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Existence of Some Classes of $N(k)$-quasi Einstein Manifolds

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ABSTRACT: The object of the present paper is to study some classes of $N(k)$-quasi Einstein manifolds. The existence of such manifolds are proved by giving non-trivial physical and geometrical examples. It is also proved that the characteristic vector field of the manifold is Killing as well as parallel unit vector field under certain curvature restrictions.

Key Words: Quasi Einstein, k-nullity distribution, $N(k)$-quasi Einstein manifold, Different curvature tensors.

Contents

1. Introduction

An $n-$dimensional semi-Riemannian or Riemannian manifold (M_n, g), $(n > 2)$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition $S = \frac{r}{n}g$, where r denotes the scalar curvature of (M_n, g). In other words, an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi Einstein manifolds arose during the study of exact solutions of Einstein field equations as well as during consideration of quasi-umbilical hypersurfaces of conformally flat spaces (e. g., see [43], [44], [45]). A non-flat n-dimensional Riemannian manifold is said to be quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \qquad X, Y \in TM \tag{1.1}$$

for smooth functions a and $b \neq 0$, where η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1 \tag{1.2}$$

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for all vector field X and the associated unit vector field ξ ([4], [5]). The $1-$form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If the generator of a quasi Einstein manifold is parallel vector field, then the manifold is locally a product manifold of the one-dimensional distribution U and $(n-1)$ dimensional distribution $U^{\perp}$, where $U^{\perp}$ is involutive and integrable [22]. In an n-dimensional quasi Einstein manifold the Ricci tensor has precisely two distinct eigen values a and $a+b$, where the multiplicity of a is $n-1$ and $a+b$ is simple [5]. A proper η-Einstein contact metric manifold is a natural example of a quasi Einstein manifold ([6], [7]). The different geometrical properties of quasi Einstein manifolds have studied by Chaki [23], Guha [25], De and Ghosh [24], Shaikh, Yoon and Hui [26], Shaikh, Kim and Hui [27], Deszcz et al. [16], Mantica and Suh [17] and others.

Let R denotes the Riemannian curvature tensor of a Riemannian manifold M_n. For a smooth function k, the $k-$nullity distribution $N(k)$ of a Riemannian manifold is defined as

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k\,[g(Y,Z)X - g(X,Z)Y]\}\,, \quad (1.3)$$

for all $X, Y, Z \in TM$ [8]. If the generator ξ belongs to $k-$nullity distribution $N(k)$, then the quasi Einstein manifold is called an $N(k)$-quasi Einstein manifold [9]. A conformally flat quasi Einstein manifolds are certain $N(k)$-quasi Einstein manifolds [9]. The deviation conditions $R(\xi,X)\cdot R = 0$, $R(\xi,X)\cdot S = 0$ have also been studied in [9], where R and S denote the curvature and Ricci tensors of the manifold respectively. In 2007, Özgür and Tripathi [10] studied the deviation conditions $\hat{Z}(\xi,X)\cdot\hat{Z} = 0$ and $\hat{Z}(\xi,X)\cdot R = 0$ on $N(k)-$quasi Einstein manifolds, where $\hat{Z}$ denotes the concircular curvature tensor. Özgür and Sular [11] continued the study of $N(k)$-quasi Einstein manifolds with conditions $R(\xi,X)\cdot C = 0$ and $R(\xi,X)\cdot\tilde{C} = 0$, where C and $\tilde{C}$ denote the Weyl conformal and quasi conformal curvature tensors respectively. Again, in 2008, Özgür [12] studied the deviation conditions $R(\xi,X)\cdot P = 0$, $P(\xi,X)\cdot S = 0$ and $P(\xi,X)\cdot P = 0$ for an $N(k)-$quasi Einstein manifold, where P denotes the projective curvature tensor and some physical examples of $N(k)-$quasi Einstein manifolds are given. In 2010, Singh et al. [13] continue the study of $N(k)-$quasi Einstein manifolds with certain deviation conditions. Several geometrical properties of $N(k)$-quasi Einstein manifolds have studied by Taleshian and Hosseinzadeh ([15], [21]), De, De and Gazi [14], Yang and Xu [20], Mallick and De [28], Crasmareanu [19], present author [18] and others. Motivated from the above studied, authors continue the study of N$(k)-$quasi Einstein manifolds.

In tune with Yano and Sawaki [42], recently Baishya and Roy Chowdhury [40] have introduced and studied the generalized quasi-conformal curvature tensor in the context of $N(k,\mu)-$manifold. The generalized quasi-conformal curvature tensor for $n-$dimensional manifold is defined as

$$\begin{aligned} W^*(X,Y)Z \;=\; & \frac{n-2}{n}\left[\{1+(n-1)a^*-b^*\} - \{1+(n-1)(a^*+b^*)\}c^*\right] C(X,Y)Z \\ & + \left[1-b^*+(n-1)a^*\right] E(X,Y)Z + (n-1)(b^*-a^*)P(X,Y)Z \\ & + \frac{n-2}{n}(c^*-1)\{1+(n-1)(a^*+b^*)\}\hat{C}(X,Y)Z \qquad (1.4) \end{aligned}$$

for all X, Y & $Z \in TM$, the set of all vector fields of the manifold M, where a^*, b^* & c^* are real constants. The beauty of generalized quasi-conformal curvature tensor lies in the fact that it has the flavour of Riemann curvature tensor R for $a^* = b^* = c^* = 0$; Conformal curvature tensor C [38] for $a^* = b^* = -\frac{1}{n-2}$ & $c^* = 1$; Conharmonic curvature tensor $\hat{C}$ [39] for $a^* = b^* = -\frac{1}{n-2}$ & $c^* = 0$; Concircular curvature tensor E ([37], p.84) for $a^* = b^* = 0$ & $c^* = 1$; Projective curvature tensor P ([37], p.84) for $a^* = -\frac{1}{n-1}$, $b^* = 0$ & $c^* = 0$ and m-projective curvature tensor H [41], for $a^* = b^* = -\frac{1}{2n-2}$ & $c^* = 0$.

