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On Nonspreading-Type Mappings in Hadamard Spaces

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abstract: In this paper, we study a new class of nonspreading-type mappings more general than the class of strictly pseudononspreading and the class of generalized nonspreading mappings. We state and prove some strong convergence theorems of the Mann-type and Ishikawa-type algorithms for approximating fixed points of our class of mappings in Hadamard spaces.

Key Words: CAT(0) spaces, Generalized k-strictly pseudononspreading mappings, k-strictly pseudononspreading mappings, Generalized asymptotically nonspreading mappings, Nonspreading mappings, Fixed points.

Contents

1. Introduction

In 2008, Kohsaka and Takahashi [11] introduced the class of *nonspreading mappings* which they defined as follows: Let C be a nonempty closed and convex subset of a real smooth, strictly convex and reflexive Banach space E. A mapping $T: C \to C$ is called *nonspreading* if

$$\phi(Tx,Ty)+\phi(Ty,Tx)\leq \phi(Tx,y)+\phi(Ty,x) \tag{1.1}$$

for all $x,y\in C$, where $\phi(x,y)=||x||^2-2\langle x,Jy\rangle+||y||^2$ for all $x,y\in E$, and J is the duality mapping on C. It is obvious that, if E is a real Hilbert space, then J is the identity mapping on C and $\phi(x,y)=||x-y||^2$ for all $x,y\in E$. Thus, for a nonempty, closed and convex subset C of a real Hilbert space, $T: C\to C$ is called *nonspreading* if

$$2||Tx-Ty||^2\leq ||Tx-y||^2+||Ty-x||^2,\ \forall x,y\in C. \tag{1.2}$$

Using this class of mappings, Kohsaka and Takahashi [11] studied the resolvents of maximal monotone operators in Banach spaces.

Later in 2011, Osilike and Isiogugu [18] introduced a new class of nonlinear mappings more general than the class of nonspreading mappings in Hilbert spaces,

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namely the class of *k-strictly pseudononspreading mappings*. Let H be a real Hilbert space, a mapping $T : D(T) \subseteq H \to H$ is called *k-strictly pseudononspreading* if there exists $k \in [0,1)$ such that

$$\begin{aligned} ||Tx - Ty||^2 &\leq ||x-y||^2 + k||x - Tx - (y - Ty)||^2 \\ &\quad +2\langle x - Tx, y - Ty\rangle \ \forall x, y \in D(T). \end{aligned} \tag{1.3}$$

It is easy to show that (1.3) is equivalent to

$$\begin{aligned} (2-k)||Tx - Ty||^2 &\leq k||x-y||^2 + (1-k)||y - Tx||^2 + (1-k)||x - Ty||^2 \\ &\quad +k||x - Tx||^2 + k||y - Ty||^2, \end{aligned}$$

for all $x, y \in D(T)$. Clearly, every nonspreading mapping is 0-strictly pseudononspreading. However, an example given in [18] shows that the converse of this statement is not always true. Furthermore, Osilike and Isiogugu [18] obtained some weak and strong convergence results for this class of mappings in Hilbert spaces.

Recently, Naraghirad [15] introduced the class of *asymptotically nonspreading* mappings in real Banach spaces, which he defined as follows: Let C be a nonempty closed and convex subset of a real Banach space E. A mapping $T : C \to C$ is called *asymptotically nonspreading* if

$$||T^n x - T^n y||^2 \leq ||x-y||^2 + 2\langle x - T^n x, J(y - T^n y)\rangle \ \forall x, y \in C \text{ and } n \in \mathbb{N}, \tag{1.4}$$

where J is the duality mapping on C. If E is a real Hilbert space, then (1.4) is equivalent to

$$2||T^n x - T^n y||^2 \leq ||T^n x - y||^2 + ||T^n y - x||^2 \ \forall x, y \in C \text{ and } n \in \mathbb{N}. \tag{1.5}$$

Clearly, if $n = 1$, then T is nonspreading. Naraghirad [15] proved some weak and strong convergence theorems for approximating fixed points of asymptotically nonspreading mappings in real Banach spaces.

Motivated by the works of Naraghirad [15], Kohsaka and Takahashi [11], Phuengrattana [19] introduced a new class of nonlinear mappings in convex metric spaces as follows: Let C be a nonempty subset of a convex metric space X. A mapping $T : C \to C$ is called *generalized asymptotically nonspreading* if there exist two functions $f, g : C \to [0, \gamma],\ \gamma < 1$ such that

$$d^2(T^n x, T^n y) \leq f(x) d^2(T^n x, y) + g(x) d^2(T^n y, x) \ \forall x, y \in C,\ n \in \mathbb{N},$$

and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in C.$$

If $n = 1$, then T is simply called *generalized nonspreading.* Observe that if $f(x) = g(x) = \frac{1}{2}$ for all $x \in C$, then T is an asymptotically nonspreading mapping. However, Phuengrattana [19] gave an example of a generalized asymptotically nonspreading mapping which is not an asymptotically nonspreading mapping. Furthermore, Phuengrattana [19] proved some existence theorems, demiclosed principle and a Δ-convergence theorem of the Mann iteration for the class of generalized asymptotically nonspreading mappings in CAT(0) spaces.

Motivated by the works of Phuengrattana [19], Osilike and Isiogugu [18], we introduce a new class of nonspreading-type mappings, called the class of (f,g)-*generalized k-strictly pseudononspreading mappings* which contains the class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings in metric spaces. Examples of our new class of mappings are given and some fixed points properties for these mappings are studied in Hadamard spaces. Furthermore, we prove some strong convergence theorems of the Mann-type and Ishikawa-type algorithms for our class of mappings in Hadamard spaces.

2. Preliminaries

In this section, we recall some definitions and some useful results that will be needed in proving our main results. We begin with the following definitions.

Let (X,d) be a metric space and $x,y\in X$. A geodesic path joining x to y is a mapping $c:[0,t]\subset\mathbb{R}\to X$ such that $c(0)=x$, $c(t)=y$ and $d(c(k),c(k'))=|k-k'|$ for all $k,k'\in[0,t]$. In this case, c is called an isometry and $d(x,y)=t$. The image of c is called a geodesic segment joining x to y. When this image is unique, it is denoted by $[x,y]$.

