivity of Maps Preserving Triple Product on $*$-ring

A. Taghavi, M. Razeghi, M. Nouri, V. Darvish and C. Li

ABSTRACT: Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-rings. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective and satisfies

$$\Phi(A \bullet_\lambda P \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P) \bullet_\lambda \Phi(P),$$

for all $A \in \mathcal{A}$ and $P \in \{I, P_1, I - P_1\}$ where P_1 is a projection in $\mathcal{A}$. The operation $\bullet_\lambda$ between two arbitrary elements S and T in $\mathcal{A}$ is defined as $S \bullet_\lambda T = ST + \lambda TS^*$ for $\lambda \in \{-1, 1\}$. Then, if $\Phi(I)$ is projection, we show that Φ is additive.

Key Words: Maps preserving triple product, Additivity, Prime $*$-ring.

Contents

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}'$ preserves Lie product $[A, B] = AB - BA$ or Jordan product $A \circ B = AB + BA$ if $\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)$ and $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$, respectively (for example, see [2,3,8,12,13,16]). The results in the mentioned papers show that, in some sense, Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are called Jordan homomorphism or Lie homomorphism. Here we only list several results [4,14,15].

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, we denote by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. This product is found playing a more and more important role in some research topics, and its study has recently attracted many author's attention.

In [5], J. Cui and C. K. Li proved a bijective map Φ on factor von Neumann algebras which preserves $[A, B]_*$ must be a $*$-isomorphism. Moreover, in [9] C. Li et al, discussed the non-linear bijective map preserving $A \bullet B$ is also $*$-ring isomorphism.

The authors in [19] weakened the conditions of the above two results. They proved that if $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras and a map Φ from $\mathcal{A}$ onto $\mathcal{B}$ that is bijective, unital and satisfies

$$\Phi(A \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P),$$

2010 *Mathematics Subject Classification:* 47B48, 46L10.
Submitted November 06, 2017. Published June 18, 2018

for all $A \in \mathcal{A}$ and $P \in \{P_1, I - P_1\}$ where P_1 is a nontrivial projection in $\mathcal{A}$ and $\lambda \in \{-1, +1\}$, then, Φ is $*$-additive.

In [11], Li, Lu and Wang assumed $\mathcal{A}$ to be a von Neumann algebra with no central abelian projections and $\mathcal{B}$ to be a $*$-algebra. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ where $[A, B]_* = AB - BA^*$ is the skew Lie product. They showed that the following holds:

1. $\Phi(I)$ is self-adjoint central element in $\mathcal{B}$ with $\Phi(I_{\mathcal{A}})^2 = I_{\mathcal{B}}$.

2. Let $\Psi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$, then there exists a central projection $E \in \mathcal{A}$ such that the restriction of Ψ to $\mathcal{A}E$ is a linear $*$-isomorphism and the restriction of Ψ to $\mathcal{A}(I - E)$ is a conjugate linear $*$-isomorphism.

Also, the authors in [10] considered the same assumptions on Φ as above, but the map Φ satisfies the following condition

$$\Phi(A \bullet B \bullet C) = \Phi(A) \bullet \Phi(B) \bullet \Phi(C)$$

where $A \bullet B = AB + BA^*$. They obtained the same results.

In this paper, motivated by the above results, we consider $\Phi : \mathcal{A} \to \mathcal{B}$ which is bijective and satisfies

$$\Phi(A \bullet_\lambda P \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P) \bullet_\lambda \Phi(P),$$

for all $A \in \mathcal{A}$ and $P \in \{I_{\mathcal{A}}, P_1, I_{\mathcal{A}} - P_1\}$ where P_1 is a projection in $\mathcal{A}$, then Φ is additive. The operation $\bullet_\lambda$ between two arbitrary elements S and T in $\mathcal{A}$ is defined as $S \bullet_\lambda T = ST + \lambda TS^*$ for $\lambda \in \{-1, 1\}$.

It is well known that ring $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. Also, for the real and imaginary part of an operator T we will use $\Re(T)$ and $\Im(T)$, respectively.

2. Main result

We need the following lemmas for our theorem:

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-rings with an unit $I_{\mathcal{A}}$ and a nontrivial projection P_1. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies the following condition*

$$\Phi(A \bullet_\lambda P \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P) \bullet_\lambda \Phi(P) \tag{2.1}$$

for each $A, P \in \mathcal{A}$ such that P is in $\{I_{\mathcal{A}}, P_1, I_{\mathcal{A}} - P_1\}$. Then Φ is unital and $\Phi(0) = 0$.

Proof. First, we show that Φ is unital. We know that

$$\Phi(I \bullet_\lambda I \bullet_\lambda I) = \Phi(I) \bullet_\lambda \Phi(I) \bullet_\lambda \Phi(I).$$

It follows that

$$\Phi(4I) = 4\Phi(I). \tag{2.2}$$

On the other hand, for $\lambda = -1$, we have

$$\Phi(A \bullet_{-1} I \bullet_{-1} I) = \Phi(A) \bullet_{-1} \Phi(I) \bullet_{-1} \Phi(I).$$

So,

$$\Phi(2A - 2A^*) = (\Phi(A)\Phi(I) - \Phi(I)\Phi(A)^*) \bullet_{-1} \Phi(I).$$

Since Φ is surjective, then there is an element A such that $\Phi(A) = I$, hence

$$\Phi(2A - 2A^*) = 0$$

it means that

$$A = A^*. \tag{2.3}$$

Also, using (2.1) for A and I, we have

$$\Phi(A \bullet_\lambda I \bullet_\lambda I) = \Phi(A) \bullet_\lambda \Phi(I) \bullet_\lambda \Phi(I)$$

it follows that

$$\Phi(2A + 2A^*) = (\Phi(A)\Phi(I) + \Phi(I)\Phi(A)^*) \bullet_1 \Phi(I).$$

We can find A such that $\Phi(A) = I$. Then

$$\Phi(2A + 2A^*) = 2\Phi(I)\Phi(I) + 2\Phi(I)\Phi(I)^*. \tag{2.4}$$

From (2.2), (2.3) and (2.4) we obtain

$$\Phi(4A) = \Phi(4I)$$

since Φ is injective then $A = I$.
Now, suppose that $A = I$ in (2.1), we have

$$\begin{aligned}\Phi(IP - PI^*P + PIP - PI^*) &= \Phi(I)\Phi(P) - \Phi(P)\Phi(I)^*\Phi(P) \\ &\quad +\Phi(P)\Phi(I)\Phi(P) - \Phi(P)\Phi(I)\end{aligned}$$

