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Infinitely Many Solutions for a Nonlocal Elliptic System of $(p_1,\dots,p_n)$-Kirchhoff Type with Critical Exponent *

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ABSTRACT: The existence of infinitely many nontrivial solutions for a nonlocal elliptic system of $(p_1,\dots,p_n)$-Kirchhoff type with critical exponent is investigated. The approach is based on variational methods and critical point theory.

Key Words: Infinitely many solutions, $(p_1,\dots,p_n)$-Kirchhoff problem, Critical exponent, Perturbed differential equation, Critical point theory, Variational methods.

Contents

1. Introduction

The aim of this paper is to ensure the existence of infinitely many weak solutions for the following perturbed system

$$\begin{cases} -\left[M_i\left(\int_\Omega |\nabla u_i|^{p_i}\mathrm{d}x\right)\right]^{p_i-1}\Delta u_i = \nu\left(\int_\Omega |u_i|^{p_i^*}\mathrm{d}x\right)^{p_i/p_i^*-1}|u_i|^{p_i^*-2}u_i \\ \quad +\lambda F_{u_i}(x,u_1,\dots,u_n)+\mu G_{u_i}(x,u_1,\dots,u_n), & \text{in }\Omega, \\ u_i|_{\partial\Omega}=0, \end{cases} \tag{1.1}$$

for $i=1,\dots,n$, where, $1<p_1,\dots,p_n<N$, $p_i^*=\frac{Np_i}{N-p_i}$, $\lambda>0$ and $\mu\geq 0$, and $M_i:\mathbb{R}^+\to\mathbb{R}$ for $i=1,\dots,n$, are continuous functions with the following condition:

(M) There exists $M_0\geq 1$ such that for all $t\geq 0$ one has $M_i(t)\geq M_0$, for $i=1,\dots,n$.

Furthermore, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that $F(x,t_1,\dots,t_n)$ is measurable in x for all $(t_1,\dots,t_n)\in\mathbb{R}^n$ and $F(x,t_1,\dots,t_n)$ is C^1 in $(t_,\dots,t_n)$ for a.e. $x\in\Omega$ and F_{u_i} denote the partial derivatives of F with respect to u_i such that there exist $i\in\{1,\dots,n\}$ such that $F_{u_i}(x,0,\dots,0)\neq 0$ in Ω.

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The system (1.1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

for $0<x<L$, $t\geq 0$ where, $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings (see [26]). Since the equations including the functions M_i depend on integrals over Ω in the system (1.1), they are no longer pointwise identity, and therefore it is often called nonlocal system. Later, the equation (1.2) was extended to

$$\frac{\partial^2 u}{\partial t^2}-K\Big(\int_a^b|\nabla u(x)|^2dx\Big)\Delta u=f(x,u)\quad\text{in }\Omega$$

where, $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ is a nonempty bounded open set with a given $\partial\Omega$ and $K:[0,+\infty[\to\mathbb{R}$ is a continuous function.

The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [2,15]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where, u describes a process which depend on the average of itself, as for example, the population density. It received great attention only after Lions [27] proposed an abstract framework for the problem. The solvability of the Kirchhoff type problems has been paid much attention to by various authors. Some early classical investigations of Kirchhoff equations can be seen in the papers [1,17,19,20,23,25,28,30] and the references therein. For example in [28] the authors obtained nontrivial solutions of a class of nonlocal quasilinear elliptic boundary value problems using the Yang index and critical groups. He and Zou in [19] were concerned with the existence of infinitely many positive solutions to a class of Kirchhoff-type problem $(a+b\int_a^b|\nabla u|^2dx)\Delta u=\lambda f(x,u)$ in Ω and $u=0$ on $\partial\Omega$ where, Ω is a smooth bounded domain of $\mathbb{R}^N$, $a,b>0$, $\lambda>0$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying some further conditions. They obtained a sequence of a.e. positive weak solutions to the above problem tending to zero in $\mathrm{L}^\infty([a,b])$ with f being more general than that of [28].

The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n>1$) were also treated in some recent papers, via variational methods like the symmetric mountain pass theorem in [16], via a three critical point theorem in [5] and in [3,4] some evolutionary higher order Kirchhoff problems were treated, mainly focusing on the qualitative properties of the solutions. Also for example Cheng et al. in [14] were concerned with the nonlocal elliptic system of (p,q)-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega|\nabla u|^p\mathrm{d}x\right)\right]^{p-1}\Delta_p u=\lambda F_u(x,u,v), & \text{in }\Omega,\\ -\left[M_2\left(\int_\Omega|\nabla v|^p\mathrm{d}x\right)\right]^{p-1}\Delta_p v=\lambda F_v(x,u,v), & \text{in }\Omega,\\ u=v=0, & \text{on }\partial\Omega.\end{cases}$$

They under bounded condition on M_i $(i = 1, 2)$ and some novel and periodic condition on F, by using of Bonanno's multiple critical points theorems without the Palais-Smale condition and Ricceri's three critical points theorem, respectively, obtained some new results of the existence of two solutions and three solutions of the above mentioned nonlocal elliptic system. Hssini in [25] studied the existence of nontrivial solution for the nonlocal elliptic system of (p, q)-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\right]^{p-1} \Delta_p u = \mu \left(\int_\Omega |u|^{p^*} \mathrm{d}x\right)^{p/p^*-1} |u|^{p^*-2}u & \\ \quad +F_u(x, u, v), & \text{in } \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q \mathrm{d}x\right)\right]^{q-1} \Delta_q v = \mu \left(\int_\Omega |v|^{q^*} \mathrm{d}x\right)^{q/q^*-1} |v|^{q^*-2}v & \\ \quad +F_v(x, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

with critical exponent. They by using Bonanno and Molica Bisci's general critical points theorem, under some conditions on M_i $(i = 1, 2)$ and F, established the existence of nontrivial solution of the above system.

The novelty of this paper is that we deal with problem (1.1) in which there exists three perturbations for the nonlinear term, which will lead to some difficulties in the proof, and as far as we know, there are very few results even for such type equations in the literature. Motivated by the above works, in the present paper, by employing a smooth version of [9, Theorem 2.1], which is a more precise version of Ricceri's Variational Principle [29, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on F and G we prove the existence of a definite interval about λ and μ in which the problem (1.1) admits a sequence of solutions which is unbounded in the space E which will be introduced later (Theorem 3.1). Furthermore, some consequences of Theorem 3.1 is listed. Replacing the conditions at infinity of the nonlinear terms, by a similar one at zero, the same results hold; see Theorem 3.8. At the end, two examples of applications are pointed out (see Examples 3.3 and 3.10).

2. Preliminaries

Our main tool to ensure the existence of infinitely many solutions for the problem (1.1) is a smooth version of Theorem 2.1 of [8] which is a more precise version of Ricceri's Variational Principle [29] that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(*a*) *for every* $r > \inf_X \Phi$ *and every* $\lambda \in]0, \frac{1}{\varphi(r)}[$, *the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X.

(*b*) *If* $\gamma < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\gamma}[$, *the following alternative holds: either*

(b_1) I_λ *possesses a global minimum,*

or

(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(*c*) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\delta}[$, *the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ,

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ.

We refer to [6,7,9,10,11,12,13,18,21,22,24] in which Theorem 2.1 has been successfully employed to the existence of infinitely many solutions for some boundary value problems.