The metric space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic and it is said to be a uniquely geodesic space if every two points of X are joined by exactly one geodesic segment. A subset C of a geodesic space X is said to be convex, if for all $x,y\in C$, the segment $[x,y]$ is in C. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consist of three points x_1,x_2,x_3 in X (known as the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i,x_j)=d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j)$ for all $i,j\in\{1,2,3\}$. A metric space (X,d) is called a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane $\mathbb{R}^2$. Furthermore, a geodesic space is a CAT(0) space if and only if it satisfies the following inequality, called the (CN) inequality of Bruhat and Titis [3] (see [2]): If x,y,z are points in X and y_0 is the midpoint of the segment $[y,z]$, then

$$d^2(x,y_0)\le\frac{1}{2}d^2(x,y)+\frac{1}{2}d^2(x,z)-\frac{1}{4}d^2(y,z). \tag{2.1}$$

Let X be a CAT(0) space. Denote the pair $(a,b)\in X\times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle .,.\rangle:(X\times X)\times(X\times X)\to\mathbb{R}$ defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}\left(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)\right)\ \forall a,b,c,d\in X$$

is called a quasilinearization mapping (see [1]). We can easily verify that $\langle\overrightarrow{ab},\overrightarrow{ab}\rangle = d^2(a,b)$, $\langle\overrightarrow{ba},\overrightarrow{cd}\rangle = -\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{ae},\overrightarrow{cd}\rangle + \langle\overrightarrow{eb},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{cd},\overrightarrow{ab}\rangle$ for all $a,b,c,d,e \in X$. A geodesic space X is said to satisfy the Cauchy-Swartz inequality if $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle \le d(a,b)d(c,d)\ \forall a,b,c,d \in X$. It has been established in [1] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality. It is generally known that CAT(0) spaces are uniquely geodesic spaces (see for example [7]) and complete CAT(0) spaces are called Hadamard spaces. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbb{R}$-trees, Hilbert ball [8], Hyperbolic spaces [20]. For more equivalent definitions and properties of CAT(0) spaces, see [2] and the references therein.

Let $\{x_n\}$ be a bounded sequence in X and $r(.,\{x_n\}) : X \to [0,\infty)$ be a continuous functional defined by $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x,\{x_n\}) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$. It is generally known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [5]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [12]. Kirk and Panyanak [10] later introduced and studied this concept in CAT(0) spaces, and proved that it is very similar to the weak convergence in Banach space setting (for more information on weak convergence in Banach space setting, see [16,17] and the references therein).

Definition 2.1. *Let C be a nonempty subset of a metric space X. A mapping $T : C \to C$ is said to be L-Lipschitzian if there exists $L > 0$ such that*

$$d(Tx,Ty) \le Ld(x,y)\ \forall x,y \in C.$$

If $L = 1$, then T is called nonexpansive.

A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if $Tx = x$. We denote the set of fixed points of T by $F(T)$.

Lemma 2.2. *[7] Let X be a CAT(0) space. Then, for all $x,y,z \in X$ and $t,s \in [0,1]$, the following hold:*

(i) $d(tx \oplus (1-t)y, z) \le td(x,z) + (1-t)d(y,z)$,

(ii) $d^2(tx \oplus (1-t)y, z) \le td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

Lemma 2.3. *[4] Let X be a CAT(0) space and $a,b,c \in X$. Then for each $t \in [0,1]$,*

$$d^2(ta \oplus (1-t)b, c) \le t^2d^2(a,c) + (1-t)^2d^2(b,c) + 2t(1-t)\langle\overrightarrow{ac},\overrightarrow{bc}\rangle.$$

Lemma 2.4. *[7] Let X be a CAT(0) space, then for each $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(z,x) = (1-t)d(x,y) \text{ and } d(z,y) = td(x,y). \tag{2.2}$$

In this case, we write $z = tx \oplus (1-t)y$.

Lemma 2.5. *[7] Every bounded sequence in a Hadamard space has a* Δ*-convergence subsequence.*

Lemma 2.6. *[6] If* $\{x_n\}$ *is a bounded sequence in a closed and convex subset* C *of a Hadamard space, then the asymptotic center of* $\{x_n\}$ *is in* C.

Let $\{x_n\}$ be a bounded sequence in a closed and convex subset C of a Hadamard space. We use the notation

$$x_n \rightharpoonup w \Longleftrightarrow \Phi(w) = \inf_{x \in C} \Phi(x),$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$. We note that $x_n \rightharpoonup w$ if and only if $A(\{x_n\}) = \{w\}$ (see [14]).

Lemma 2.7. *[14] If* $\{x_n\}$ *is a bounded sequence in a closed and convex subset* C *of a Hadamard space, then* $\Delta\text{-}\lim_{n\to\infty} x_n = w$ *implies that* $x_n \rightharpoonup w$.

Lemma 2.8. *[9] Let* X *be a Hadamard space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *Then* $\{x_n\}$ Δ*-converges to* x *if and only if* $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0$ $\forall y \in X$.

Lemma 2.9. *[21]. Let* $\{a_n\}$ *be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n + \gamma_n, \ \ n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*
(i) $\{\alpha_n\} \subset [0,1]$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 0)$, $\Sigma_{n=0}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *[13]. Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_j\}$ *of* $\{n\}$ *with* $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3. Main Results

We first introduce a new class of nonspreading-type mappings, called the class of *(f, g)-generalized k-strictly pseudononspreading mappings.*

Definition 3.1. *Let X be a metric space. We say that a mapping $T : D(T) \subseteq X \to X$ is (f,g)-generalized (or simply generalized) k-strictly pseudononspreading if there exist two functions $f, g : D(T) \subseteq X \to [0, \gamma]$, $\gamma < 1$ and $k \in [0,1)$ such that*

$$\begin{aligned}(1-k)d^2(Tx,Ty) \quad &\leq \quad kd^2(x,y) + [f(x)-k]\,d^2(Tx,y) + [g(x)-k]\,d^2(x,Ty) \\ &\quad +kd^2(x,Tx) + kd^2(y,Ty)\ \forall x,y \in D(T),\end{aligned}$$

and

$$0 < f(x) + g(x) \leq 1\ \forall x \in D(T).$$

Remark 3.2.

(i) Clearly, every generalized nonspreading mapping is a generalized 0-strictly pseudononspreading mapping.

(ii) Every k-strictly pseudononspreading mapping is a generalized k-strictly pseudononspreading mapping. Indeed, if T is a k-strictly pseudononspreading mapping, then for all $x, y \in D(T)$, there exists $k \in [0,1)$ such that

$$\begin{aligned}(2-k)d^2(Tx,Ty) \quad &\leq \quad kd^2(x,y) + (1-k)d^2(Tx,y) + (1-k)d^2(x,Ty) \\ &\quad +kd^2(x,Tx) + kd^2(y,Ty),\end{aligned}$$

which implies

$$\begin{aligned}\left(1-\frac{k}{2}\right)d^2(Tx,Ty) \quad &\leq \quad \frac{k}{2}d^2(x,y) + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(y,Tx) + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(x,Ty) \\ &\quad +\frac{k}{2}d^2(x,Tx) + \frac{k}{2}d^2(y,Ty).\end{aligned}$$

That is

$$\begin{aligned}(1-k')\,d^2(Tx,Ty) \quad &\leq \quad k'd^2(x,y) + (f(x)-k')\,d^2(Tx,y) + (g(x)-k')\,d^2(x,Ty) \\ &\quad +k'd^2(x,Tx) + k'd(y,Ty),\end{aligned}$$

where $f(x) = g(x) = \frac{1}{2}$, $\forall x \in D(T)$ and $k' = \frac{k}{2} \in [0,1)$. Hence, T is a generalized k-strictly pseudononspreading mapping.