then

$$\Phi(0) = \Phi(P) - \Phi(P)^2 + \Phi(P)^2 - \Phi(P) = 0.$$

□

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-rings with an unit $I_{\mathcal{A}}$ and a nontrivial projection P_1. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which satisfies*

$$\Phi(A \bullet_\lambda P \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P) \bullet_\lambda \Phi(P)$$

for all $A, P \in \mathcal{A}$ such that P is in $\{I_{\mathcal{A}}, P_1, I_{\mathcal{A}} - P_1\}$. Let A, B and T be in $\mathcal{A}$ such that $\Phi(T) = \Phi(A) + \Phi(B)$. Then we have

$$\begin{aligned}\Phi(TP + PT^*P + PTP + PT^*) &= \Phi(AP + PA^*P + PAP + PA^*) \\ &\quad + \Phi(BP + PB^*P + PBP + PB^*)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\Phi(TP - PT^*P + PTP - PT^*) &= \Phi(AP - PA^*P + PAP - PA^*) \\ &\quad + \Phi(BP - PB^*P + PBP - PB^*)\end{aligned} \tag{2.6}$$

Proof. We just show relation (2.5). Other relation (2.6) can be obtain in a similar way.

Let

$$\Phi(T) = \Phi(A) + \Phi(B). \tag{2.7}$$

Equivalently, we have

$$\Phi(T)^* = \Phi(A)^* + \Phi(B)^*. \tag{2.8}$$

From (2.7), it follows

$$\begin{aligned}\Phi(T)\Phi(P)^2 + \Phi(P)\Phi(T)\Phi(P)^* &= \Phi(A)\Phi(P)^2 + \Phi(P)\Phi(A)\Phi(P)^* \\ &\quad + \Phi(B)\Phi(P)^2 + \Phi(P)\Phi(B)\Phi(P)^*.\end{aligned} \tag{2.9}$$

Similarly, by (2.8) we can write

$$\begin{aligned}\Phi(P)^2\Phi(T)^* + \Phi(P)\Phi(T)^*\Phi(P) &= \Phi(P)^2\Phi(A) + \Phi(P)\Phi(A)^*\Phi(P) \\ &\quad + \Phi(P)^2\Phi(B) + \Phi(P)\Phi(B)^*\Phi(P).\end{aligned} \tag{2.10}$$

By adding (2.9) and (2.10) we have

$$\begin{aligned}\Phi(TP + PT^*P + PTP + PT^*) &= \Phi(AP + PA^*P + PAP + PA^*) \\ &\quad + \Phi(BP + PB^*P + PBP + PB^*).\end{aligned}$$

□

Main Theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-rings with an unit $I_{\mathcal{A}}$ and a nontrivial projection P_1. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective map which satisfies

$$\Phi(A \bullet_\lambda P \bullet_\lambda P) = \Phi(A) \bullet_\lambda \Phi(P) \bullet_\lambda \Phi(P) \tag{2.11}$$

for all $A, P \in \mathcal{A}$ where P is in $\{I_{\mathcal{A}}, P_1, I_{\mathcal{A}} - P_1\}$. Then, Φ is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j,\ i, j = 1, 2$, as a sub-ring of $\mathcal{A}$. Then for every $A \in \mathcal{A}$ we may write $A = P_1AP_1 + P_1AP_2 + P_2AP_1 + P_2AP_2$. One could call it Peirce decomposition. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

To show additivity of Φ on $\mathcal{A}$, we use above decomposition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i, j = 1, 2$.

Claim 1. *For each $A_{11} \in \mathcal{A}_{11}$ and $A_{12} \in \mathcal{A}_{12}$ we have*

$$\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).$$

Proof. Since Φ is surjective, we can find $T \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(A_{12}). \tag{2.12}$$

Using (2.5) to (2.12) for P_1, we get

$$\begin{aligned}&\Phi(TP_1 + P_1T^*P_1 + P_1TP_1 + P_1T^*)\\ &= \Phi(A_{11}P_1 + P_1A_{11}^*P_1 + P_1A_{11}P_1 + P_1A_{11}^*)\\ &+\Phi(A_{12}P_1 + P_1A_{12}^*P_1 + P_1A_{12}P_1 + P_1A_{12}^*).\end{aligned}$$

From that $T = T_{11} + T_{12} + T_{21} + T_{22}$, it follows that

$$\Phi(T_{11} + T_{21} + T_{11}^* + T_{11} + T_{11}^* + T_{21}^*) = \Phi(A_{11} + A_{11}^* + A_{11} + A_{11}^*) + \Phi(0).$$

So, we have

$$\Phi(2T_{11} + 2T_{11}^* + T_{21} + T_{21}^*) = \Phi(2A_{11} + 2A_{11}^*).$$

Since Φ is injective , we obtain

$$2T_{11} + 2T_{11}^* + T_{21} + T_{21}^* = 2A_{11} + 2A_{11}^*.$$

In fact,

$$4\Re(T_{11}) = 4\Re(A_{11}) \quad \text{and} \quad \Re(T_{21}) = 0. \tag{2.13}$$

Similarly, by applying (2.6) to (2.12) for P_1, we obtain

$$\begin{aligned}&\Phi(TP_1 - P_1T^*P_1 + P_1TP_1 - P_1T^*)\\ &= \Phi(A_{11}P_1 - P_1A_{11}^*P_1 + P_1A_{11}P_1 - P_1A_{11}^*)\\ &+\Phi(A_{12}P_1 - P_1A_{12}^*P_1 + P_1A_{12}P_1 - P_1A_{12}^*).\end{aligned}$$

By knowing that $T = T_{11} + T_{12} + T_{21} + T_{22}$, it follows that

$$\Phi(T_{11} + T_{21} - T_{11}^* + T_{11} - T_{11}^* - T_{21}^*) = \Phi(A_{11} - A_{11}^* + A_{11} - A_{11}^*) + \Phi(0).$$

So, we have

$$\Phi(2T_{11} - 2T_{11}^* + T_{21} - T_{21}^*) = \Phi(2A_{11} - 2A_{11}^*).$$

Since Φ is injective , we obtain

$$2T_{11} - 2T_{11}^* + T_{21} - T_{21}^* = 2A_{11} - 2A_{11}^*.$$

Therefore,

$$4\Im(T_{11}) = 4\Im(A_{11}) \quad \text{and} \quad \Im(T_{21}) = 0. \tag{2.14}$$

By considering (2.13) and (2.14), we have $T_{11} = A_{11}$ and $T_{21} = 0$.