We let E be the Cartesian product of n Sobolev spaces $W_0^{1,p_1}(\Omega)$, $W_0^{1,p_2}(\Omega), \ldots$ and $W_0^{1,p_n}(\Omega)$, i.e.,

$$E := W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_n}(\Omega),$$

equipped with the norm

$$\|u\| = \sum_{i=1}^{n} \|\nabla u_i\|_{p_i},$$

where, $u = (u_1, \ldots, u_n)$ and

$$\|\nabla u_i\|_{p_i} = \left(\int_\Omega |\nabla u_i|^{p_i} \mathrm{d}x\right)^{1/p_i}, \quad i = 1, \cdots, n.$$

Following we consider the well-know inequalities

$$\left(\int_\Omega |u|^{p_i^*} \mathrm{d}x\right)^{1/p_i^*} \leq \frac{1}{S_{p_i}^{1/p_i}} \left(\int_\Omega |\nabla u|^{p_i} \mathrm{d}x\right)^{1/p_i}, \quad \forall u \in W_0^{1,p_i}(\Omega),\ i = 1, \ldots, n \tag{2.1}$$

where, S_{p_i}, $i = 1, \ldots, n$ is the best constant in the Sobolev inclusion $W_0^{1,p_i}(\Omega) \hookrightarrow L^{p_i^*}(\Omega)$, $i = 1, \ldots, n$. Fixing $\nu \in [0, \underline{S})$ with $\underline{S} =: \min\{S_{p_1}, \ldots, S_{p_n}\}$, and set

$$m(\nu, p_1, \ldots, p_n) := \min\left\{1 - \frac{\nu}{S_{p_1}}, \ldots, 1 - \frac{\nu}{S_{p_n}}\right\}.$$

Theorem 2.2 ([25, Theorem 2.1]). *If $\mu \in [0, S_{p_i}[$, then the functional*

$$S_\mu(u) = \frac{1}{p_i}\int_\Omega |\nabla u|^{p_i}\mathrm{d}x - \frac{\nu}{p_i}\left(\int_\Omega |u|^{p_i^\star}\mathrm{d}x\right)^{p_i/p_i^\star}$$

is sequentially weakly lower semi-continuous in $W_0^{1,p_i}(\Omega)$.

Let us recall that $u = (u_1, \dots, u_n) \in E$ is called a weak solution of system (1.1) if

$$\sum_{i=1}^n \left[M_i\left(\int_\Omega |\nabla u_i(x)|^{p_i}\mathrm{d}x\right)\right]^{p_i-1}\int_\Omega |\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)\mathrm{d}x$$
$$- \sum_{i=1}^n \frac{\nu}{p_i^*}\left(\int_\Omega |u_i(x)|^{p^*}\mathrm{d}x\right)^{p_i/p*-1}\int_\Omega |u_i(x)|^{p_i^*-2}u_i(x)v_i(x)\mathrm{d}x$$
$$- \lambda\sum_{i=1}^n \int_\Omega F_{u_i}(x, u_1, \dots, u_n)v_i(x)\mathrm{d}x - \mu\sum_{i=1}^n \int_\Omega G_{u_i}(x, u_1, \dots, u_n)v_i\mathrm{d}x = 0$$

for every $v = (v_1, \dots, v_n) \in E$.

Here and in the sequel "meas(Ω)" denotes the Lebesgue measure of the set Ω. A special case of our main result is the following theorem.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^2$, $\mathrm{meas}(\Omega) \geq 1$, $F \in \mathrm{C}(\mathbb{R}^2, \mathbb{R})$ and*

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t_1|+|t_2|<\xi}F(t_1,t_2)}{\xi^{\min\{p_1,p_2\}}} = 0 \quad \text{and} \quad \limsup_{(\xi_1,\xi_2)\to(+\infty,+\infty)}\frac{F(\xi_1,\xi_2)}{\xi_1^{p_1}+\xi_2^{p_2}} = +\infty.$$

Then, for every nonnegative arbitrary function $G \in \mathrm{C}(\mathbb{R}^2, \mathbb{R})$ satisfying the condition

$$G_\star := \lim_{\xi\to+\infty}\frac{\sup_{|t_1|+|t_2|<\xi}G(t_1,t_2)}{\xi^{\min\{p_1,p_2\}}} < +\infty,$$

for every $\mu \in [0, \mu_{\star,\lambda}[$ where,

$$\mu_{\star,\lambda} := \frac{1}{G_\star\left(\sum_{i=1}^2\left(p_i\frac{(\mathrm{meas}(\Omega))^{\max\{p_1,p_2\}-1}\mathrm{m}(\nu,p_1,p_2)}{2^{\min\{p_1,p_2\}}}\right)^{\frac{1}{p_i}}\right)^{\min\{p_1,p_2\}}}$$
$$- \frac{\mathrm{meas}(\Omega)}{G_\star}\liminf_{\xi\to+\infty}\frac{\sup_{|t_1|^{p_1}+|t_2|^{p_2}\leq\xi}F(t_1,t_2)}{\xi^{\min\{p_1,p_2\}}}$$

and for every $\nu \in [0, \min(S_{p_1}, S_{p_2})[$, the problem

$$\begin{cases} -\Delta_{p_1}u_1 = \nu\left(\int_\Omega |u_1|^{p_1^*}\mathrm{d}x\right)^{p_1/p_1^*-1}|u_1|^{p_1^*-2}u_1 & \\ \qquad\qquad +F_{u_1}(u_1,u_2) + \mu G_{u_1}(u_1,u_2), & \text{in } \Omega, \\ -\Delta_{p_2}u_2 = \nu\left(\int_\Omega |u_2|^{p_2^*}\mathrm{d}x\right)^{p_2/p_2^*-1}|u_2|^{p_2^*-2}u_2 & \\ \qquad\qquad +F_{u_2}(u_1,u_2) + \mu G_{u_2}(u_1,u_2), & \text{in } \Omega, \\ u_1 = u_2 = 0, & \text{on } \partial\Omega. \end{cases}$$

has an unbounded sequence of pairwise distinct weak solutions.

3. Main results

In this section we formulate our main results. For this let

$$D := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$$

and denote, as usual, with Γ the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}\mathrm{d}z, \text{ for all } t > 0.$$

We present our main result as follows.

For all $\xi > 0$ we denote by $K(\xi)$ the set

$$\left\{(t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |t_i|^{p_i} \leq \xi\right\}.$$

Let $\underline{p} = \min\{p_i\,;\, i = 1\ldots, n\}$ and $\overline{p} = \max\{p_i\,;\, i = 1\ldots, n\}$. Put

$$A := \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F(x, t_1, \ldots, t_n)\,\mathrm{d}x}{\xi^{\underline{p}}}$$

and

$$B := \limsup_{(\xi_1,\ldots,\xi_n)\to(+\infty,\ldots,+\infty)} \frac{\int_{B(x_0,D/2)} F(x, \xi_1, \ldots, \xi_n)\mathrm{d}x}{\sum_{i=1}^{n} H_i(\xi_i)}$$

where $H_i(\xi_i) = \frac{1}{p_i}\widehat{M_i}\left(\frac{2\xi_i}{D}\right)^{p_i}$ with

$$\widehat{M_i}(t) = \int_0^t [M_i(s)]^{p_i-1}\mathrm{d}s \tag{3.1}$$

for $i = 1, \ldots, n$, and

$$S := \left(\sum_{i=1}^{n} \left(\underline{S}\, m(\nu, p_1, \ldots, p_n)\, p_i\right)^{\frac{1}{p_i}}\right)^{\underline{p}}.$$

Our main result is the following theorem.