The following examples show that the class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings are properly contained in the class of generalized k-strictly pseudononspreading mappings.

First, we give an example of a generalized k-strictly pseudononspreading mapping which is not k-strictly pseudononspreading.

Example 3.3. *Let $T : [0,\infty) \to [0,\infty)$ be defined by*

$$Tx = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Then, T is not k-strictly pseudononspreading. In fact, if we take $x = 1$ and $y = 0.5$, then
$|Tx - Ty|^2 = 0.82644$, $|x - y|^2 = 0.25$, $k|x - Tx - (y - Ty)|^2 = 0.16736k$, $2\langle x - Tx, y - Ty\rangle = 0.09091$.
Hence,
$|Tx - Ty|^2 = 0.82644 > 0.34091 + 0.16736k = |x - y|^2 + k|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle$, *for all* $k \in [0, 1)$.

However, T is a generalized k-strictly pseudononspreading mapping with $k = 0$. To see this, let $f, g : [0,\infty) \to [0, 0.9]$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ 0.9, & \text{if } x \in [0,1) \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{(x+\frac{1}{10})^2}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Case 1: *If $x \geq 1$ and $y \in [0, 1)$, then $Tx = \frac{1}{x+\frac{1}{10}}$, $Ty = 0$, $f(x) = 0$ and $g(x) = \frac{1}{(x+\frac{1}{10})^2}$. Thus, we obtain*

$$|Tx - Ty|^2 = \frac{1}{(x + \frac{1}{10})^2} \leq 0 + g(x)x^2 = f(x)|y - Tx|^2 + g(x)|x - Ty|^2.$$

Case 2: *If $x \in [0, 1)$ and $y \geq 1$, then $Tx = 0$, $Ty = \frac{1}{y+\frac{1}{10}}$, $f(x) = 0.9$ and $g(x) = 0$. Thus, we obtain*

$$|Tx - Ty|^2 = \frac{1}{(y + \frac{1}{10})^2} < f(x)y^2 + 0 = f(x)|y - Tx|^2 + g(x)|x - Ty|^2.$$

Case 3: *If $x \geq 1$ and $y \geq 1$, then $Tx = \frac{1}{x+\frac{1}{10}}$, $Ty = \frac{1}{y+\frac{1}{10}}$, $f(x) = 0$ and $g(x) = \frac{1}{(x+\frac{1}{10})^2}$. Thus, we obtain*

$$|Tx - Ty|^2 = \Big|\frac{1}{x + \frac{1}{10}} - \frac{1}{y + \frac{1}{10}}\Big|^2 = \frac{(x - y)^2}{(x + \frac{1}{10})^2(y + \frac{1}{10})^2}$$

and

$$f(x)|y - Tx|^2 + g(x)|x - Ty|^2 = \frac{1}{(x + \frac{1}{10})^2}\Big|x - \frac{1}{y + \frac{1}{10}}\Big|^2 = \frac{(1 - xy - \frac{x}{10})^2}{(x + \frac{1}{10})^2(y + \frac{1}{10})^2}.$$

Since $(x - y)^2 - (1 - xy - \frac{x}{10})^2 < 0$, we conclude that $|Tx - Ty|^2 < f(x)|y - Tx|^2 + g(x)|x - Ty|^2$, for all $x \geq 1$ and $y \geq 1$.

For the case where $x, y \in [0, 1)$, we have that $|Tx - Ty|^2 = 0 < f(x)|y - Tx|^2 + g(x)|x - Ty|^2$. Thus,

$$|Tx - Ty|^2 \leq f(x)|y - Tx|^2 + g(x)|x - Ty|^2 \ \forall x, y \in [0,\infty).$$

Hence, T is a generalized nonspreading mapping. It then follows that T is a generalized k-strictly pseudononspreading mapping with $k = 0$.

We now give an example of a generalized k-strictly pseudononspreading mapping which is neither k-strictly pseudononspreading nor generalized nonspreading.

Example 3.4. *Let* $T : [0, \infty) \to \mathbb{R}$ *defined by*

$$Tx = \begin{cases} -3x, & \text{if } x \in [0, 1], \\ \frac{1}{x}, & \text{if } x \in (1, \infty). \end{cases}$$

We first show that T is not k-strictly pseudononspreading. Indeed, if $x = \frac{11}{10}$ and $y = \frac{1}{3}$, then
$|Tx - Ty|^2 = 3.64463,\ |x - y|^2 = 0.58778,\ k|x - Tx - (y - Ty)|^2 = 1.30513k,\ 2\langle x - Tx, y - Ty\rangle = 0.50909$.
Hence,
$|Tx - Ty|^2 = 3.64463 > 1.09687 + 1.30513k = |x - y|^2 + k|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle\ \forall k \in [0, 1)$.
Therefore, T is not k-strictly pseudononspreading.

Next, we show that T is not generalized nonspreading. Suppose for contradiction that T is a generalized nonspreading mapping, then we can always find two functions $f, g : [0, \infty) \to [0, \gamma],\ \gamma < 1$ such that

$$|Tx - Ty|^2 \le f(x)|Tx - y|^2 + g(x)|Ty - x|^2\ \forall x, y \in [0, \infty)$$

and

$$0 < f(x) + g(x) \le 1\ \forall x \in [0, \infty).$$

In particular, for $x = 0$ and $y = 1$, we have that

$$9 = |Tx - Ty|^2 \le f(x)|Tx - y|^2 + g(x)|Ty - x|^2 = f(x) + 9g(x).$$

That is,

$$9 \le f(x) + 9g(x). \tag{3.1}$$

If $f(x) = 0$, then we have that $9 \le 9g(x) < 9$ and this is a contradiction. Now, suppose $f(x) \neq 0$, then we obtain from (3.1) that $f(x) \ge 9(1 - g(x)) \ge 9f(x)$ (since $f(x) + g(x) \le 1$). This implies that $1 \ge 9$ and this is a contraction. Therefore, T is not generalized nonspreading.