On the other hand, by applying (2.5) to (2.12) for P_2 we have

$$\begin{aligned}&\Phi(TP_2 + P_2T^*P_2 + P_2TP_2 + P_2T^*)\\&= \Phi(A_{11}P_2 + P_2A_{11}^*P_2 + P_2A_{11}P_2 + P_2A_{11}^*)\\&+\Phi(A_{12}P_2 + P_2A_{12}^*P_2 + P_2A_{12}P_2 + P_2A_{12}^*).\end{aligned}$$

It follows that

$$\Phi(T_{22} + T_{12} + T_{22}^* + T_{22} + T_{12}^* + T_{22}^*) = \Phi(A_{12} + A_{12}^*) + \Phi(0).$$

So, we have

$$\Phi(2T_{22} + 2T_{22}^* + T_{12} + T_{12}^*) = \Phi(A_{12} + A_{12}^*).$$

Since Φ is injective , we obtain

$$2T_{22} + 2T_{22}^* + T_{12} + T_{12}^* = A_{12} + A_{12}^*.$$

Hence,

$$\Re(T_{22}) = 0 \quad \text{and} \quad \Re(T_{12}) = \Re(A_{12}). \tag{2.15}$$

Similarly, by applying (2.6) to (2.12) for P_2, we obtain

$$\begin{aligned}&\Phi(TP_2 - P_2T^*P_2 + P_2TP_2 - P_2T^*)\\&= \Phi(A_{11}P_2 - P_2A_{11}^*P_2 + P_2A_{11}P_2 - P_2A_{11}^*)\\&+\Phi(A_{12}P_2 - P_2A_{12}^*P_2 + P_2A_{12}P_2 - P_2A_{12}^*).\end{aligned}$$

It leads to

$$\Phi(T_{22} + T_{12} - T_{22}^* + T_{22} - T_{22}^* - T_{12}^*) = \Phi(A_{12} - A_{12}^*) + \Phi(0).$$

So, we have

$$\Phi(2T_{22} - 2T_{22}^* + T_{12} - T_{12}^*) = \Phi(A_{12} - A_{12}^*).$$

Since Φ is injective , we obtain

$$2T_{22} - 2T_{22}^* + T_{12} - T_{12}^* = A_{12} - A_{12}^*.$$

Therefore,

$$\Im(T_{22}) = 0 \quad \text{and} \quad \Im(T_{12}) = \Im(A_{12}). \tag{2.16}$$

Combining (2.15) and (2.16), we have $T_{12} = A_{12}$ and $T_{22} = 0$. $\square$

Claim 2. *For each $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21}).$$

Proof. By surjectivity of Φ, we can find $T \in \mathcal{A}$, such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(A_{21}). \tag{2.17}$$

By using (2.5) for (2.17) for P_1, we have

$$\begin{aligned}&\Phi(TP_1 + P_1T^*P_1 + P_1TP_1 + P_1T^*)\\&= \Phi(A_{12}P_1 + P_1A_{12}^*P_1 + P_1A_{12}P_1 + P_1A_{12}^*)\\&+\Phi(A_{21}P_1 + P_1A_{21}^*P_1 + P_1A_{21}P_1 + P_1A_{21}^*)\end{aligned}$$

It follows that

$$\Phi(T_{11} + T_{21} + T_{11}^* + T_{11} + T_{11}^* + T_{21}^*) = \Phi(A_{21} + A_{21}^*) + \Phi(0).$$

So, we have

$$\Phi(2T_{11} + 2T_{11}^* + T_{21} + T_{21}^*) = \Phi(A_{21} + A_{21}^*)$$

Since Φ is injective, we have

$$2T_{11} + 2T_{11}^* + T_{21} + T_{21}^* = A_{21} + A_{21}^*.$$

Therefore, we have

$$4\Re(T_{11}) = 0 \quad \text{and} \quad \Re(T_{21}) = \Re(A_{21}). \tag{2.18}$$

In a similar way applying (2.6) to (2.17) for P_1, we have

$$4\Im(T_{11}) = 0 \quad \text{and} \quad \Im(T_{21}) = \Im(A_{21}). \tag{2.19}$$

By (2.18) and (2.19), we obtain $T_{11} = 0$ and $T_{21} = A_{21}$.
By using(2.5) for P_2 in (2.17) , we have

$$\begin{aligned}&\Phi(T_{22}P_2 + T_{12}P_2 + P_2T_{22}^* + P_2T_{12}^* + P_2T_{22}P_2 + P_2T_{22}^*P_2)\\&= \Phi(A_{12}P_2 + P_2A_{12}^* + P_2A_{12}P_2 + P_2A_{12}^*P_2)\\&+\Phi(A_{21}P_2 + P_2A_{21}^* + P_2A_{21}P_2 + P_2A_{21}^*P_2).\end{aligned}$$

So, we have

$$\Phi(2T_{22} + T_{12} + 2T_{22}^* + T_{12}^*) = \Phi(A_{12} + A_{12}^*).$$

Since Φ is injective, we have

$$2T_{22} + T_{12} + 2T_{22}^* + T_{12}^* = A_{12} + A_{12}^*$$

it follows that

$$T_{22} + T_{22}^* + P_1TP_2 + P_2T^*P_1 + P_2TP_2 + P_2T^*P_2 = A_{12} + A_{12}^*.$$

So,

$$T_{22} + T_{22}^* + (P_1 + P_2)TP_2 + P_2T^*(P_1 + P_2) = A_{12} + A_{12}^*.$$

Hence, we obtain

$$T_{22} + T_{22}^* + ITP_2 + P_2T^*I = A_{12} + A_{12}^*.$$

Multiplying the both sides of the above equality by P_1 we obtain $T_{12} = A_{12}$. Since $T_{12} = A_{12}$, from the equality

$$2T_{22} + T_{12} + 2T_{22}^* + T_{12}^* = A_{12} + A_{12}^*$$

we have $T_{22} + T_{22}^* = 0$. So, $2\Re(T_{22}) = 0$.
If we use (2.6) for P_2 in (2.17), we have $\Im(T_{22}) = 0$. So, we obtain $T_{22} = 0, T_{12} = A_{12}$. It means that $T = A_{12} + A_{21}$. $\square$

Lemma 2.3. $\Phi(4\Re(A)) = 4\Re(\Phi(A))$.