Theorem 3.1. *Assume that*

(A1) $F(x, t_1, \ldots, t_n) \geq 0$ *for all* $x \in B(x_0, \frac{D}{2})$ *and* $(t_1, \ldots, t_n) \in \mathbb{R}^n$;

(A2)

$$A < \frac{2^N\Gamma(1 + N/2)}{D^N\pi^{N/2}(2^N - 1)} SB.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ where,

$$\lambda_1 := \frac{D^N \pi^{N/2}(2^N - 1)}{2^N \Gamma(1+N/2)} \frac{1}{B} \quad and \quad \lambda_2 := \frac{S}{A},$$

and for every nonnegative arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ which is measurable in Ω and of class $\mathrm{C}^1(\mathbb{R}^n)$ satisfying the condition

$$G_\infty := \lim_{\xi \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} G(x,t_1,\dots,t_n)\mathrm{d}x}{\xi^{\underline{p}}} < +\infty, \tag{3.2}$$

for every $\mu \in [0, \mu_{G,\lambda}[$ where,

$$\mu_{G,\lambda} := \frac{1-\lambda S A}{S G_\infty}$$

and for every $\nu \in [0, \min(S_{p_1}, \dots, S_{p_n})[$, the problem (1.1) has an unbounded sequence of weak solutions in E.

Proof: Our goal is to apply Theorem 2.1. For this, fix λ, μ and G satisfying our assumptions. Now, set $Q(x,t_1,\dots,t_n) = F(x,t_1,\dots,t_n) + \frac{\mu}{\lambda} G(x,t_1,\dots,t_n)$ for all $(x,t_1,\dots,t_n) \in \Omega \times \mathbb{R}^n$. Take $X = E$ and define in X two functionals J and I by setting, for each $u = (u_1,\dots,u_n) \in X$, as follows

$$J(u) = \sum_{i=1}^n \frac{1}{p_i} \widehat{M_i}\left(\int_\Omega |\nabla u_i|^{p_i}\mathrm{d}x\right) - \sum_{i=1}^n \frac{\nu}{p_i}\left(\int_\Omega |u_i|^{p_i^*}\mathrm{d}x\right)^{p_i/p_i^*},$$

and

$$I(u) = \int_\Omega F(x,u_1(x),\dots,u_n(x))\mathrm{d}x + \frac{\mu}{\lambda}\int_\Omega G(x,u_1(x),\dots,u_n(x))\mathrm{d}x.$$

It is well known that I is a Gâteaux differentiable functional and sequentially weakly upper semi-continuous whose Gâteaux derivative at the point $u = (u_1,\dots,u_n) \in X$ is the functional $I'(u) \in X^*$, given by

$$I'(u)v = \sum_{i=1}^n \int_\Omega F_{u_i}(x,u_1,\dots,u_n)v_i\mathrm{d}x + \frac{\mu}{\lambda}\sum_{i=1}^n \int_\Omega G_{u_i}(x,u_1,\dots,u_n)v_i\mathrm{d}x$$

for every $v = (v_1,\dots,v_n) \in X$, and $I' : X \to X^*$ is a compact operator. Moreover, J is a Gâteaux differentiable functional which Gâteaux derivative at the point $u = (u_1,\dots,u_n) \in X$ is the functional $J'(u) \in X^*$, given by

$$J'(u)(v) = \sum_{i=1}^n \left[M_i\left(\int_\Omega |\nabla u_i|^{p_i}\mathrm{d}x\right)\right]^{p_i-1}\int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i \nabla v_i \mathrm{d}x$$
$$- \sum_{i=1}^m \frac{\nu}{p_i^*}\left(\int_\Omega |u_i|^{p^*}\mathrm{d}x\right)^{p_i/p*-1}\int_\Omega |u_i|^{p^*-2}u_i v_i\mathrm{d}x$$

for every $v = (v_1, \dots, v_n) \in X$. Furthermore, the fact that $\widehat{M_i}$ is continuous and monotone for $i = 1, \dots, n$, by (M) and Theorem 2.2 we get that J is sequentially weakly lower semi-continuous for $\nu \in [0, \min(S_{p_1}, \dots, S_{p_n})[$ and it is also a coercive functional. Now from the definition of J and (2.1), for every $u = (u_1, \dots, u_n) \in X$ we have

$$J(u) \geq m(\nu, p_1, \dots, p_n) \sum_{i=1}^{n} \frac{\|\nabla u_i\|_{p_i}^{p_i}}{p_i}. \tag{3.3}$$

Put $\Upsilon_\lambda = J - \lambda I$. By the assumption (M), it is standard to see that $\Upsilon_\lambda \in C^1(X, \mathbb{R})$, and a critical point of Υ_λ corresponds to a weak solution of problem (1.1). So, our end is to apply Theorem 2.1 to J and I. Now, we wish to prove that $\gamma < +\infty$ where, γ is defined in Theorem 2.1. Let $\{\xi_k\}$ be a real sequence such that $\xi_k > 0$ for all $k \in \mathbb{N}$ and $\xi_k \to +\infty$ as $k \to \infty$ and

$$\begin{aligned} &\lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1, \dots, t_n) \in K(\xi_k)} Q(x, t_1, \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} \\ &= \liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{(t_1, \dots, t_n) \in K(\xi)} F(x, t_1, \dots, t_n) \mathrm{d}x}{\xi^{\underline{p}}} \end{aligned}$$

for all $k \in \mathbb{N}$. Put $r_k = \frac{\xi_k^{\underline{p}}}{\underline{S}}$ for all $k \in \mathbb{N}$. By taking (2.1) into account, we have

$$\begin{aligned} \max_{x \in \Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} &\leq \sum_{i=1}^{n} \frac{\left(\int_\Omega |u_i|^{p_i^*} \mathrm{d}x\right)^{p_i/p_i^*}}{p_i} \leq \sum_{i=1}^{n} \frac{\int_\Omega |\nabla u_i|^{p_i} \mathrm{d}x}{p_i S_{p_i}} \\ &\leq \frac{1}{\min\{S_{p_1}, \dots, S_{p_n}\}} \sum_{i=1}^{n} \frac{\|\nabla u_i\|_{p_i}^{p_i}}{p_i} \end{aligned} \tag{3.4}$$

for each $u = (u_1, \dots, u_n) \in X$. Since ξ_k is a positive sequence, $r_k > 0$ for all $k \in \mathbb{N}$. Thus by taking (3.3) and (3.4) in to account, we have

$$\begin{aligned} &J^{-1}(]-\infty, r_k[) \\ &\subseteq \left\{ u \in X;\ \max_{x \in \Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq \underline{S}\, m(\mu, p_1, \dots, p_n)\, r_k, \text{ for each } x \in \Omega \right\}. \end{aligned} \tag{3.5}$$

Hence, taking into account that $I(0, \dots, 0) = J(0, \dots, 0) = 0$, for every k large enough, one has

$$\begin{aligned}\varphi(r_k) &= \inf_{u \in J^{-1}(]-\infty, r_k[)} \frac{(\sup_{v \in J^{-1}(]-\infty, r_k[)} I(v)) - I(u)}{r_k - J(u)} \le \frac{\sup_{v \in J^{-1}(]-\infty, r_k[)} I(v)}{r_k} \\ &\le \frac{S \int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} Q(x, t_1, \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} \\ &= \frac{S \int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} \left(F(x, t_1, \dots, t_n) + \frac{\mu}{\lambda} G(x, t_1, \dots, t_n)\right) \mathrm{d}x}{\xi_k^{\underline{p}}} \\ &\le \frac{S \int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} F(x, t_1 \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} \\ &\quad + \frac{\mu}{\lambda} \frac{S \int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} G(x, t_1, \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}}.\end{aligned}$$

