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On the Derivative of a Polynomial

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ABSTRACT: Let $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$, be a polynomial of degree at most n having no zeros in $|z| < k$, $k \le 1$, and $Q(z) = z^n \overline{P(1/\overline{z})}$, it is proved by Dewan et al. [5] that if $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \le \frac{n}{1 + k^{n-\mu+1}} \{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\}.$$

In this paper, we generalize the above inequality for the polynomials of type $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$.

Key Words: Polynomial, Inequality, Maximum modulus, Restricted zeros.

Contents

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n, then according to the well known Bernstein's inequality on the derivative of a polynomial, we have

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all its zeros at the origin.
For polynomials having no zeros in $|z| < 1$, Erdös conjectured and later Lax [8] proved that if $P(z) \ne 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

With equality for those polynomials, which have all their zeros on $|z| = 1$.
In the literature, there already exists various refinements and generalizations of

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(1.2), for example (see Aziz [1], Bidkham et.al [2,3,4], Khojastehnezhad and Bidkham [7], Zireh [14] etc).
As an extension of (1.2) Malik [12] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Further Govil [9] proved that for the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ which has no zeros in $|z| < k$, $k \leq 1$, if $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Whereas the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ having all its zeros on $|z| = k$, $k \leq 1$, Govil [10] proved

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Recently Dewan and Hans [5] obtained a generalization of (1.4) and proved for $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ that having no zeros in $|z| < k$, $k \leq 1$, if $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^{n-\mu+1}} \{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\}. \tag{1.6}$$

For $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ that having all its zeros on $|z| = k$, $k \leq 1$, Dewan [5] also proved

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)|. \tag{1.7}$$

In this paper, first we obtain the following result

Theorem 1.1. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^{n+\mu-1}} \max_{|z|=1} |P(z)|. \tag{1.8}$$

Remark 1.2. *If we take $\mu = 1$ in Theorem 1.1, then inequality (1.8) reduces to inequality (1.4) due to Govil.*

Next we prove the following interesting result which is a refinement of inequality(1.8).

Theorem 1.3. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \leq 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ becomes maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^{n+\mu-1}} \{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\}. \tag{1.9}$$

Remark 1.4. *If we take* $\mu = 1$ *in Theorem 1.3, then inequality (1.9) reduces to the following result which proved by Aziz and Ahmad [1].*

Corollary 1.5. *Let* $P(z)$ *is a polynomial of degree* n*, having no zeros in* $|z| < k$*,* $k \leq 1$ *and* $Q(z) = z^n\overline{P(1/\overline{z})}$*. If* $|P'(z)|$ *and* $|Q'(z)|$ *becomes maximum at the same point on* $|z| = 1$*, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n}\{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\}. \tag{1.10}$$

Finally we prove the following result.

Theorem 1.6. *Let* $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$*,* $1 \leq \mu \leq n$ *is a polynomial of degree* n*, having all its zeros on* $|z| = k$*,* $k \leq 1$*, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n+\mu-1} + k^{n+\mu-2}} \max_{|z|=1} |P(z)|. \tag{1.11}$$

Remark 1.7. *If we take* $\mu = 1$ *in Theorem 1.6, then inequality (1.11) reduces to inequality (1.5) due to Govil.*

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. *[13] Let* $P(z)$ *be a polynomial of degree* n*, then for* $R \geq 1$

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{2.1}$$

Lemma 2.2. *Let* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$*,* $1 \leq \mu \leq n$ *be a polynomial of degree* n*, having all its zeros in* $|z| \leq k$*,* $k \geq 1$*, then for* $|z| = 1$

$$k^{n-\mu-1}|Q'(z)| \leq |P'(k^2 z)|, \tag{2.2}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$*.*

Proof: Let $F(z) = P(kz)$, then $F(z)$ has all its zeros in $|z| \leq 1$. If $G(z) = z^n\overline{F(1/\overline{z})} = z^n\overline{P(k/\overline{z})} = k^n Q(z/k)$, then all the zeros of $G(z)$ lie in $|z| \geq 1$. Since $|F(z)| = |G(z)|$ on $|z| = 1$, we can say that an application of maximum modulus principle to the function $\frac{G(z)}{F(z)}$ will yield $|G(z)| \leq |F(z)|$, $|z| \geq 1$. Therefore the polynomial $G(z) - \lambda F(z)$, will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$. Gauss-Lucas theorem will then imply that polynomial $G'(z) - \lambda F'(z)$ will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$ and therefore $|G'(z)| \leq |F'(z)|$, $|z| \geq 1$. Substituting for $F'(z)$ and $G'(z)$, we get

$$k^{n-1}|Q'(z/k)| \leq k|P'(kz)|, \tag{2.3}$$

where $|z| \geq 1$.
Since $Q(z) = \overline{c}_n + \sum_{\nu=\mu}^{n} \overline{c}_{n-\nu} z^{\nu}$, then

$$k^{n-1}|\sum_{\nu=\mu}^{n} \nu \overline{c}_{n-\nu} (\frac{z}{k})^{\nu-1}| \leq k|P'(kz)|.$$

i.e,

$$k^{n-\mu}|\sum_{\nu=\mu}^{n} \nu \overline{c}_{n-\nu} (\frac{z}{k})^{\nu-\mu}| \leq k|P'(kz)|, \tag{2.4}$$