Proof. By applying (2.11) for I and knowing that Φ is unital, we have

$$\begin{aligned}
\Phi(A + A^* + A + A^*) &= \Phi(A \bullet_1 I \bullet_1 I) \\
&= \Phi(A) \bullet_1 \Phi(I) \bullet_1 \Phi(I) \\
&= (\Phi(A)\Phi(I) + \Phi(I)\Phi(A)^*)) \bullet_1 \Phi(I) \\
&= \Phi(A) + \Phi(A)^* + \Phi(A) + \Phi(A)^*.
\end{aligned}$$

So, $\Phi(4\Re(A)) = 4\Re(\Phi(A))$. Similarly, we can prove $\Phi(4i\Im(A)) = 4i\Im(\Phi(A))$. $\square$

Claim 3. *For each $A_{11} \in \mathcal{A}_{11}, A_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{11} + A_{21}) = \Phi(A_{11}) + \Phi(A_{21}).$$

Proof. We prove $\Phi(2A_{11} + 2A_{21}) = \Phi(2A_{11}) + \Phi(2A_{21})$. Since Φ is surjective, there exists $T \in \mathcal{A}$ such that

$$\Phi(2T) = \Phi(2A_{11}) + \Phi(2A_{21}). \tag{2.20}$$

By using (2.5) for $P = P_1$ in (2.20), Claim 1 and Lemma 2.3, we have

$$\begin{aligned}
&\Phi(2TP_1 + 2P_1T^* + 2P_1TP_1 + 2P_1T^*P_1) \\
&= \Phi(2A_{11}P_1 + 2P_1A_{11}^* + 2P_1A_{11}P_1 + 2P_1A_{11}^*P_1) \\
&+ \Phi(2A_{21}P_1 + 2P_1A_{21}^* + 2P_1A_{21}P_1 + 2P_1A_{21}^*P_1).
\end{aligned}$$

So, we have

$$\begin{aligned}
&\Phi(2T_{11}P_1 + 2T_{21}P_1 + 2P_1T_{11}^* + 2P_1T_{21}^* + 2P_1T_{11}P_1 + 2P_1T_{11}^*P_1) \\
&= \Phi(2A_{11}P_1 + 2P_1A_{11}^* + 2P_1A_{11}P_1 + 2P_1A_{11}^*P_1) \\
&+ \Phi(2A_{21}P_1 + 2P_1A_{21}^* + 2P_1A_{21}P_1 + 2P_1A_{21}^*P_1).
\end{aligned}$$

On the other hand, by Lemma 2.3, we can write

$$\begin{aligned}
\Phi(4T_{11}+2T_{21}+4T_{11}^*+2T_{21}^*) &= \Phi(4A_{11}+4A_{11}^*)+\Phi(2A_{21}+2A_{21}^*) \\
&= \Phi(4\Re 2A_{11})+\Phi(4\Re A_{21}^*) \\
&= 4\Re(\Phi(2A_{11})+\Phi(A_{21}^*)) \\
&= 4\Re(\Phi(2A_{11}+A_{21}^*)) \\
&= \Phi(4\Re 2A_{11}+4\Re A_{21}^*) \\
&= \Phi(8\Re A_{11}+4\Re A_{21})
\end{aligned}$$

therefore

$$\Phi(8\Re T_{11}+4\Re T_{21}) = \Phi(8\Re A_{11}+4\Re A_{21}).$$

Since Φ is injective, we have

$$8\Re T_{11}+4\Re T_{21} = 8\Re A_{11}+4\Re A_{21}.$$

In a similar way using (2.6) for $P_i = P_1$ in (2.20) we have

$$8\Im T_{11}+4\Im T_{21} = 8\Im A_{11}+4\Im A_{21}.$$

So, we have

$$8T_{11}+4T_{21} = 8A_{11}+4A_{21}.$$

Multiplying the left side of the above equality by P_1, we have $T_{11} = A_{11}$, $T_{21} = A_{21}$. By using (2.5) in (2.20) for $P = P_2$ we have

$$\begin{aligned}
&\Phi(2TP_2+2P_2T^*+2P_2T^*P_2+2P_2TP_2) \\
&= \Phi(2A_{11}P_2+2P_2A_{11}^*+2P_2A_{11}P_2+2P_2A_{11}^*P_2) \\
&+\Phi(2A_{21}P_2+2P_2A_{21}^*+2P_2A_{21}P_2+2P_2A_{21}^*P_2)
\end{aligned}$$

it follows that

$$\Phi(2T_{22}P_2+2T_{12}P_2+2P_2T_{22}^*+2P_2T_{12}^*+2P_2T_{22}^*P_2+2P_2T_{22}P_2) = 0.$$

So,

$$\Phi(4T_{22}+2T_{12}+4T_{22}^*+2T_{12}^*) = 0.$$

Since Φ is injective $8\Re T_{22}+4\Re T_{12} = 0$.
In a similar way by using (2.6) for $P = P_2$ in (2.20), we have $8\Im T_{22}+4\Im T_{12} = 0$.
So, we have

$$8\Re T_{22}+8i\Im T_{22}+4\Re T_{12}+4i\Im T_{12} = 0.$$

Therefore, $8T_{22}+4T_{12} = 0$. Multiplying the above equality by P_1 on the left, we have $T_{12} = 0$ and $T_{22} = 0$. Hence, we have $T = A_{11}+A_{12}$.

In a same way, we prove that $\Phi(A_{12}+A_{22}) = \Phi(A_{12})+\Phi(A_{22})$. □

Claim 4. $\Phi(A_{ij}+B_{ij}) = \Phi(A_{ij})+\Phi(B_{ij})$.