Finally, we show that T is a generalized k-strictly pseudononspreading mapping with $k = \frac{9}{10}$. To see this, let $f, g : [0, \infty) \to [0, 0.9]$ be defined by

$$f(x) = \begin{cases} \frac{9}{10}, & \text{if } x \in [0, 1], \\ \frac{1}{10}, & \text{if } x \in (1, \infty) \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{10}, & \text{if } x \in [0, 1], \\ \frac{9}{10}, & \text{if } x \in (1, \infty). \end{cases}$$

Case 1: If $x, y \in [0,1]$, then $Tx = -3x,\ Ty = -3y,\ f(x) = \frac{9}{10}$ and $g(x) = \frac{1}{10}$. So that,
$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{9}{10}|x-y|^2,\ [f(x)-\frac{9}{10}]|Tx-y|^2 = 0,\ [g(x)-\frac{9}{10}]|x-Ty|^2 = \frac{-8x^2-48xy-72y^2}{10},\ \frac{9}{10}|x-Tx|^2 = \frac{144x^2}{10},\ \frac{9}{10}|y-Ty|^2 = \frac{144y^2}{10}$.

Hence,

$$\begin{aligned}
& \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
& + \frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\
= \ & \frac{9}{10}|x-y|^2 + \frac{136x^2+72y^2-48xy}{10} \\
\geq \ & \frac{9}{10}|x-y|^2 = \left(1-\frac{9}{10}\right)|Tx-Ty|^2.
\end{aligned}$$

Case 2: If $x, y \in (1,\infty)$, then $Tx = \frac{1}{x},\ Ty = \frac{1}{y},\ f(x) = \frac{1}{10}$ and $g(x) = \frac{9}{10}$. So that,
$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{x^2+y^2-2xy}{10x^2y^2},\ \frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10},\ [f(x)-\frac{9}{10}]|Tx-y|^2 = \frac{-8x^2y^2-8+16xy}{10x^2},\ [g(x)-\frac{9}{10}]|x-Ty|^2 = 0,\ \frac{9}{10}|x-Tx|^2 = \frac{9x^4-18x^2+9}{10x^2},\ \frac{9}{10}|y-Ty|^2 = \frac{9y^4-18y^2+9}{10y^2}$.

Hence,

$$\begin{aligned}
& \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
& + \frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\
& = \frac{18x^4y^2+10x^2y^4+16xy^3+y^2+9x^2-18x^3y^3-36x^2y^2}{10x^2y^2}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \frac{18x^4y^2+10x^2y^4+16xy^3+y^2+9x^2-18x^3y^3-36x^2y^2}{10x^2y^2} - \frac{x^2+y^2-2xy}{10x^2y^2} \\
& = \frac{18x^4y^2+10x^2y^4+16xy^3+8x^2+2xy-18x^3y^3-36x^2y^2}{10x^2y^2} \geq 0,
\end{aligned}$$

for all $x, y \in (1,\infty)$.
Hence, we conclude that

$$\begin{aligned}
(1-\frac{9}{10})|Tx-Ty|^2 \ \leq \ & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
& + \frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2,\ \forall x, y \in (1,\infty)
\end{aligned}$$

Case 3: If $x \in (1, \infty)$ and $y \in [0, 1]$, then $Tx = \frac{1}{x}$, $Ty = -3y$, $f(x) = \frac{1}{10}$ and $g(x) = \frac{9}{10}$. So that,
$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{1+6xy+9x^2y^2}{10x^2}$, $\frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10}$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = \frac{-8x^2y^2-8+16xy}{10x^2}$, $[g(x)-\frac{9}{10}]|x-Ty|^2 = 0$, $\frac{9}{10}|x-Tx|^2 = \frac{9x^4-18x^2+9}{10x^2}$, $\frac{9}{10}|y-Ty|^2 = \frac{144y^2}{10}$. Hence,

$$\begin{aligned}
&\frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
&+\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\
&= \frac{18x^4+145x^2y^2+16xy+1-18x^3y-18x^2}{10x^2}.
\end{aligned}$$

Observe that
$\frac{18x^4+145x^2y^2+16xy+1-18x^3y-18x^2}{10x^2} - \frac{1+6xy+9x^2y^2}{10x^2} = \frac{18x^4+136x^2y^2+10xy-18x^3y-18x^2}{10x^2} \geq 0$
for all $x \in (1, \infty)$ and $y \in [0, 1]$. Hence, we conclude that

$$\begin{aligned}
(1-\frac{9}{10})|Tx-Ty|^2 \;\leq\; & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
&+\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \;\forall x \in (1, \infty),\; y \in [0, 1].
\end{aligned}$$

Case 4: If $x \in [0, 1]$ and $y \in (1, \infty)$, then $Tx = -3x$, $Ty = \frac{1}{y}$, $f(x) = \frac{9}{10}$ and $g(x) = \frac{1}{10}$. So that,
$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{1+6xy+9x^2y^2}{10y^2}$, $\frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10}$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = 0$, $[g(x)-\frac{9}{10}]|x-Ty|^2 = \frac{-8x^2y^2-8+16xy}{10y^2}$, $\frac{9}{10}|x-Tx|^2 = \frac{144x^2}{10}$, $\frac{9}{10}|y-Ty|^2 = \frac{9y^4-18y^2+9}{10y^2}$. Hence,

$$\begin{aligned}
&\frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|y-Tx|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
&+\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\
&= \frac{18y^4+145x^2y^2+16xy+1-18xy^3-18y^2}{10y^2}.
\end{aligned}$$

By similar argument as in **Case 3**, we obtain that

$$\begin{aligned}
(1-\frac{9}{10})|Tx-Ty|^2 \;\leq\; & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\
&+\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \forall x \in [0, 1],\; y \in (1, \infty).
\end{aligned}$$

Therefore, T is a generalized k-strictly pseudononspreading mapping with $k = \frac{9}{10}$.

Proposition 3.5. *The class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings are independent. That is, the class of generalized nonspreading mappings is not a subclass of the class of k-strictly pseudononspreading mappings, and the class of k-strictly pseudononspreading mappings is not a subclass of the class of generalized nonspreading mappings.*

Proof:
First, we recall that the mapping defined in Example 3.3 is a generalized nonspreading mapping but not a k-strictly pseudononspreading mapping. However, if we consider the mapping $T : [0, 1] \to \mathbb{R}$ defined by $Tx = -3x$. Then, T is k-strictly pseudononspreading but not generalized nonspreading. To see that T is k-strictly pseudononspreading, observe that
$|Tx-Ty|^2 = 9|x-y|^2$, $|x-Tx-(y-Ty)|^2 = 16|x-y|^2$ and $2\langle x-Tx, y-Ty\rangle = 32xy$.
Thus,

$$\begin{aligned} |Tx - Ty|^2 &= |x-y|^2 + 8|x-y|^2 \\ &= |x-y|^2 + \frac{8}{16}|x - Tx - (y - Ty)|^2 \\ &\leq |x-y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle, \end{aligned}$$

since $32xy \geq 0$ $\forall x, y \in [0, 1]$.

But, if we take $x = 0$ and $y = 1$, by the same argument as in Example 3.4, we obtain that T is not a generalized nonspreading mapping. Hence, our proof is complete.