where $|z| \geq 1$.
If we take kz instead of z in inequality (2.4), then we have

$$k^{n-\mu}|\sum_{\nu=\mu}^{n} \nu \overline{c}_{n-\nu} z^{\nu-\mu}| \leq k|P'(k^2 z)|, \tag{2.5}$$

where $|z| \geq 1/k$.
Since $1/k \leq 1$, we have in particular,

$$k^{n-\mu}|\sum_{\nu=\mu}^{n} \nu \overline{c}_{n-\nu} z^{\nu-\mu}| \leq k|P'(k^2 z)|, \tag{2.6}$$

where $|z| \geq 1$.
This implies

$$k^{n-\mu}|\sum_{\nu=\mu}^{n} \nu \overline{c}_{n-\nu} z^{\nu-1}| \leq k|P'(k^2 z)|, \tag{2.7}$$

where $|z| = 1$.
This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ be a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |Q'(z)| \leq k^{n+\mu-1} \max_{|z|=1} |P'(z)|, \tag{2.8}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof: On applying Lemma 2.2 we have

$$k^{n-\mu-1}|Q'(z)| \leq |P'(k^2 z)|. \tag{2.9}$$

Now using Lemma 2.1 for the polynomial $P'(k^2 z)$, of degree $n-1$. We have

$$\max_{|z|=k^2} |P'(z)| \leq k^{2n-2} \max_{|z|=1} |P'(z)|. \tag{2.10}$$

Combining (2.9) and (2.10), we have desired result. □

Lemma 2.4. *Let* $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, \quad 1 \leq \mu \leq n$ *be a polynomial of degree* n*, has no zeros in* $|z| < k$*,* $k \leq 1$*, then*

$$k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|, \tag{2.11}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof: Since $P(z)$ has no zeros in $|z| < k$, then $Q(z) = z^n \overline{P(1/\overline{z})}$ has all its zeros in $|z| \leq 1/k, \ 1/k \geq 1$. On applying Lemma 2.3 to the polynomial $Q(z)$, we have

$$k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|.$$

□

The following lemma is due to Malik [6].

Lemma 2.5. *Let* $P(z)$ *be a polynomial of degree* n*, has no zero in* $|z| < k$*,* $k \geq 1$*, then for* $|z| = 1$

$$k|P'(z)| \leq |Q'(z)| \tag{2.12}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 2.6. *Let* $P(z)$ *be a polynomial of degree* n*, having all its zeros on* $|z| = k$*,* $k \leq 1$*, then for* $|z| = 1$

$$|Q'(z)| \leq k|P'(z)| \tag{2.13}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof: Since $P(z)$ has all its zeros on $|z| = k$, then $Q(z) = z^n \overline{P(1/\overline{z})}$ has all its zeros in $|z| = 1/k, \ 1/k \geq 1$. On applying Lemma 2.5 to the polynomial $Q(z)$, we have

$$1/k|Q'(z)| \leq |P'(z)|.$$

□

The following lemma is a special case of a result due to Govil and Rahman [11].

Lemma 2.7. *Let* $P(z)$ *be a polynomial of degree* n*, then for* $|z| = 1$

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)| \tag{2.14}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

3. Proofs of the theorems

Proof of Theorem 1.1.
Since $|P'(z)|$ and $|Q'(z)|$ attained maximum at the same point on $|z| = 1$. This implies there exist a point z_0 such that $|P'(z_0)| = \max_{|z|=1} |P'(z)| = \max_{|z|=1} |Q'(z)| = |Q'(z_0)|$. On the other hand by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$

On applying Lemma 2.4, we have

$$|P'(z_0)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

This implies

$$\max_{|z|=1} |P'(z)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. Let $m = \min_{|z|=k} |P(z)|$. For α with $|\alpha| < 1$, we have $|\alpha m| < m \leq |P(z)|$, where $|z| = k$.
Therefore by implying Rouche's theorem, the polynomial $G(z) = P(z) - \alpha m$ has no zeros in $|z| < k$. On applying Theorem 1.1 to the polynomial $G(z)$, we have

$$\max_{|z|=1} |G'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |G(z)|,$$

i.e,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |P(z) - \alpha m|.$$

If we choose a point z_0 on $|z| = 1$ such that $\max_{|z|=1} |P(z)| = |P(z_0)|$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} |P(z_0) - \alpha m|.$$

Now by suitable choice of argument of α, we get

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \{|P(z_0)| - |\alpha| m\}.$$

By making $|\alpha| \to 1$, the result follows. □

Proof of Theorem 1.6. If z_0 is a point on $|z| = 1$ such that $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$. Then by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$

On applying Lemma 2.6, we have

$$\frac{1}{k} |Q'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$

i.e,

$$(\frac{1}{k}+1)\max_{|z|=1}|Q'(z)| \leq n\max_{|z|=1}|P(z)|.$$

Now applying Lemma 2.4, we have

$$(\frac{1}{k}+1)k^{n+\mu-1}\max_{|z|=1}|P'(z)| \leq n\max_{|z|=1}|P(z)|.$$

This completes the proof of Theorem 1.6. □

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