Proof. Above claim is equivalent to $\Phi(2A_{ij}+2B_{ij}) = \Phi(2A_{ij})+\Phi(2B_{ij})$ Since Φ is surjective we can find T such that

$$\Phi(2T) = \Phi(2A_{ij}) + \Phi(2B_{ij}) \tag{2.21}$$

We should show that $T = A_{ij} + B_{ij}$.
Applying (2.5) to P_i, we have

$$\begin{aligned}&\Phi(2TP_i + 2P_iT^*P_i + 2P_iTP_i + 2P_iT^*)\\&= \Phi(2A_{ij}P_i + 2P_iA_{ij}^*P_i + 2P_iA_{ij}P_i + 2P_iA_{ij}^*)\\&+\Phi(2B_{ij}P_i + 2P_iB_{ij}^*P_i + 2P_iB_{ij}P_i + 2P_iB_{ij}^*,\end{aligned}$$

it follows that

$$\Phi(2T_{ii} + 2T_{ji} + 2T_{ii}^* + 2T_{ii} + 2T_{ii}^* + 2T_{ji}^*) = \Phi(0).$$

So, we have

$$\Phi(8\Re(T_{ii}) + 4\Re(T_{ji}) = 0,$$

by injectivity, we have $\Re(T_{ii}) = 0$ and $\Re(T_{ji}) = 0$. Similarly we can show that $\Im(T_{ii}) = 0$ and $\Im(T_{ji}) = 0$.
Now By (2.5) for P_j, we have

$$\begin{aligned}&\Phi(2TP_j + 2P_jT^*P_j + 2P_jTP_j + 2P_jT^*)\\&= \Phi(2A_{ij}P_j + 2P_jA_{ij}^*P_j + 2P_jA_{ij}P_j + 2P_jA_{ij}^*)\\&+\Phi(2B_{ij}P_j + 2P_jB_{ij}^*P_j + 2P_jB_{ij}P_j + 2P_jB_{ij}^*.\end{aligned}$$

It follows that

$$\Phi(2T_{jj} + 2T_{ij} + 2T_{jj}^* + 2T_{jj} + 2T_{jj}^* + 2T_{ij}^*) = \Phi(2A_{ij} + 2A_{ij}^*) + \Phi(2B_{ij} + 2B_{ij}^*),$$

so, we have

$$\Phi(8\Re(T_{jj}) + 4\Re(T_{ij})) = \Phi(4\Re(A_{ij})) + \Phi(4\Re(B_{ij}^*)).$$

By Lemma 2.3, we have

$$\begin{aligned}\Phi(8\Re(T_{jj}) + 4\Re(T_{ij})) &= \Phi(4\Re(A_{ij})) + \Phi(4\Re(B_{ij}^*))\\&= 4\Re(\Phi(A_{ij})) + 4\Re(\Phi(B_{ij}^*))\\&= 4\Re\left(\Phi(A_{ij}) + \Phi(B_{ij}^*)\right)\\&= 4\Re\left(\Phi(A_{ij} + B_{ij}^*)\right) \qquad \text{By Claim 2}\\&= \Phi(4\Re(A_{ij} + B_{ij}^*)) \qquad \text{By Lemma 2.3}\\&= \Phi(4\Re(A_{ij}) + 4\Re(B_{ij}^*))\\&= \Phi(4\Re(A_{ij}) + 4\Re(B_{ij})).\end{aligned}$$

By injectivity $\Re(T_{jj}) = 0$ and $\Re(T_{ij}) = \Re(A_{ij}) + \Re(B_{ij})$. Similarly, we can show that $\Im(T_{ij}) = \Im(A_{ij}) + \Im(B_{ij})$. It leads us to $\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})$. □

Claim 5. *For each $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Proof. We prove the following identity

$$\Phi(2A_{11} + 2A_{12} + 2A_{21} + 2A_{22}) = \Phi(2A_{11}) + \Phi(2A_{12}) + \Phi(2A_{21}) + \Phi(2A_{22}),$$

since Φ is surjective, there exists

$$T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A} \tag{2.22}$$

such that

$$\Phi(2T) = \Phi(2A_{11}) + \Phi(2A_{12}) + \Phi(2A_{21}) + \Phi(2A_{22}).$$

By using (2.5)for $P = P_1$ in (2.22) and using Claim 3, we have

$$\begin{aligned}
&\Phi(2TP_1 + 2P_1T^* + 2P_1TP_1 + 2P_1T^*P_1) \\
&= \Phi(2A_{11}P_1 + 2P_1A_{11}^* + 2P_1A_{11}P_1 + 2P_1A_{11}^*P_1) \\
&+\Phi(2A_{22}P_1 + 2P_1A_{22}^* + 2P_1A_{22}P_1 + 2P_1A_{22}^*P_1) \\
&+\Phi(2A_{12}P_1 + 2P_1A_{12}^* + 2P_1A_{12}P_1 + 2P_1A_{12}^*P_1) \\
&+\Phi(2A_{21}P_1 + 2P_1A_{21}^* + 2P_1A_{21}P_1 + 2P_1A_{21}^*P_1)
\end{aligned}$$

then

$$\begin{aligned}
\Phi(4T_{11} + 2T_{21} + 4T_{11}^* + 2T_{21}^*) &= \Phi(4A_{11} + 4A_{11}^*) + \Phi(2A_{21} + 2A_{21}^*) \\
&= \Phi(4\Re 2A_{11}) + \Phi(4\Re A_{21}) \\
&= 4\Re(\Phi(2A_{11}) + \Phi(A_{21})) \\
&= 4\Re(\Phi(2A_{11} + A_{21})) \\
&= \Phi(4\Re 2A_{11} + \Re A_{21}).
\end{aligned}$$