Moreover, it follows from Assumption $(A2)$ that

$$\liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi)} F(x, t_1, \dots, t_n) \mathrm{d}x}{\xi^{\underline{p}}} < +\infty,$$

which concludes

$$\lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} F(x, t_1, \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} < +\infty. \tag{3.6}$$

Then, in view of (3.2) and (3.6), we have

$$\begin{aligned}&\lim_{k \to \infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} F(x, t_1 \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} \\ &+ \frac{\mu}{\lambda} \lim_{k \to \infty} \frac{\int_\Omega \sup_{(t_1 \dots,t_n) \in K(\xi_k)} G(x, t_1, \dots, t_n) \mathrm{d}x}{\xi_k^{\underline{p}}} < +\infty,\end{aligned}$$

which implies

$$\lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} \left(F(x, t_1, \dots, t_n) + \frac{\mu}{\lambda} G(x, t_1, \dots, t_n)\right) \mathrm{d}x}{\xi_k^{\underline{p}}} < +\infty.$$

Therefore,

$$\begin{aligned}\gamma &\le \liminf_{k \to +\infty} \varphi(r_k) \\ &\le S \lim_{k \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n) \in K(\xi_k)} \left(F(x, t_1, \dots, t_n) + \frac{\mu}{\lambda} G(x, t_1, \dots, t_n)\right) \mathrm{d}x}{\xi_k^{\underline{p}}} < +\infty.\end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}&\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} Q(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}}\\ &\le \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} F(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}}\\ &\quad+\frac{\mu}{\lambda}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} G(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}},\end{aligned}$$

taking (3.2) into account, one has

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} Q(x,t_1,\dots,t_n)\mathrm{d}x}{\xi^{\underline{p}}}\le A+\frac{\mu}{\lambda}G_\infty. \tag{3.8}$$

Thus

$$\gamma\le\liminf_{k\to+\infty}\varphi(r_k)\le SA+S\frac{\mu}{\lambda}G_\infty<+\infty.$$

Now, we consider a sequence $\{(\gamma_{k_1},\dots,\gamma_{k_n})\}_{k_i=1}^{\infty}\subseteq\mathbb{R}^n$, $i=1,\dots,n$, such that $\gamma_{k\,i}>0$ for all $k_i\in\mathbb{N}$ and for all $i=1,\dots,n$, and

$$\lim_{(k_1,\dots,k_n)\to(+\infty,\dots,+\infty)}\sum_{i=1}^n(\gamma_{k\,i})^{\frac{1}{2}}=+\infty.$$

Now, we consider a sequence $\{(w_{k_1},\dots,w_{k_n})\}_{k_i=1}^{\infty}\subseteq X$ for all $i=1,\dots,n$ defined by

$$w_{k_i}=\begin{cases}0 & \text{if } x\in\Omega\backslash B(x_0,D),\\ \frac{2\gamma_{k_i}}{D}(D-|x-x_0|) & \text{if } x\in B(x_0,D)\backslash B(x_0,D/2),\\ \gamma_{k_i} & \text{if } x\in B(x_0,\frac{D}{2}).\end{cases} \tag{3.9}$$

One has

$$\begin{aligned}J(w_{k_1},\dots,w_{k_n})&=\sum_{i=1}^n\frac{1}{p_i}\widehat{M}_i\left(\int_\Omega|\nabla w_k|^{p_i}\mathrm{d}x\right)-\sum_{i=1}^n\frac{\mu}{p_i}\left(\int_\Omega|w_k|^{p^*}\mathrm{d}x\right)^{p/p^*}\\ &\le\sum_{i=1}^n\frac{1}{p_i}\widehat{M}_i\left(\int_{B(x_0,D)\backslash B(x_0,D/2)}\frac{(2\gamma_{k\,i})^{p_i}}{D^{p_i}}\mathrm{d}x\right)\\ &=(\mathrm{meas}(B(x_0,D))-\mathrm{meas}(B(x_0,D/2)))\sum_{i=1}^n\frac{1}{p_i}\widehat{M}_i\left(\frac{2\gamma_{k\,i}}{D}\right)^{p_i}\\ &=\frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(D^N-(D/2)^N\right)\sum_{i=1}^n\frac{1}{p_i}\widehat{M}_i\left(\frac{2\gamma_{k\,i}}{D}\right)^{p_i}\\ &=\frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(D^N-(D/2)^N\right)\sum_{i=1}^n H_i(\gamma_{k\,i}).\end{aligned} \tag{3.10}$$

On the other hand, since G is nonnegative, we observe

$$I(w_{k_1},\dots,w_{k_n}) = \int_\Omega F(x, w_{k_1}(x),\dots,w_{k_n}(x))\mathrm{d}x \geq \int_{B(x_0,\frac{D}{2})} F(x,\gamma_{k_1},\dots,\gamma_{k_n})\mathrm{d}x. \tag{3.11}$$

So, from (3.13), (3.10) and (3.11) we conclude

$$\begin{aligned} J(w_{k_1},\dots,w_{k_n}) - \lambda I(w_{k_1},\dots,w_{k_n}) &\leq \frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}\sum_{i=1}^n H_i(\gamma_{ki}) \\ &\quad - \lambda\int_{B(x_0,D/2)} F(x,\gamma_{k_1},\dots,\gamma_{k_n})\mathrm{d}x. \end{aligned}$$

Now, we consider the cases $B<+\infty$ and let

$$\tau \in \left]0, \frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}B - \frac{1}{\lambda}\right[.$$

From this, there exists $(v_{1_\tau},\dots,v_{n_\tau})\subseteq\mathbb{R}^n$ such that

$$\int_{B(x_0,\frac{D}{2})} F(x,\gamma_{k_1},\dots,\gamma_{k_n})\mathrm{d}x > \left(\frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}B-\tau\right)\sum_{i=1}^n H_i(\gamma_{ki})$$

for all $k_i > v_{i_\tau}$, $i=1,\dots,n$, and so

$$\begin{aligned} \Upsilon_\lambda(w_{k_1},\dots,w_{k_n}) &< \sum_{i=1}^n H_i(\gamma_{ki}) - \lambda\left(\frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}B-\tau\right)\sum_{i=1}^n H_i(\gamma_{ki}) \\ &= \left[1-\lambda\left(\frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}B-\tau\right)\right]\sum_{i=1}^n H_i(\gamma_{ki}). \end{aligned}$$

Since

$$1-\lambda\left(\frac{\pi^{N/2}\left(D^N-(D/2)^N\right)}{\Gamma(1+N/2)}B-\tau\right) < 0$$

and taking (3.3) and (3.10) into account one has

$$\lim_{k\to+\infty} I_\lambda(w_{k_1},\dots,w_{k_n}) = -\infty.$$

If $B=+\infty$, fix $M>\frac{1}{\lambda}$, from this there exists v_M such that

$$\int_{B(x_0,\frac{D}{2})} F(x,\gamma_{k_1},\dots,\gamma_{k_n})\mathrm{d}x > M\sum_{i=1}^n H_i(\gamma_{ki})$$

for all $k_i > v_{i_M}$, $i = 1, \dots, n$, and moreover

$$\begin{aligned}\Upsilon_\lambda(w_{k_1}, \dots, w_{k_n}) &< \frac{\pi^{N/2}\left(D^N - (D/2)^N\right)}{\Gamma(1+N/2)} \sum_{i=1}^{n} H_i(\gamma_{ki}) \\ &\quad - \lambda \int_{B(x_0,D/2)} F(x, \gamma_{k_1}, \dots, \gamma_{k_n})\mathrm{d}x \\ &< (1-\lambda M) \sum_{i=1}^{n} H_i(\gamma_{ki}).\end{aligned}$$