Remark 3.6. *Observe that if T is (f, g)-generalized k-strictly pseudononspreading with $F(T) \neq \emptyset$ and $f(p) \neq 0$ $\forall p \in F(T)$, then for each $p \in F(T)$ and $y \in D(T)$, we have*

$$d^2(p, Ty) \leq f(p)d^2(p, y) + g(p)d^2(p, Ty) + kd^2(y, Ty),$$

which implies

$$(1 - g(p))d^2(p, Ty) \leq f(p)d^2(p, y) + kd^2(y, Ty).$$

Since $f(p) + g(p) \leq 1$, we obtain

$$d^2(p, Ty) \leq d^2(p, y) + \frac{k}{f(p)}d^2(y, Ty). \tag{3.2}$$

Proposition 3.7. *Let C be a nonempty closed and convex subset of a Hadamard space X and $T : C \to C$ be (f, g)-generalized k-strictly pseudononspreading mapping with $k \in [0, 1)$, where $f, g : C \to [0, \gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in C$. Suppose that $F(T) \neq \emptyset$ and $f(p) \neq 0$, with $\frac{k}{f(p)} \leq \beta < 1$ for each $p \in F(T)$, then $F(T)$ is closed and convex.*

Proof:
We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$

converges to $x^* \in C$. Since T is (f,g)-generalized k-strictly pseudononspreading mapping, then from (3.2), we obtain

$$\begin{aligned} d^2(x_n, Tx^*) &\leq d^2(x_n, x^*) + \frac{k}{f(x_n)} d^2(x^*, Tx^*) \\ &\leq \left[d(x_n, x^*) + \sqrt{\frac{k}{f(x_n)}} d(x^*, Tx^*) \right]^2 . \end{aligned}$$

Thus,

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_n) + d(x_n, Tx^*) \\ &\leq 2d(x^*, x_n) + \sqrt{\frac{k}{f(x_n)}} d(x^*, Tx^*) \\ &\leq 2d(x^*, x_n) + \sqrt{\beta} d(x^*, Tx^*), \end{aligned} \tag{3.3}$$

which implies

$$1 - \sqrt{\beta} d(x^*, Tx^*) \leq 2d(x^*, x_n) \to 0, \text{ as } n \to \infty.$$

Since $\sqrt{\beta} < 1$, it follows that $d(x^*, Tx^*) = 0$. Therefore, $x^* \in F(T)$.
Next, we show that $F(T)$ is convex. Let $z = tx \oplus (1-t)y$ for each $x, y \in F(T)$ and $t \in [0,1]$, then from
Lemma 2.2, Lemma 2.4 and (3.2), we obtain

$$\begin{aligned} d^2(z, Tz) &= d^2(tx \oplus (1-t)y, Tz) \\ &\leq td^2(x, Tz) + (1-t)d^2(y, Tz) - t(1-t)d^2(x,y) \\ &\leq t\left[d^2(x,z) + \frac{k}{f(x)} d^2(z, Tz) \right] \\ &\quad +(1-t)\left[d^2(y,z) + \frac{k}{f(y)} d^2(z,Tz) \right] - t(1-t)d^2(x,y) \\ &= t\left[(1-t)^2 d^2(x,y) + \frac{k}{f(x)} d^2(z,Tz) \right] \\ &\quad +(1-t)\left[t^2 d^2(x,y) + \frac{k}{f(y)} d^2(z,Tz) \right] - t(1-t)d^2(x,y) \\ &= \left(t\frac{k}{f(x)} + (1-t)\frac{k}{f(y)} \right) d^2(z,Tz) \\ &\leq Md^2(z,Tz), \end{aligned}$$

where $M := \max\left\{ \frac{k}{f(x)}, \frac{k}{f(y)} \right\}$. Thus,

$$(1-M)d^2(z,Tz) \leq 0.$$

Since $M < 1$, it follows that $z \in F(T)$, which completes our proof.

Lemma 3.8. *(Demiclosed Principle). Let C be a nonempty closed and convex subset of a Hadamard space X and $T : C \to C$ be (f,g)-generalized k-strictly pseudononspreading mapping with $k \in [0,1)$, where $f,g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in C$. Suppose $k < \frac{1-f(x)}{2}$ for all $x \in C$, and $\{x_n\}$ is a bounded sequence in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in F(T)$.*

Proof:
Since $\Delta\text{-}\lim_{n\to\infty} x_n = z$, we have from Lemma 2.7 that $x_n \rightharpoonup z$. Thus, by Lemma 2.6, we obtain that $A(\{x_n\}) = \{z\}$. Hence, since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have that $\Phi(z) := \limsup_{n\to\infty} d^2(x_n, z) = \limsup_{n\to\infty} d^2(Tx_n, z)$, which implies that $\Phi(Tz) = \limsup_{n\to\infty} d^2(x_n, Tz) = \limsup_{n\to\infty} d^2(Tx_n, Tz)$. Now, since T is (f,g)-generalized k-strictly pseudononspreading, we obtain

$$\begin{aligned}
(1-k)d^2(x_n, Tz) &\leq (1-k)\left(d(x_n, Tx_n) + d(Tz, Tx_n)\right)^2 \\
&= (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) \\
&\quad +(1-k)d^2(Tz, Tx_n) \\
&\leq (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) \\
&\quad +kd^2(z, x_n) + [f(z)-k]d^2(Tz, x_n) + [g(z)-k]d^2(z, Tx_n) \\
&\quad +kd^2(z, Tz) + kd^2(x_n, Tx_n) \\
&\leq (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) \\
&\quad +kd^2(z, x_n) + [f(z)-k]d^2(Tz, x_n) + kd^2(z, Tz) \\
&\quad +kd^2(x_n, Tx_n) + [g(z)-k]\left(d(z, x_n) + d(x_n, Tx_n)\right)^2 \\
&= (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) \\
&\quad +kd^2(z, x_n) + [f(z)-k]d^2(Tz, x_n) \\
&\quad +[g(z)-k]d^2(z, x_n) + 2[g(z)-k]d(z, x_n)d(x_n, Tx_n) \\
&\quad +[g(z)-k]d^2(x_n, Tx_n) + kd^2(z, Tz) + kd^2(x_n, Tx_n),
\end{aligned}$$

which implies

$$\begin{aligned}
(1-f(z))d^2(x_n, Tz) &\leq d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) \\
&\quad +g(z)d^2(z, x_n) + 2[g(z)-k]d(z, x_n)d(x_n, Tx_n) \\
&\quad +[g(z)-k]d^2(x_n, Tx_n) + kd^2(z, Tz) \\
&\leq g(z)d^2(z, x_n) + kd^2(z, Tz) + d^2(x_n, Tx_n) \\
&\quad +2M(1-k)d(x_n, Tx_n) + 2M[g(z)-k]d(x_n, Tx_n) \\
&\quad +[g(z)-k]d^2(x_n, Tx_n),
\end{aligned}$$

where $M := \sup_{n\geq 1}\{d(x_n, z), d(Tx_n, Tz)\}$. Taking $\limsup$ on both sides of the inequality above, we obtain

$$\limsup_{n\to\infty}(1-f(z))d^2(x_n, Tz) \leq \limsup_{n\to\infty}\left[g(z)d^2(x_n, z) + kd^2(z, Tz)\right]. \quad (3.4)$$