Since Φ is injective, we have

$$8\Re T_{11} + 4\Re T_{21} = 8\Re A_{11} + 4\Re A_{21}.$$

In a similar way, by using (2.6)for $P = P_1$ in (2.22), we have

$$8\Im T_{11} + 4\Im T_{21} = 8\Im A_{11} + 4\Im A_{21}.$$

So, we have

$$8\Re T_{11} + 8i\Im T_{11} + 4\Re T_{21} + 4i\Im T_{21} = 8\Re A_{11} + 8i\Im A_{11} + 4\Re A_{21} + 4i\Im A_{21}$$

then we have

$$8T_{11} + 4T_{21} = 8A_{11} + 4A_{21}.$$

Multiplying the above equality by P_2 on the left, we have $T_{21} = A_{21}$. So, $T_{11} = A_{11}$. By using (2.5) for $P = P_2$ in (2.22), we have

$$\begin{aligned}
&\Phi(2TP_2 + 2P_2T^* + 2P_2TP_2 + 2P_2T^*P_2)\\
&= \Phi(2A_{11}P_2 + 2P_2A_{11}^*) + 2P_2A_{11}P_2 + 2P_2A_{11}^*P_2\\
&+\Phi(2A_{22}P_2 + 2P_2A_{22}^* + 2P_2A_{22}P_2 + 2P_2A_{22}^*P_2)\\
&+\Phi(2A_{12}P_2 + 2P_2A_{12}^* + 2P_2A_{12}P_2 + 2P_2A_{12}^*P_2)\\
&+\Phi(2A_{12}P_2 + 2P_2A_{21}^* + 2P_2A_{21}P_2 + 2P_2A_{21}^*P_2).
\end{aligned}$$

Therefore, from Claim 3 and Lemma 2.3, we have

$$\begin{aligned}
\Phi(4T_{22} + 2T_{12} + 4T_{22}^* + 2T_{12}^*) &= \Phi(4A_{22} + 4A_{22}^*) + \Phi(2A_{12} + 2A_{12}^*)\\
&= \Phi(4\Re 2A_{22}) + \Phi(4\Re A_{12})\\
&= 4\Re\Phi(2A_{22}) + 4\Re\Phi(A_{12})\\
&= 4\Re(\Phi(2A_{22}) + \Phi(A_{12}))\\
&= 4\Re(\Phi(2A_{22} + A_{12}))\\
&= \Phi(4\Re(2A_{22} + A_{12})),
\end{aligned}$$

so, we have

$$\Phi(8\Re T_{22} + 4\Re T_{12}) = \Phi(8\Re A_{22} + 4\Re A_{12}).$$

Since Φ is injective, we obtain

$$8\Re T_{22} + 4\Re T_{12} = 8\Re A_{22} + 4\Re A_{12}.$$

In a same way by using (2.6)for $P = P_2$ in (2.22), Lemma 2.3 and Claim 3, we have

$$8\Im T_{22} + 4\Im T_{12} = 8\Im A_{22} + 4\Im A_{12}.$$

So,

$$8\Re T_{22} + 8i\Im T_{22} + 4\Re T_{12} + 4i\Im T_{12} = 8\Re A_{22} + 8i\Im A_{22} + 4\Re A_{12} + 4i\Im A_{12}.$$

Hence, we have

$$8T_{22} + 4T_{12} = 8A_{22} + 4A_{12}.$$

Multiplying the above equality by P_2 on the left, we have $T_{22} = A_{22}$, $T_{12} = A_{12}$. So, $T = A_{11} + A_{22} + A_{12} + A_{21}$. □

Claim 6. *Φ preserves the orthogonal projections P_1 and P_2 on the both sides.*

Proof. By (2.11) for P_i and $\lambda = 1$ we have

$$\begin{aligned}
\Phi(AP_i + P_iA^*P_i + P_iAP_i + P_iA^*) &= \Phi(A)\Phi(P_i)^2 + \Phi(P_i)\Phi(A)^*\Phi(P_i)\\
&+\Phi(P_i)\Phi(A)\Phi(P_i) + \Phi(P_i)\Phi(P_i)^*\Phi(A)^*,
\end{aligned}$$

and for $\lambda = -1$, we have

$$\begin{aligned}\Phi(AP_i - P_iA^*P_i + P_iAP_i - P_iA^*) &= \Phi(A)\Phi(P_i)^2 - \Phi(P_i)\Phi(A)^*\Phi(P_i) \\ &\quad + \Phi(P_i)\Phi(A)\Phi(P_i) - \Phi(P_i)\Phi(P_i)^*\Phi(A)^*.\end{aligned}$$

By adding the two above relations, using Claim 5 and Lemma 2.3 we have

$$\begin{aligned}2\Phi(A)\Phi(P_i)^2 + 2\Phi(P_i)\Phi(A)\Phi(P_i) &= \Phi(AP_i + P_iA^*P_i + P_iAP_i + P_iA^*) \\ &\quad + \Phi(AP_i - P_iA^*P_i + P_iAP_i - P_iA^*) \\ &= \Phi(A_{ii} + A_{ji} + A_{ii}^* + A_{ji}^* + A_{ii} + A_{ii}^*) \\ &\quad + \Phi(A_{ii} + A_{ji} - A_{ii}^* - A_{ji}^* + A_{ii} - A_{ii}^*) \\ &= \Phi(4\Re A_{ii}) + \Phi(4i\Im A_{ii}) + \Phi(A_{ji}) \\ &\quad + \Phi(A_{ji}^*) + \Phi(A_{ji}) - \Phi(A_{ji}^*) \\ &= 4\Re\Phi(A_{ii}) + 4i\Im\Phi(A_{ii}) + 2\Phi(A_{ji}) \\ &= 4\Phi(A_{ii}) + 2\Phi(A_{ji}) \\ &= 2\Phi(A_{ii}) + 2(\Phi(A_{ii}) + \Phi(A_{ji})) \\ &= 2\Phi(A_{ii}) + 2\Phi(A_{ii} + A_{ij}) \\ &= 2\Phi(A_{ii}) + 2\Phi(AP_i).\end{aligned}$$

So, we obtain

$$\Phi(A)\Phi(P_i)^2 + \Phi(P_i)\Phi(A)\Phi(P_i) = \Phi(AP_i) + \Phi(A_{ii}). \tag{2.23}$$

It follows that

$$\Phi(A)\Phi(P_i)^2 + \Phi(P_i)\Phi(A)\Phi(P_i) = \Phi(AP_i) + \Phi(P_iAP_i).$$

By considering $A = I$, we have

$$\Phi(P_i)^2 + \Phi(P_i)\Phi(P_i) = \Phi(P_i) + \Phi(P_i).$$

Hence,

$$\Phi(P_i)^2 = \Phi(P_i)$$

So, we have proved $\Phi(P_i)$ is idempotent. Now, we prove that $\Phi(P_i)$ self-adjoint.