Since $1 - \lambda M < 0$, and arguing as before, we have

$$\lim_{(k_1,\dots,k_n)\to(+\infty,\dots,+\infty)} I_\lambda(w_{k_1}, \dots, w_{k_n}) = -\infty.$$

Taking into account that

$$\left]\frac{D^N\pi^{N/2}(2^N-1)}{2^N\Gamma(1+N/2)B}, \frac{S}{A}\right[\subseteq \left]0, \frac{1}{\gamma}\right[$$

and that Υ_λ does not possess a global minimum, from part (b) of Theorem 2.1, there exists an unbounded sequence $\{u_k\} = \{(u_{1_k}, \dots, u_{n_k})\}$ of critical points which are the weak solutions of (1.1). So, our conclusion is achieved. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, Theorem 2.1 assures that there is a sequence $\{u_m\} = \{(u_{1_m}, \dots, u_{n_m})\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{m\to\infty} \Phi(u_m) = +\infty$, which from (3.3) it follows that $\lim_{m\to\infty} \|u_m\| = +\infty$. Hence, we have the conclusion. Moreover, since G is nonnegative, we have

$$\limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,D/2)} Q(x, \xi_1, \dots, \xi_n)\mathrm{d}x}{\sum_{i=1}^{n} H_i(\xi_i)} \geq B. \tag{3.12}$$

Therefore, from (3.8) and (3.12), and from Assumption ($A3$) and (3.7) one has

$$\overline{\lambda} \in]\nu_1, \nu_2[\subseteq \left]\frac{D^N\pi^{N/2}(2^N-1)}{2^N\Gamma(1+N/2)B}, \frac{S}{A}\right[\subseteq \left]0, \frac{1}{\gamma}\right[.$$

For the fixed $\overline{\lambda}$, the inequality (3.7) assures that the condition (b) of Theorem 2.1 can be used and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_m\} = \{(u_{1_m}, \dots, u_{n_m})\}$ of solutions of the problem (1.1) such that $\lim_{m\to\infty} \|u_m\| = +\infty$. The other step is to verify that the functional $J - \overline{\lambda} I$ has no global minimum. Since

$$\begin{aligned}\frac{1}{\overline{\lambda}} &< \frac{2^N\Gamma(1+N/2)}{D^N\pi^{N/2}(2^N-1)} B \\ &= \frac{2^N\Gamma(1+N/2)}{D^N\pi^{N/2}(2^N-1)} \limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,D/2)} F(x, \xi_1, \dots, \xi_n)\mathrm{d}x}{\sum_{i=1}^{n} H_i(\xi_i)},\end{aligned}$$

and

$$\frac{1}{\underline{\lambda}} < \tau < \frac{2^N\Gamma(1+N/2)}{D^N\pi^{N/2}(2^N-1)}\frac{\int_{B(x_0,D/2)}F(x,\gamma_{k_1},\dots,\gamma_{k_1})\mathrm{d}x}{\sum_{i=1}^n H_i(\gamma_{ki})} \tag{3.13}$$

for each $k\in\mathbb{N}$ large enough. □

Remark 3.2. *Under the conditions*

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega \max_{(t_1,\dots,t_n)\in K(\xi)}F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\xi^{\underline{p}}}=0,$$

and

$$\limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)}\frac{\int_{B(x_0,D/2)}F(x,\xi_1,\dots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)}=+\infty.$$

Theorem 3.1 assures that for every $\lambda>0$ and for each $\mu\in[0,\frac{1}{G_\infty}[$ the problem (1.1) admits infinitely many weak solutions. Moreover, if $G_\infty=0$, the result holds for every $\lambda>0$ and $\mu\geq 0$.

Now, we give an application of Theorem 3.1.

Example 3.3. *Let $n=N=2$, $\Omega=\{(x_1,x_2)\in\mathbb{R}^2;\ x_1^2+x_2^2\leq 1\}\subset\mathbb{R}^2$, $1<p_1,p_2<2$ and $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ be the function defined by*

$$F(x_1,x_2,t_1,t_2)$$
$$=\begin{cases} f^*(x_1,x_2)(t_1^2+t_2^2)e^{\sqrt{t_1^2+t_2^2}} \\ \times\left(1-\sin(\ln(\sqrt{t_1^2+t_2^2}))\right) & \text{if } (x_1,x_2,t_1,t_2)\in\Omega\times(\mathbb{R}^2\setminus\{(0,0)\}),\\ 0 & \text{if } (x_1,x_2,t_1,t_2)\in\Omega\times\{(0,0)\}\end{cases}$$

where, $f^*:\Omega\to\mathbb{R}$ is a nonnegative continuous function. Since

$$\liminf_{\xi\to+\infty}\sup_{|t_1|^{p_1}+|t_2|^{p_2}\leq\xi}F(x_1,x_2,t_1,t_2)=0$$

and

$$\liminf_{(\xi_1,\xi_2)\to(+\infty,+\infty)}F(x_1,x_2,\xi_1,\xi_2)=(\xi_1^2+\xi_2^2)e^{\sqrt{\xi_1^2+\xi_2^2}}$$

we have

$$\liminf_{\xi\to+\infty}\frac{\iint_\Omega \sup_{|t_1|^{p_1}+|t_2|^{p_2}\leq\xi}F(x_1,x_2,t_1,t_2)\mathrm{d}x_1\mathrm{d}x_2}{\xi^{\min\{p_1,p_2\}}}=0$$

and

$$\limsup_{(\xi_1,\xi_2)\to(+\infty,+\infty)}\frac{\iint_{x_1^2+x_2^2\leq\frac{1}{2}}F(x_1,x_2,\xi_1,\xi_2)\mathrm{d}x_1\mathrm{d}x_2}{\xi_1^{p_1}+\xi_2^{p_2}}=+\infty.$$

Hence, by using Theorem 3.1, the problem (1.1) with

$$G(x_1,x_2,t_1,t_2)=e^{(x_1^2+x_2^2)-(t_1^2+t_2^2)}$$

for all $(x_1, x_2, t_1, t_2) \in \Omega \times \mathbb{R}^2$, *for every*

$$(\lambda, \mu, \nu) \in]0, +\infty[\times[0, +\infty[\times[0, \min(S_{p_1}, S_{p_2})[$$

has an unbounded sequence of weak solutions.

Remark 3.4. *Assumption* $(A2)$ *in Theorem 3.1 could be replaced by the following more general condition*

$(A'2)$ *there exist two sequence* $\{\theta_k\}_{k=1}^{\infty} = \{(\theta_{k1}, \dots, \theta_{kn})\} \subset \mathbb{R}^n$ *and* $\{\eta_k\} \subset \mathbb{R}$ *with* $\eta_k > 0$ *for every* $k \in \mathbb{N}$ *and* $\sum_{i=1}^n H_i(\theta_{ki}) < \frac{\eta_k^p}{S}$ *for all* $k \in \mathbb{N}$ *and* $\lim_{k\to+\infty} \eta_k = +\infty$ *such that*

$$\lim_{k\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in K(\eta_k)} F(x, t_1, \dots, t_n)\mathrm{d}x - \int_{B(x_0,D/2)} F(x, \theta_{k1}, \dots, \theta_{kn})\mathrm{d}x}{\frac{\eta_k^p}{S} - \sum_{i=1}^n H_i(\theta_{ki})}$$
$$< \limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,D/2)} F(x, \xi_1, \dots, \xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)}.$$