That is,

$$(1-f(z))\Phi(Tz) \leq g(z)\Phi(z) + kd^2(z,Tz). \tag{3.5}$$

Now, by letting $t = \frac{1}{2}$ in Lemma 2.2 (ii), we obtain

$$d^2\left(x_n, \frac{z \oplus Tz}{2}\right) \leq \frac{1}{2}d^2(x_n,z) + \frac{1}{2}d^2(x_n,Tz) - \frac{1}{4}d^2(z,Tz).$$

Taking lim sup on both sides of the inequality above and noting that $A(\{x_n\}) = \{z\}$, we obtain

$$\Phi(z) \leq \Phi\left(\frac{z \oplus Tz}{2}\right) \leq \frac{1}{2}\Phi(z) + \frac{1}{2}\Phi(Tz) - \frac{1}{4}d^2(z,Tz).$$

That is,

$$d^2(z,Tz) \leq 2\Phi(Tz) - 2\Phi(z). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$d^2(z,Tz) \leq \frac{2g(z)}{1-f(z)}\Phi(z) + \frac{2k}{1-f(z)}d^2(z,Tz) - 2\Phi(z),$$

which implies

$$\frac{1-f(z)-2k}{1-f(z)}d^2(z,Tz) \leq \frac{2(g(z)+f(z)-1)}{1-f(z)}\Phi(z). \tag{3.7}$$

Since $g(z)+f(z) \leq 1$, we obtain from (3.7) that

$$(1-f(z)-2k)\ d^2(z,Tz) \leq 0.$$

Since $k < \frac{1-f(z)}{2}$, it follows that $z \in F(T)$. Hence, our proof is complete.

Theorem 3.9. *Let C be a nonempty closed and convex subset of a Hadamard space X and T be (f,g)-generalized k-strictly pseudononspreading mapping on C with constant $k \in [0,1)$, where $f,g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x)+g(x) \leq 1$ for all $x \in C$. Suppose $F(T) \neq \emptyset$ and $k < \min\{f(x), \frac{1-f(x)}{2}\}$ for each $x \in C$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1-t_n)x_n \oplus t_n u, \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n Ty_n, \ n \geq 1, \end{cases} \tag{3.8}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$, *C3:* $0 < a \leq \alpha_n \leq 1 - \frac{k}{f(p)}$ *for each* $p \in F(T)$*.*

Then $\{x_n\}$ *converges strongly to an element of* $F(T)$*.*

Proof: Let $p \in F(T)$, then from (3.2), (3.8) and Lemma 2.2, we obtain

$$\begin{aligned} d^2(p, x_{n+1}) &\leq (1-\alpha_n)d^2(p, y_n) + \alpha_n d^2(p, Ty_n) - \alpha_n(1-\alpha_n)d^2(y_n, Ty_n) \\ &\leq (1-\alpha_n)d^2(p, y_n) + \alpha_n \left[d^2(p, y_n) + \frac{k}{f(p)} d^2(y_n, Ty_n)\right] \\ &\quad -\alpha_n(1-\alpha_n)d^2(y_n, Ty_n) \\ &= d^2(p, y_n) - \alpha_n \left[\left(1 - \frac{k}{f(p)}\right) - \alpha_n\right] d^2(y_n, Ty_n) \\ &\leq d^2(p, (1-t_n)x_n \oplus t_n u) \qquad (3.9) \\ &\leq (1-t_n)d^2(p, x_n) + t_n d^2(p, u) \\ &\leq \max\{d^2(p, x_n), d^2(p, u)\} \\ &\vdots \\ &\leq \max\{d^2(p, x_1), d^2(p, u)\}. \end{aligned}$$

Therefore, $\{d^2(p, x_n)\}$ is bounded. Consequently, $\{x_n\}$ and $\{y_n\}$ are bounded. Again, from (3.8), we obtain

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} t_n d(x_n, u) = 0. \qquad (3.10)$$

We divide our proof into two cases.
Case 1: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-increasing, then

$$\lim_{n\to\infty} \{d^2(p, x_n)\}$$

exists. Consequently,

$$\lim_{n\to\infty} \left[d^2(p, x_n) - d^2(p, x_{n+1})\right] = 0. \qquad (3.11)$$

Thus, from (3.9), we have

$$\begin{aligned} \alpha_n \left[\left(1 - \frac{k}{f(p)}\right) - \alpha_n\right] d^2(y_n, Ty_n) &\leq d^2(p, y_n) - d^2(p, x_{n+1}) \\ &\leq (1-t_n)d^2(p, x_n) + t_n d^2(p, u) \\ &\quad -d^2(p, x_{n+1}) \\ &= d^2(p, x_n) - d^2(p, x_{n+1}) \\ &\quad +t_n \left[d^2(p, u) - d^2(p, x_n)\right] \to 0, \end{aligned}$$

as $n \to \infty$. By condition C3, we obtain that

$$\lim_{n\to\infty} d^2(y_n, Ty_n) = 0. \qquad (3.12)$$

Since $\{x_n\}$ is bounded and X is a Hadamard space, then from Lemma 2.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. It follows from (3.10) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} y_{n_k} = z$. Thus, from (3.12) and Lemma 3.8, we obtain that $z \in F(T)$.
Furthermore, for arbitrary $u \in X$, we have from Lemma 2.8 that

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \leq 0, \tag{3.13}$$

which implies from condition C1 that

$$\limsup_{n\to\infty} \left(t_n d^2(z,u) + 2(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \right) \leq 0. \tag{3.14}$$

We now show that $\{x_n\}$ converges strongly to z. From (3.8) and Lemma 2.3, we obtain

$$\begin{aligned} d^2(z, x_{n+1}) &\leq d^2(z, (1-t_n)x_n \oplus t_n u) \\ &\leq (1-t_n)^2 d^2(z,x_n) + t_n^2 d^2(z,u) + 2t_n(1-t_n)\langle \overrightarrow{zx_n}, \overrightarrow{zu} \rangle \\ &\leq (1-t_n) d^2(z,x_n) + t_n \left(t_n d^2(z,u) + 2(1-t_n)\langle \overrightarrow{zx_n}, \overrightarrow{zu} \rangle \right) \end{aligned} \tag{3.15}$$