$$\begin{aligned}\Phi(P_i) &= \Phi\left(\frac{P_i}{4}\bullet_1 I\bullet_1 I\right)\\ &= \Phi\left(\frac{P_i}{4}\right)\bullet_1 \Phi(I)\bullet_1 \Phi(I)\\ &= \left(\Phi\left(\frac{P_i}{4}\right)\Phi(I)+\Phi(I)\Phi\left(\frac{P_i}{4}\right)^*\right)\bullet_1 \Phi(I)\\ &= \Phi\left(\frac{P_i}{4}\right)\Phi(I)^2+\Phi(I)\Phi\left(\frac{P_i}{4}\right)^*\Phi(I)+\Phi(I)\Phi(I)^*\Phi\left(\frac{P_i}{4}\right)^*\\ &\quad +\Phi(I)\Phi\left(\frac{P_i}{4}\right)\Phi(I)^*\\ &= 2\Phi\left(\frac{P_i}{4}\right)+2\Phi\left(\frac{P_i}{4}\right)^*\end{aligned}$$

So,

$$\Phi(P_i)=2\Phi\left(\frac{P_i}{4}\right)+2\Phi\left(\frac{P_i}{4}\right)^*, \tag{2.24}$$

it follows that $\Phi(P_i)=\Phi(P_i)^*$. □

Note that $\Phi(P_i)\Phi(P_j)=0$ for $i\neq j$.

Claim 7. *For every $A_{ij}\in\mathcal{A}_{ij}$, $B_{ij}\in\mathcal{B}_{ij}$ and for $i\neq j$ we have $\Phi(A_{ij})=B_{ij}$.*

Proof. For $A_{ij}\in\mathcal{A}_{ij}$, we can write

$$\Phi(A_{ij}\bullet_\lambda P_j\bullet_\lambda P_j)=\Phi(A_{ij})\bullet_\lambda \Phi(P_j)\bullet_\lambda \Phi(P_j).$$

For $\lambda=1$ the equation above converts to

$$\begin{aligned}\Phi(A_{ij}+A_{ij}^*) &= \Phi(A_{ij})\Phi(P_j)^2+\Phi(P_j)\Phi(A_{ij})^*\Phi(P_j)\\ &\quad +\Phi(P_j)\Phi(A_{ij})\Phi(P_j)+\Phi(P_j)\Phi(P_j)^*\Phi(A_{ij})^*.\end{aligned}\tag{2.25}$$

For $\lambda=-1$, we can write

$$\begin{aligned}\Phi(A_{ij}-A_{ij}^*) &= \Phi(A_{ij})\Phi(P_j)^2-\Phi(P_j)\Phi(A_{ij})^*\Phi(P_j)\\ &\quad +\Phi(P_j)\Phi(A_{ij})\Phi(P_j)-\Phi(P_j)\Phi(P_j)^*\Phi(A_{ij})^*.\end{aligned}\tag{2.26}$$

From (2.25), (2.26) and Lemma 2.3 we have

$$\Phi(A_{ij})=\Phi(A_{ij})\Phi(P_j)^2+\Phi(P_j)\Phi(A_{ij})\Phi(P_j). \tag{2.27}$$

The above equation gives us

$$\Phi(P_j)\Phi(A_{ij})\Phi(P_i)=0,$$

$$\Phi(P_j)\Phi(A_{ij})\Phi(P_j) = 0$$

and

$$\Phi(P_i)\Phi(A_{ij})\Phi(P_i) = 0.$$

So, we have

$$\Phi(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}.$$

Since Φ^{-1} has the same property as Φ, we have

$$\mathcal{B}_{ij} \subseteq \Phi(\mathcal{A}_{ij}).$$

Similarly, one can prove that $\Phi(A_{ji}) = B_{ji}$. □

Claim 8. *For every $A_{ii} \in \mathcal{A}_{ii}$ we have $\Phi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$.*

Proof. We know

$$\Phi(A_{ii} \bullet_\lambda P_i \bullet_\lambda P_i) = \Phi(A_{ii}) \bullet_\lambda \Phi(P_i) \bullet_\lambda \Phi(P_i).$$

For $\lambda = 1$, we have

$$\begin{aligned} \Phi(2A_{ii} + 2A_{ii}) &= \Phi(A_{ii})\Phi(P_i) + \Phi(P_i)\Phi(A_{ii})^*\Phi(P_i) \\ &\quad + \Phi(P_i)\Phi(A_{ii})\Phi(P_i) + \Phi(P_i)\Phi(A_{ii})^*. \end{aligned} \tag{2.28}$$

For $\lambda = -1$, we have

$$\begin{aligned} \Phi(2A_{ii} - 2A_{ii}^*) &= \Phi(A_{ii})\Phi(P_i) - \Phi(P_i)\Phi(A_{ii})^*\Phi(P_i) \\ &\quad + \Phi(P_i)\Phi(A_{ii})\Phi(P_i) - \Phi(P_i)\Phi(A_{ii})^*. \end{aligned} \tag{2.29}$$

By adding (2.28) and (2.29), we obtain

$$4\Re(\Phi(A_{ii})) + 4i\Im(\Phi(A_{ii})) = 2\Phi(A_{ii})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ii})\Phi(P_i).$$

It follows that

$$4\Phi(A_{ii}) = 2\Phi(A_{ii})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ii})\Phi(P_i). \tag{2.30}$$

By the above equation we have

$$\Phi(P_i)\Phi(A_{ii})\Phi(P_j) = 0,$$

$$\Phi(P_j)\Phi(A_{ii})\Phi(P_i) = 0$$

and

$$\Phi(P_j)\Phi(A_{ii})\Phi(P_j) = 0.$$

Therefore, we have

$$\Phi(A_{ii}) = \sum_{i,j=1}^{2} \Phi(P_i)\Phi(A_{ii})\Phi(P_j) = \Phi(P_i)\Phi(A_{ii})\Phi(P_i) \in \mathcal{B}_{ii}.$$

Hence, $\Phi(\mathcal{A}_{ii}) \subseteq \mathcal{B}_{ii}$. Since Φ^{-1} has the same properties as Φ then we have the result. □