Indeed, clearly, by choosing $\theta_k = 0$ *for all* $k \in \mathbb{N}$ *from* $(A'2)$ *we obtain* $(A2)$. *Moreover, if we assume* $(A'2)$ *instead of* $(A2)$ *and choose* $r_k = \frac{\eta_k^p}{S}$ *for all* $n \in \mathbb{N}$, *by the same arguing as inside in Theorem 3.1, we obtain*

$$\varphi(r_k) \le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])} \Psi(v) - \int_\Omega F(x, w_{k_1}(x), \dots, w_{k_n}(x))\mathrm{d}x}{r_k - \sum_{i=1}^n \frac{1}{p_i}\widehat{M}_i\left(\int_\Omega |\nabla w_{ki}|^{p_i}\mathrm{d}x\right) + \sum_{i=1}^n \frac{\mu}{p_i}\left(\int_\Omega |w_{ki}|^{p_i^*}\mathrm{d}x\right)^{p_i/p_i^*}}$$
$$\le \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\eta_k)} F(x, t_1, \dots, t_n)\,\mathrm{d}x - \int_{B(x_0,D/2)} F(x, \theta_{k1}, \dots, \theta_{kn})\mathrm{d}x}{\frac{\eta_k^p}{S} - \sum_{i=1}^n H_i(\theta_{ki})}$$

where, $w_k(t)$ *is the same as* (3.9) *but* γ_k *replaced by* θ_k. *We have the same conclusion as in Theorem 3.1 with the interval* $]\lambda_1, \lambda_2[$ *replaced by the interval*

$$\Lambda' = \left] \frac{1}{\lim_{k\to+\infty} \frac{\int_{B(x_0,D/2)} F(x,\theta_{k1},\dots,\theta_{kn})\mathrm{d}x}{\sum_{i=1}^n H_i(\theta_{ki})}}, \frac{1}{S \lim_{k\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\eta_k)} F(x,t_1,\dots,t_n)\,\mathrm{d}x - \int_{B(x_0,D/2)} F(x,\theta_{k1},\dots,\theta_{kn})\mathrm{d}x}{\eta_k^p - S\sum_{i=1}^n H_i(\theta_{ki})}} \right[.$$

Here, we point out a simple consequence of Theorem 3.1.

Corollary 3.5. *Assume that Assumption* $(A1)$ *holds. Furthermore, suppose that*

$(B1)$ $\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\xi^p} < \frac{1}{S}$

$(B2)$ $\limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,D/2)} F(x,\xi_1,\dots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)} > 1.$

Then, for every nonnegative arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *which is measurable in* Ω *and of class* $\mathrm{C}^1(\mathbb{R}^n)$ *satisfying the condition* (3.2) *and for every* $\mu \in [0, \mu_{G,1}[$ *where*

$$\mu_{G,1} := \frac{1 - SA}{SG_\infty},$$

and for every $\nu \in [0, \min(S_{p_1}, \ldots, S_{p_n})[$, *the system*

$$\begin{cases} -\left[M_i\left(\int_\Omega |\nabla u_i|^{p_i} \mathrm{d}x\right)\right]^{p_i-1} \Delta_{p_i} u_i = \nu \left(\int_\Omega |u_i|^{p_i^*} \mathrm{d}x\right)^{p_i/p_i^*-1} |u_i|^{p_i^*-2} u_i & \\ +F_{u_i}(x, u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n), & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega \end{cases}$$

for $i = 1, \ldots, n$ *has an unbounded sequence of weak solutions in* E.

Remark 3.6. *Theorem 2.3 is an immediately consequence of Corollary 3.5 when* $\mu = 0$.

We here give the following consequence of the main result.

Corollary 3.7. *Let* $F_1 : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *be a function such that* $F_1(x, t_1, \ldots, t_n)$ *is measurable in* x *for all* $(t_1, \ldots, t_n) \in \mathbb{R}^N$ *and* $F_1(x, t_1, \ldots, t_n)$ *is* C^1 *in* $(t_, \ldots, t_n)$ *for a.e.* $x \in \Omega$. *Assume that*

$(D1)$ $\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F_1(x,t_1,\ldots,t_n)\, \mathrm{d}x}{\xi^{\underline{p}}} < +\infty$;

$(D2)$ $\limsup_{(\xi_1,\ldots,\xi_n)\to(+\infty,\ldots,+\infty)} \frac{\int_{B(x_0,D/2)} F_1(x,\xi_1,\ldots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)} = +\infty$.

Then, for every function $F_j : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *such that* $F_j(x, t_1, \ldots, t_n)$ *is measurable in* x *for all* $(t_1, \ldots, t_n) \in \mathbb{R}^N$ *and* $F_j(x, t_1, \ldots, t_n)$ *is* C^1 *in* $(t_, \ldots, t_n)$ *for a.e.* $x \in \Omega$, *satisfying*

$$\min\left\{ \inf_{(x,\xi)\in(\Omega\setminus B(x_0,D/2))\times\mathbb{R}^N} F_j(x, \xi);\ 2 \le j \le k \right\} \ge 0$$

and

$$\min\left\{ \liminf_{\xi\to+\infty} \frac{F_j(x, t_1, \ldots, t_n)}{\xi^{\underline{p}}};\ (t_1, \ldots, t_n) \in K(\xi),\ 2 \le j \le k \right\} < +\infty,$$

for each

$$\lambda \in \left]0, \frac{1}{S \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F_1(x,t_1,\ldots,t_n)\, \mathrm{d}x}{\xi^{\underline{p}}}}\right[,$$

for every nonnegative arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *which is measurable in* Ω *and of class* $\mathrm{C}^1(\mathbb{R}^n)$ *satisfying the condition* (3.2), *for every* $\mu \in [0, \mu_{G,\lambda}[$ *where,*

$$\mu_{G,\lambda} := \frac{1 - \lambda S A_1}{SG_\infty}$$

with,

$$A_1 = \liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F_1(x,t_1,\dots,t_n)\,dx}{\xi^{\underline{p}}},$$

and for every $\nu \in [0, \min(S_{p_1},\dots,S_{p_n})[$, *the system*

$$\begin{cases} -\left[M_i\left(\int_\Omega |\nabla u_i|^{p_i} dx\right)\right]^{p_i-1} \Delta_{p_i} u_i = \nu \left(\int_\Omega |u_i|^{p_i^*} dx\right)^{p_i/p_i^*-1} |u_i|^{p_i^*-2} u_i & \\ +\lambda \sum_{j=1}^k (F_j)_{u_i}(x,u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n), & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega \end{cases}$$

for $i = 1,\dots,n$ *has an unbounded sequence of weak solutions in* E.