Hence, from (3.14) and Lemma 2.9, we conclude that $\{x_n\}$ converges strongly to z.
Case 2: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{p, d^2(x_{n_i})\}$ of $\{p, d^2(x_n)\}$ such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \leq d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \leq d^2(p, x_{m_k+1}) \ \forall k \in \mathbb{N}. \tag{3.16}$$

Thus, from (3.8) and (3.16), we obtain

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k}) \right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, x_{n+1}) - d^2(p, x_n) \right) \\ &\leq \limsup_{n\to\infty} \left((1-t_n) d^2(p, x_n) + t_n d^2(p,u) - d^2(p, x_n) \right) \\ &\leq \limsup_{n\to\infty} \left[t_n \left(d^2(p,u) - d^2(p,x_n) \right) \right] = 0, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k}) \right) = 0. \tag{3.17}$$

Following the same line of argument as in **Case 1**, we can show that

$$\lim_{k\to\infty} \left(t_{m_k} d^2(z,u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}} \rangle \right) \leq 0. \tag{3.18}$$

Also, from (3.15) we have

$$d^2(z, x_{m_k+1}) \leq (1 - t_{m_k})d^2(z, x_{m_k}) + t_{m_k}\left(t_{m_k}d^2(z, u) + 2(1 - t_{m_k})\langle\overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z, x_{m_k}) \leq d^2(z, x_{m_k+1})$, we obtain

$$d^2(z, x_{m_k}) \leq \left(t_{m_k}d^2(z, u) + 2(1 - t_{m_k})\langle\overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, from (3.18) we get

$$\lim_{k\to\infty} d^2(z, x_{m_k}) = 0. \tag{3.19}$$

It then follows from (3.16), (3.17) and (3.19) that $\lim_{k\to\infty} d^2(z, x_k) = 0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges to $z \in F(T)$.
In view of Remark 3.2, we obtain the following corollaries.

Corollary 3.10. *Let C be a nonempty closed and convex subset of a Hadamard space X and T be a generalized nonspreading mapping on C with $F(T) \neq \emptyset$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1 - t_n)x_n \oplus t_n u, \\ x_{n+1} = (1 - \alpha_n)y_n \oplus \alpha_n T y_n, \ n \geq 1, \end{cases} \tag{3.20}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$, satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq b < 1$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Corollary 3.11. *Let C be a nonempty closed and convex subset of a Hadamard space X and T be a k-strictly pseudononspreading mapping on C with $F(T) \neq \emptyset$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1 - t_n)x_n \oplus t_n u, \\ x_{n+1} = (1 - \alpha_n)y_n \oplus \alpha_n T y_n, \ n \geq 1, \end{cases} \tag{3.21}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$, satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq 1 - k$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Theorem 3.12. *Let C be a nonempty closed and convex subset of a Hadamard space X. Let T be an L-Lipschitzian and (f,g)-generalized k-strictly pseudononspreading mapping on C with constant $k \in (0,1)$, where $f,g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x)+g(x) \leq 1$ for all $x \in C$. Suppose $F(T) \neq \emptyset$ and $k < \min\{f(x), \frac{1-f(x)}{2}\}$ for each $x \in C$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = (1-t_n)x_n \oplus t_n u, \\ y_n = (1-\beta_n)u_n \oplus \beta_n T u_n, \\ x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T y_n, \ n \geq 1, \end{cases} \tag{3.22}$$

where $\{t_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions:

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq \frac{k}{f(p)}\beta_n < \beta_n \leq b < \frac{2}{(1+\frac{f(p)}{k})+\sqrt{(1+\frac{f(p)}{k})^2+4L^2}}$,
for each $p \in F(T)$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof: Let $p \in F(T)$, since T is L-Lipschitzian and generalized k-strictly pseudononspreading, we obtain from (3.2), (3.22) and Lemma 2.2 that

$$\begin{aligned} d^2(p,Ty_n) &\leq d^2(p,y_n) + \frac{k}{f(p)} d^2(y_n,Ty_n) \\ &= d^2(p,(1-\beta_n)u_n \oplus \beta_n T u_n) + \frac{k}{f(p)} d^2((1-\beta_n)u_n \oplus \beta_n T u_n, Ty_n) \\ &\leq (1-\beta_n)d^2(p,u_n) + \beta_n d^2(p,Tu_n) - \beta_n(1-\beta_n)d^2(u_n,Tu_n) \\ &\quad + \frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n) + \frac{k}{f(p)} L^2\beta_n^3 d^2(u_n,Tu_n) \\ &\quad - \frac{k}{f(p)}\beta_n(1-\beta_n)d^2(u_n,Tu_n) \\ &\leq (1-\beta_n)d^2(p,u_n) + \beta_n\left[d^2(p,u_n) + \frac{k}{f(p)}d^2(u_n,Tu_n)\right] \\ &\quad -\beta_n(1-\beta_n)d^2(u_n,Tu_n) + \frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n) \\ &\quad + \frac{k}{f(p)}L^2\beta_n^3 d^2(u_n,Tu_n) - \frac{k}{f(p)}\beta_n(1-\beta_n)d^2(u_n,Tu_n) \\ &= d^2(p,u_n) + \frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n) \\ &\quad -\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)}) - \frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right] d^2(u_n,Tu_n). \end{aligned} \tag{3.23}$$

Also, from (3.22), (3.23) and condition C3 , we obtain

$$\begin{aligned}
d^2(p,x_{n+1}) &\leq (1-\alpha_n)d^2(p,u_n)+\alpha_n d^2(p,Ty_n)-\alpha_n(1-\alpha_n)d^2(u_n,Ty_n)\\
&\leq (1-\alpha_n)d^2(p,u_n)+\alpha_n d^2(p,u_n)+\frac{k}{f(p)}\alpha_n(1-\beta_n)d^2(u_n,Ty_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n)\\
&\quad -\alpha_n(1-\alpha_n)d^2(u_n,Ty_n)\\
&\leq d^2(p,u_n)-\alpha_n\left[(1-\frac{k}{f(p)})+(\frac{k}{f(p)}\beta_n-\alpha_n)\right]d^2(u_n,Ty_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n)\\
&\leq d^2(p,u_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n)\\
&\leq d^2(p,(1-t_n)x_n\oplus t_n u) \qquad (3.24)\\
&\leq (1-t_n)d^2(p,x_n)+t_n d^2(p,u)\\
&\leq \max\{d^2(p,x_n),d^2(p,u)\}\\
&\vdots\\
&\leq \max\{d^2(p,x_1),d^2(p,u)\}.
\end{aligned}$$

Therefore, $\{d^2(p,x_n)\}$ is bounded. Consequently, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are all bounded.
From (3.22) and condition C1, we have that

$$\lim_{n\to\infty} d(u_n,x_n)\leq \lim_{n\to\infty} t_n d(u,x_n)=0. \qquad (3.25)$$