Claim 9. *For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Proof. For $\Phi(T)_{ij} = \Phi(P_i)\Phi(T)\Phi(P_j) \in \mathcal{B}_{ij}$, we can write

$$\Phi(T)_{ij}\Phi(AP_i + BP_i) = \Phi(P_i)\Phi(T)\Phi(P_j)\Phi(AP_i + BP_i).$$

We multiply the right side of the above equation by $\Phi(P_i)$ to obtain

$$\begin{aligned}
\Phi(T)_{ij}\Phi(AP_i + BP_i)\Phi(P_i) &= \Phi(P_i)\Phi(T)\Phi(P_j)\Phi(AP_i + BP_i)\Phi(P_i) \\
&= \Phi(P_i)\Phi(T)\Phi(P_j(AP_i + BP_i)P_i) \quad (\Phi(\mathcal{A}_{ji}) = \mathcal{B}_{ji}) \\
&= \Phi(P_i)\Phi(T)\Phi(P_jAP_i + P_jBP_i) \\
&= \Phi(P_i)\Phi(T)\Phi(A_{ji} + B_{ji}) \\
&= \Phi(P_i)\Phi(T)(\Phi(A_{ji}) + \Phi(B_{ji})) \\
&= \Phi(P_i)\Phi(T)(\Phi(P_jAP_i) + \Phi(P_jBP_i)) \\
&= \Phi(P_i)\Phi(T)(\Phi(P_j)\Phi(AP_i)\Phi(P_i) \\
&\quad +\Phi(P_j)\Phi(BP_i)\Phi(P_i)) \\
&= \Phi(T)_{ij}(\Phi(AP_i) + \Phi(BP_i))\Phi(P_i).
\end{aligned}$$

We showed that

$$\Phi(T)_{ij}(\Phi(AP_i + BP_i) - \Phi(AP_i) - \Phi(BP_i))\Phi(P_i) = 0$$

for all $\Phi(T)_{ij} \in \mathcal{B}_{ij}$, since $\mathcal{B}$ is prime, we have

$$\Phi(AP_i + BP_i)\Phi(P_i) = \Phi(AP_i)\Phi(P_i) + \Phi(BP_i)\Phi(P_i).$$

Multiplying the above equation from the left side by $\Phi(P_i)$ and knowing that $\Phi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$ we obtain

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

□

So, by Claims 4, 5, 9, Φ is additive.

Acknowledgments: The authors would like to thank anonymous referee for a thorough and detailed report with many helpful comments and suggestions.

References

1. R. L. An, J. C. Hou, Characterizations of Jordan †-skew multiplicative maps on operator algebras of indefinite inner product spaces, Chin. Ann. Math. **26** (2005) 569-582.

2. Z. F. Bai, S.P. Du, Multiplicative Lie isomorphism between prime rings, Comm. Algebra **36** (2008) 1626-1633.

3. Z. F. Bai, S. P. Du, Multiplicative $*$-Lie isomorphism between factors, J. Math. Anal. Appl. **346** (2008) 327-335.

4. K. I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale, On Herstein's Lie map conjecture (II), J. Algebra **238** (2001) 239-264.

5. J. Cui, C. K. Li, Maps preserving product $XY - YX^*$ on factor von Neumann algebras, Linear Algebra Appl. **431** (2009) 833-842.

6. L. Dai, F. Lu, Nonlinear maps preserving Jordan $*$-products, J. Math. Anal. Appl. **409** (2014) 180-188.

7. H. Gao, $*$-Jordan-triple multiplicative surjective maps on $B(H)$, J. Math. Anal. Appl. **401** (2013) 397–403.

8. P. Ji, Z. Liu, Additivity of Jordan maps on standard Jordan operator algebras, Linear Algebra Appl. **430** (2009) 335-343.

9. C. Li, F. Lu, X. Fang, Nonlinear mappings preserving product $XY + YX^*$ on factor von Neumann algebras, Linear Algebra Appl. **438** (2013) 2339-2345.

10. C. Li, F. Lu, Nonlinear maps preserving the Jordan triple 1-$*$-product on von Neumann algebras, Complex Analysis and Operator Theory **11** (2017), 109-117.

11. C. Li, F. Lu, T. Wang, Nonlinear maps preserving the Jordan triple $*$-product on von Neumann algebras, Ann. Funct. Anal. **7** (2016), 496-507.

12. F. Lu, Additivity of Jordan maps on standard operator algebras, Linear Algebra Appl. **357** (2002) 123-131.

13. F. Lu, Jordan triple maps, Linear Algebra Appl. **375** (2003) 311-317.

14. W. S. Martindale III, When are multiplicative mappings additive? Proc. Amer. Math. Soc. **21** (1969) 695-698.

15. C. R. Mires, Lie isomorphisms of factors, Trans. Amer. Math. Soc. **147** (1970) 5-63.

16. L. Molnár, On isomorphisms of standard operator algebras, Studia Math. **142** (2000) 295-302.

17. A. Taghavi, Maps preserving Jordan triple product on the self-adjoint elements of C^*-algebras, Asian-European Journal of Mathematics. 10(2) 1750022 (2017) 1-7.

18. A. Taghavi, H. Rohi, V. Darvish, Additivity of maps preserving Jordan η_*-products on C^*-algebras, Bulletin of the Iranian Mathematical Society. **41** (7) (2015) 107–116

19. A. Taghavi, V. Darvish, H. Rohi, Additivity of maps preserving products $AP \pm PA^*$ on C^*-algebras, Mathematica Slovaca. **67** (1) (2017) 213–220.

20. V. Darvish, H. M. Nazari, H. Rohi, A. Taghavi, Maps preserving η-product $A^*B + \eta BA^*$ on C^*-algebras, Journal of Korean Mathematical Society. **54** (3) (2017) 867–876.

A. Taghavi, M. Razeghi,
M. Nouri, V. Darvish,
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-146
Babolsar, Iran.
E-mail address: `taghavi@umz.ac.ir, razeghi.mehran19@yahoo.com`
E-mail address: `mojtaba.nori2010@gmail.com, vahid.darvish@mail.com`

and

C. Li,
School of Mathematical Sciences
Shandong Normal University
Jinan 250014
People's Republic of China.
E-mail address: `lcjbxh@163.com`