Proof: Set $F(t,\xi_1,\dots,\xi_n) = \sum_{j=1}^k F_j(t,\xi_1,\dots,\xi_n)$ for all $(\xi_1,\dots,\xi_n) \in \mathbb{R}^n$. From the assumption $(D2)$ and the condition

$$\min\left\{ \inf_{(x,\xi_1,\dots,\xi_n)\in(\Omega\setminus B(x_0,D/2))\times\mathbb{R}^n} F_j(x,\xi_1,\dots,\xi_n);\ 2 \le j \le n \right\} \ge 0$$

we conclude

$$\begin{aligned} &\limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,D/2)} F(x,\xi_1,\dots,\xi_n)dx}{\sum_{i=1}^n H_i(\xi_i)} \\ &= \limsup_{(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)} \frac{\sum_{j=1}^k \int_{B(x_0,D/2)} F_j(x,\xi_1,\dots,\xi_n)dx}{\sum_{i=1}^n H_i(\xi_i)} = +\infty. \end{aligned}$$

Moreover, from the assumption $(D1)$ and the condition

$$\min\left\{ \liminf_{\xi\to+\infty} \frac{F_j(x,t_1,\dots,t_n)}{\xi^{\underline{p}}};\ (t_1,\dots,t_n) \in K(\xi),\ 2 \le j \le n \right\} < +\infty,$$

we obtain

$$\begin{aligned} &\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F(x,t_1,\dots,t_n)\,dx}{\xi^{\underline{p}}} \\ &\le \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F_1(x,t_1,\dots,t_n)\,dx}{\xi^{\underline{p}}} < +\infty. \end{aligned}$$

Hence, the conclusion follows from Theorem 3.1. □

Arguing as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.1 instead of (b), one establishes the following result. For this we set

$$A_0 := \liminf_{\xi\to0^+} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F(x,t_1,\dots,t_n)\,dx}{\xi^{\underline{p}}},$$

$$B_0 := \limsup_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{\int_{B(x_0,D/2)} F(x,\xi_1,\dots,\xi_n)dx}{\sum_{i=1}^n H_i(\xi_i)}$$

Theorem 3.8. *Assume that Assumption (A1) holds. Furthermore, suppose that*

(*E*1)

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F_1(x,t_1,\dots,t_n)\,\mathrm{d}x}{\xi^{\underline{p}}} < \frac{2^N\Gamma(1+N/2)}{D^N\pi^{N/2}(2^N-1)} S \limsup_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{\int_{B(x_0,D/2)} F_1(x,\xi_1,\dots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)}.$$

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ *where,*

$$\lambda_3 := \frac{D^N\pi^{N/2}(2^N-1)}{2^N\Gamma(1+N/2)}\frac{1}{B_0} \quad \text{and} \quad \lambda_4 := \frac{S}{A_0},$$

and for every nonnegative arbitrary function $G : \Omega\times\mathbb{R}^n \to \mathbb{R}$ *which is measurable in* Ω *and of class* $C^1(\mathbb{R}^n)$ *satisfying the condition*

$$G_0 := \lim_{\xi\to 0} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} G(x,t_1,\dots,t_n)\mathrm{d}x}{\xi^{\underline{p}}} < \infty, \tag{3.14}$$

for every $\mu \in [0, \overline{\mu}_{G,\lambda}[$ *where,*

$$\overline{\mu}_{G,\lambda} := \frac{1-\lambda S A_0}{S G_0}$$

and for every $\nu \in [0, \min(S_{p_1},\dots,S_{p_n})[$, *the problem* (1.1) *has an unbounded sequence of weak solutions in* E.

Proof: Fix $\overline{\lambda} \in]\lambda_3, \lambda_4[$ and let G is the function satisfying the condition (3.14). Since, $\overline{\lambda} < \lambda_4$, one has $\mu_{g,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in]0, \mu_{G,\overline{\lambda}}[$ and set $\nu_3 := \lambda_3$ and $\nu_4 := \frac{\lambda_2}{1+\frac{\mu}{\lambda}\lambda_4 S G_0}$. If $G_0 = 0$, clearly, $\nu_3 = \lambda_3$, $\nu_4 = \lambda_4$ and $\lambda \in]\nu_3, \nu_4[$. If $G_0 \neq 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, one has

$$\frac{\overline{\lambda}}{\lambda_4} + \overline{\mu} G_0 < 1,$$

and so

$$\frac{\lambda_2}{1+\frac{\mu}{\lambda}\lambda_4 S G_0} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_4$. Hence, recalling that $\overline{\lambda} > \lambda_3 = \nu_3$, one has $\overline{\lambda} \in]\nu_3, \nu_4[$. Put $Q(x,t_1,\dots,t_n) = F(x,t_1,\dots,t_n) + \frac{\overline{\mu}}{\lambda} G(x,t_1,\dots,t_n)$ for all $(x,t_1,\dots,t_n) \in \Omega\times\mathbb{R}^n$. Since

$$\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} Q(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}} \le \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} F(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}} + \frac{\mu}{\lambda}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} G(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}},$$

taking (3.14) into account, one has

$$\liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} Q(x,t_1,\dots,t_n)\mathrm{d}x}{\xi^{\underline{p}}} \le A_0+\frac{\mu}{\lambda}G_0. \tag{3.15}$$

Moreover, since G is nonnegative, from Assumption $(E1)$ we have

$$\limsup_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)}\frac{\int_{B(x_0,D/2)} Q(x,\xi_1,\dots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)} \ge B_0. \tag{3.16}$$

Therefore, from (3.15) and (3.16), we obtain

$$\overline{\lambda}\in]\nu_3,\nu_4[\subseteq\left]\frac{1}{\limsup_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)}\frac{\int_{B(x_0,D/2)} Q(x,\xi_1,\dots,\xi_n)\mathrm{d}x}{\sum_{i=1}^n H_i(\xi_i)}}\right.$$
$$\left.,\ \frac{1}{S\liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} Q(x,t_1,\dots,t_n)\mathrm{d}x}{\xi^{\underline{p}}}}\right[\subseteq\,]\lambda_3,\lambda_4[\,.$$

We take X, J, I and $\Upsilon_{\overline{\lambda}}$ as in the proof of Theorem 3.1. We prove that $\delta<+\infty$. For this, let $\{\xi_k\}$ be a real sequence such that $\xi_k>0$ for all $k\in\mathbb{N}$ and $\xi_k\to 0^+$ as $k\to\infty$ and

$$\liminf_{k\to+\infty}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_k)} F(x,t_1,\dots,t_n)\mathrm{d}x}{\xi_k^{\underline{p}}}<+\infty.$$

Put $r_k=\frac{\xi_k^{\underline{p}}}{S}$ for all $n\in\mathbb{N}$. Let us show that the functional $\Upsilon_{\overline{\lambda}}$ has not a local minimum at zero. For this, let $\{\gamma_{k1},\dots,\gamma_{kn}\}_{k_i=1}^{\infty}$ for $i=1\dots,n$, such that $\gamma_{ki}>0$ for all $k_i\in\mathbb{N}$ and $i=1,\dots,n$, and a constant τ such that

$$\lim_{(k_1,\dots,k_n)\to(+\infty,\dots,+\infty)}\sum_{i=1}^n(\gamma_{ki})^{\frac{1}{2}}=0^+$$

and

$$\frac{1}{\overline{\lambda}}<\tau<\frac{2^N\Gamma(1+N/2)}{D^N\pi^{N/2}(2^N-1)}\frac{\int_{B(x_0,D/2)} F(x,\gamma_k,\dots,\gamma_k)\mathrm{d}x}{\sum_{i=1}^n H_i(\gamma_{ki})} \tag{3.17}$$

for each $n\in\mathbb{N}$ large enough. Let $\{(w_{k_1},\dots,w_{k_n})\}$ be a sequence in X defined by (3.9). So, owing to (3.10), (3.11) and (3.17) we obtain

$$\Upsilon_{\overline{\lambda}}(w_k)\le\sum_{i=1}^n H_i(\gamma_{ki})-\overline{\lambda}\int_{B(x_0,D/2)} F(x,\gamma_k,\dots,\gamma_k)dt<(1-\overline{\lambda}\tau)\sum_{i=1}^n H_i(\gamma_{ki})<0$$

for every $k\in\mathbb{N}$ large enough. Since $\Upsilon_{\overline{\lambda}}(0)=0$, that means that 0 is not a local minimum of the functional $\Upsilon_{\overline{\lambda}}$. Hence, the part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_k\}$ in X of critical points of $\Upsilon_{\overline{\lambda}}$ such that $\|u_k\|\to 0$ as $k\to\infty$, and the proof is complete. □

Remark 3.9. *Applying Theorem 3.8, results similar to Remark 3.4, Corollaries 3.5 and 3.7 can be obtained.*

In the sequel we give the following example as an application of Theorem 3.8.