We now consider two cases for our proof.
Case 1: Suppose that $\{d^2(p,x_n)\}$ is monotonically non-increasing, then

$$\lim_{n\to\infty}\{d^2(p,x_n)\}$$

exists. Hence,

$$\lim_{n\to\infty}\left[d^2(p,x_{n+1})-d^2(p,x_n)\right]=0. \qquad (3.26)$$

Let $P_n=\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]$, then we obtain from (3.24) that

$$\begin{aligned}
P_n d^2(u_n,Tu_n) &\leq d^2(p,u_n)-d^2(p,x_{n+1})\\
&\leq (1-t_n)d^2(p,x_n)+t_n d^2(p,u)-d^2(p,x_{n+1})\\
&= d^2(p,x_n)-d^2(p,x_{n+1})\\
&+ t_n\left[d^2(p,u)-d^2(p,x_n)\right]\to 0,\ \text{as } n\to\infty. \qquad (3.27)
\end{aligned}$$

From condition C3, we obtain that $2-b\left(1+\frac{f(p)}{k}\right) > b\sqrt{\left(1+\frac{f(p)}{k}\right)^2+4L^2}$. Which implies that

$$2\frac{k}{f(p)} - b\left(1+\frac{k}{f(p)}\right) > b\frac{k}{f(p)}\sqrt{\left(1+\frac{f(p)}{k}\right)^2+4L^2}.$$

That is,

$$\left[2\frac{k}{f(p)} - b\left(1+\frac{k}{f(p)}\right)\right]^2 > 4\left(\frac{k}{f(p)}\right)^2 b^2L^2 + b^2\left(1+\frac{k}{f(p)}\right)^2,$$

which after simplification yields

$$\frac{k}{f(p)} - b\frac{k}{f(p)} - \frac{k}{f(p)}b^2L^2 - b > 0.$$

Thus,

$$\begin{aligned} P_n &= \alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)}) - \frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right] \\ &> a^2\left[(1+\frac{k}{f(p)}) - \frac{k}{f(p)} - \beta_n(1+\frac{k}{f(p)}) - \frac{k}{f(p)}L^2\beta_n^2\right] \\ &> a^2\left[(1+\frac{k}{f(p)}) - 1 - b(1+\frac{k}{f(p)}) - \frac{k}{f(p)}L^2b^2\right] > 0. \end{aligned}$$

Hence, we obtain from (3.27) that

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0. \tag{3.28}$$

Since $\{x_n\}$ is bounded and X is a Hadamard space, then from Lemma 2.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. It then follows from (3.25), (3.28) and Lemma 3.8, that $z \in F(T)$.
Furthermore, for arbitrary $u \in X$, we have from Lemma 2.8 that

$$\limsup_{n\to\infty}\langle\overrightarrow{zu}, \overrightarrow{zx_n}\rangle \le 0, \tag{3.29}$$

which implies from condition C1 that

$$\limsup_{n\to\infty}\left(t_nd^2(z,u) + 2(1-t_n)\langle\overrightarrow{zu}, \overrightarrow{zx_n}\rangle\right) \le 0. \tag{3.30}$$

Next, we show that $\{x_n\}$ converges strongly to z. From (3.24) and Lemma 2.3, we obtain

$$\begin{aligned} d^2(z, x_{n+1}) &\le d^2(z, (1-t_n)x_n \oplus t_nu) \\ &\le (1-t_n)^2d^2(z,x_n) + t_n^2d^2(z,u) + 2t_n(1-t_n)\langle\overrightarrow{zx_n}, \overrightarrow{zu}\rangle \\ &\le (1-t_n)d^2(z,x_n) + t_n\left(t_nd^2(z,u) + 2(1-t_n)\langle\overrightarrow{zx_n}, \overrightarrow{zu}\rangle\right) \end{aligned} \tag{3.31}$$

Hence, from (3.30) and Lemma 2.9, we obtain that $\{x_n\}$ converges strongly to z.
Case 2: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{p, d^2(x_{n_i})\}$ of $\{p, d^2(x_n)\}$ such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \leq d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \leq d^2(p, x_{m_k+1}) \ \forall k \in \mathbb{N}. \tag{3.32}$$

Thus, from (3.24) and (3.32), we obtain

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, x_{n+1}) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left((1-t_n)d^2(p, x_n) + t_n d^2(p, u) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left[t_n \left(d^2(p, u) - d^2(p, x_n)\right)\right] = 0, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right) = 0. \tag{3.33}$$

Following the same line of argument as in **Case 1**, we can show that

$$\lim_{k\to\infty} \left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right) \leq 0. \tag{3.34}$$

Also, from (3.31) we have

$$d^2(z, x_{m_k+1}) \leq (1-t_{m_k})d^2(z, x_{m_k}) + t_{m_k}\left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z, x_{m_k}) \leq d^2(z, x_{m_k+1})$, we obtain

$$d^2(z, x_{m_k}) \leq \left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, from (3.34) we get

$$\lim_{k\to\infty} d^2(z, x_{m_k}) = 0. \tag{3.35}$$

It then follows from (3.32), (3.33) and (3.35) that $\lim_{k\to\infty} d^2(z, x_k) = 0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges to $z \in F(T)$.
Also, by Remark 3.2, we obtain the following corollaries.

Corollary 3.13. *Let C be a nonempty closed and convex subset of a Hadamard space X and T be a generalized nonspreading mapping on C with $F(T) \neq \emptyset$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = (1-t_n)x_n \oplus t_n u, \\ y_n = (1-\beta_n)u_n \oplus \beta_n T u_n, \\ x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T y_n, \ n \geq 1, \end{cases} \tag{3.36}$$

where $\{t_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$, satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq b < 1$ *and* $0 < a \leq \beta_n \leq b < 1$.

Then $\{x_n\}$ *converges strongly to an element of* $F(T)$.

Corollary 3.14. *Let* C *be a nonempty closed and convex subset of a Hadamard space* X. *Let* T *be an* L*-Lipschitzian and* k*-strictly pseudononspreading mapping on* C *with constant* $k \in (0,1)$. *Suppose* $F(T) \neq \emptyset$ *and for arbitrary* $u, x_1 \in C$, *the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} u_n = (1-t_n)x_n \oplus t_n u, \\ y_n = (1-\beta_n)u_n \oplus \beta_n T u_n, \\ x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T y_n, \ n \geq 1, \end{cases} \tag{3.37}$$

where $\{t_n\}$, $\{\beta_n\}$ *and* $\{\alpha_n\}$ *are sequences in* $[0,1]$, *satisfying the following conditions:*

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+1}{k}\right)+\sqrt{\left(\frac{k+1}{k}\right)^2+4L^2}}$.

Then $\{x_n\}$ *converges strongly to an element of* $F(T)$.

Declaration

The authors declare that they have no competing interests.

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