Example 3.10. *Let* $n = N = 2$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\ |x_1| + |x_2| \le 1\} \subset \mathbb{R}^2$, $p_1 = \frac{3}{2}$, $p_2 = \frac{7}{4}$ *and* $F : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be the function defined by*

$$F(x_1, x_2, t_1, t_2) = \begin{cases} e^{x_1+x_2}\Big[(t_1^2 + t_2^2)\ln(\ln(\frac{1}{(t_1^2+t_2^2)})) \\ \times \sin^2(\ln(\ln(\ln(\frac{1}{(t_1^2+t_2^2)})))) \\ +4(t_1^2 + t_2^2)\ln^{-1}(\frac{1}{(t_1^2+t_2^2)})\Big], & \text{if } (x_1, x_2, t_1, t_2) \in \Omega \times (\mathbb{R}^2 \setminus \{(0,0)\}), \\ 0 & \text{if } (x_1, x_2, t_1, t_2) \in \Omega \times \{(0,0)\}. \end{cases}$$

Thus

$$\liminf_{\xi \to 0^+} \frac{\iint_\Omega \sup_{|t_1|^{\frac{3}{2}} + |t_2|^{\frac{7}{4}} \le \xi} F(x_1, x_2, t_1, t_2) dx_1 dx_2}{\xi^{\frac{3}{2}}} = 0$$

and

$$\limsup_{(\xi_1,\xi_2)\to(0^+,0^+)} \frac{\iint_{x_1^2 + x_2^2 \le \frac{1}{2}} F(x_1, x_2, \xi_1, \xi_2) dx_1 dx_2}{\xi_1^{\frac{3}{2}} + \xi_2^{\frac{7}{4}}} = +\infty.$$

Hence, by using Theorem 3.8 the problem (1.1) in this case, with

$$G(x_1, x_2, t_1, t_2) = x_1^2 + x_2^2(|t_1| + |t_2|)^3$$

for all $(x_1, x_2, t_1, t_2) \in \Omega \times \mathbb{R}^2$*, for every* $(\lambda, \mu, \nu) \in]0, +\infty[\times [0, \frac{12}{\pi}[\times [0, \min(S_{\frac{3}{2}}, S_{\frac{7}{4}})[$ *has an unbounded sequence of weak solutions.*

Let $n = 1$. As an application of the results, we consider the problem

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^{p_1} dx\right)\right]^{p_1-1} \Delta_{p_1} u = \nu\left(\int_\Omega |u|^{p_1^*} dx\right)^{p_1/p_1^*-1} |u|^{p_1^*-2}u \\ +\lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

where, $M_1 : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfy in condition (M) and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

Now we let $\widehat{M_1} : W_0^{1,p_1} \to \mathbb{R}$ be as in (3.1). Put

$$F(x,t) = \int_0^t f(x,\xi) d\xi, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}$$

and

$$G(x,t) = \int_0^t g(x,\xi) d\xi, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

The following two corollaries are consequences of Theorems 3.1 and 3.8.

Corollary 3.11. *Assume that*

$(A1')$ $F(x,t) \geq 0$ *for all* $x \in B(x_0, \frac{D}{2})$ *and* $t \in \mathbb{R}$*;*

$(A2')$

$$\limsup_{\xi \longrightarrow +\infty} \frac{\int_{B(x_0,D/2)} F(x,\xi)\mathrm{d}x}{\widehat{M_1}\left(\frac{2\xi}{D}\right)^p} < \frac{2^N\Gamma(1+N/2)(\mathrm{meas}(\Omega))^{P_1-1}m(\nu,p_1)}{2^{p_1}D^N\pi^{N/2}(2^N-1)} \times \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\,\mathrm{d}x}{\xi^{\underline{p}}}.$$

Then, for each

$$\lambda \in \left] \frac{D^N\pi^{N/2}(2^N-1)}{2^N\Gamma(1+N/2)} \frac{1}{p_1 \limsup_{\xi\to+\infty} \frac{\int_{B(x_0,D/2)} F(x,\xi_1,\ldots,\xi_n)\mathrm{d}x}{\widehat{M_1}(\frac{2\xi}{D})^p}}, \frac{p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)}{2^{p_1}\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\,\mathrm{d}x}{\xi^{p_1}}} \right[$$

for every L^1*-Carathéodory function* $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *whose* $G(t,x) = \int_0^\xi g(t,\xi)\mathrm{d}\xi$ *for every* $(t,x) \in \overline{\Omega} \times \mathbb{R}$*, is a nonnegative function satisfying the condition*

$$g_\infty := \lim_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} G(x,t)\mathrm{d}x}{\xi^{p_1}} < +\infty,$$

for every $\mu \in [0, \mu_{g,\lambda}[$ *where*

$$\mu_{g,\lambda} := \frac{2^{p_1} - \lambda p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t\,\mathrm{d}x}{\xi^{p_1}}}{p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)g_\infty}$$

and for every $\nu \in [0, S_{p_1}[$*, the problem* (3.18) *has an unbounded sequence of weak solutions in* W_0^{1,p_1}*.*

Corollary 3.12. *Assume that the assumption* $(A1')$ *holds. Furthermore, suppose that*

$$\liminf_{\xi\longrightarrow 0^+} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x}{\xi^{p_1}} < \frac{2^{N-p_1}\Gamma(1+N/2)p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)}{D^N\pi^{N/2}(2^N-1)} \times \limsup_{\xi\to0^+} \frac{\int_{B(x_0,D/2)} F(x,\xi)\mathrm{d}x}{\widehat{M_1}\left(\left(\frac{2\xi}{D}\right)^{p_1}\right)}.$$

Then, for each

$$\lambda \in \left[\frac{D^N\pi^{N/2}(2^N-1)}{2^N\Gamma(1+N/2)}\frac{1}{\limsup_{\xi\to 0^+}\frac{\int_{B(x_0,D/2)}F(x,\xi)\mathrm{d}x}{\widehat{M_1}\left(\left(\frac{2\xi}{D}\right)^{p_1}\right)}},\right.$$
$$\left.\frac{p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)}{2^{p_1}\liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{|t|\le\xi}F(x,t)\mathrm{d}x}{\xi^{p_1}}}\right[,$$

for every L^1-Carathéodory function $g:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ whose $G(t,x)=\int_0^\xi g(t,\xi)\mathrm{d}\xi$ for every $(t,x)\in\overline{\Omega}\times\mathbb{R}$, is a nonnegative function satisfying the condition

$$g_0 := \lim_{\xi\to 0}\frac{\int_\Omega \sup_{|t|\le\xi}G(x,t)\mathrm{d}x}{\xi^{p_1}} < +\infty,$$

for every $\mu\in[0,\overline{\mu}_{g,\lambda}[$ where

$$\overline{\mu}_{g,\lambda} := \frac{2^{p_1}-\lambda p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)\liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{|t|\le\xi}F(x,t)\,\mathrm{d}x}{\xi^{p_1}}}{p_1(\mathrm{meas}(\Omega))^{p_1-1}m(\nu,p_1)g_0}$$

and for every $\nu\in[0,S_{p_1}[$, the problem (3.18) has an unbounded sequence of weak solutions.

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