The present work is structured as follows: Section 2 is preliminaries which deals with the some known results of $N(k)-$quasi Einstein manifolds and different curvature tensors. In section 3, we construct some physical as well as structural examples which support the existence of some classes of $N(k)-$quasi Einstein manifolds. In next section, we consider the curvature condition $R(\xi, X).W^* = 0$ and prove that the characteristic vector field is Killing as well as unit parallel vector field. Section 5 is concerned with $\mathcal{Z}-$pseudosymmetric condition and obtain some geometric results. In last section, we have to prove that there does not exist an $N(k)-$quasi Einstein manifold under certain curvature restriction.

2. Preliminaries

In consequence of (1.1), (1.2) and (1.3), we get

$$S(X, \xi) = (a + b)\eta(X), \tag{2.1}$$

$$k = \frac{a + b}{n - 1} \tag{2.2}$$

and

$$r = na + b, \tag{2.3}$$

where r denotes the scalar curvature of the Riemannian manifold (M_n, g). In an $n-$dimensional $N(k)-$quasi Einstein manifold (M_n, g), the following relations hold ([9], [10])

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.4}$$

$$R(X, \xi)Y = k[\eta(Y)X - g(X, Y)\xi] = -R(\xi, X)Y, \tag{2.5}$$

$$R(\xi, X)\xi = k[\eta(X)\xi - X], \tag{2.6}$$

$$Q\xi = k(n - 1)\xi, \tag{2.7}$$

$$\eta(R(X, Y)Z) = k[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] \tag{2.8}$$

for arbitrary vector fields X, Y and Z. The equation (1.4) can also be written as

$$\begin{aligned} W^*(X, Y)Z \quad = \quad & R(X, Y)Z + a^*[S(Y, Z)X - S(X, Z)Y] \\ & + b^*[g(Y, Z)QX - g(X, Z)QY] \\ & - \frac{c^* r}{n}\left(\frac{1}{n - 1} + a^* + b^*\right)[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{2.9}$$

for arbitrary vector fields X, Y, Z; where $S(X,Y) = g(QX,Y)$, Q denotes the Ricci operator and r being the scalar curvature of the manifold.
Recently, Mantica and Molinari [36] defined a generalized $(0,2)$ type tensor $\mathcal{Z}$ as

$$\mathcal{Z}(X,Y) = S(X,Y) + fg(X,Y), \tag{2.10}$$

for arbitrary vector fields X and Y, where f is a smooth function. In consequence of (1.1), (1.2) and (2.10), we have

$$\mathcal{Z}(X,Y) = (a+f)g(X,Y) + b\eta(X)\eta(Y). \tag{2.11}$$

The curvature conditions $R \cdot W^*$ and $W^* \cdot R$ are defined by

$$\begin{aligned}(R(W,X)\cdot W^*)(Y,Z)U &= R(W,X)W^*(Y,Z)U - W^*(R(W,X)Y,Z)U \\ &\quad -W^*(Y,R(W,X)Z)U \\ &\quad -W^*(Y,Z)R(W,X)U \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}(W^*(W,X)\cdot R)(Y,Z)U &= W^*(W,X)R(Y,Z)U - R(W^*(W,X)Y,Z)U \\ &\quad -R(Y,W^*(W,X)Z)U \\ &\quad -R(Y,Z)W^*(W,X)U, \end{aligned} \tag{2.13}$$

respectively, for all vector fields W, X, Y, Z, U, where $R(W,X)$ acts on W^* and $W^*(W,X)$ acts on R as a deviation.

3. Examples of $N(k)-$quasi Einstein manifolds

In this section, we give some examples of the $N(k)-$quasi Einstein manifolds which support the existence of such manifolds.

In general relativity, a dust solution is an exact solution of the Einstein field equation in which the gravitational field is generated completely by the mass, momentum and stress density of a perfect fluid which has positive mass density and vanishing pressure. Since dust particle is a pressure less perfect fluid and therefore it interact with each other only gravitationally. Due to this property, dust models are often employed in cosmology as models of a toy universe, in which the dust particles are considered as highly idealized model of galaxies, clusters or superclusters. In astrophysics, dust solutions have been employed as models of gravitational collapse. We can also observed the well known dust models as: model finite rotating disks of dust gains; Steller model comprising a ball of fluid surrounded by vacuum; model an accretion disk around a massive object; Friedmann-Lemaître-Robertson-Walker dusts (homogeneous and isotropic solution often refered to as the matter-dominated FLRW model); Kasner dusts (the simplest cosmological model exhibiting anisotropic expansion); Bianchi dust models (generalizations of FLRW and Kasner models, exhibiting various types of Lie algebras of Killing vector fields); Lemaître–Tolman–Bondi (LTB) dusts (some of the simplest inhomogeneous cosmological models, often employed as models of gravitational collapse); Kantowski-Sachs dusts (cosmological models which exhibit perturbations from FLRW models); Van Stockum dust (a cylindrically symmetric rotating dust); the Neugebauer-Meinel dust (which models a rotating disk of dust matched to an axisymmetric vacuum exterior; this solution has been called, with some justice, the most remarkable exact solution discovered since the Kerr vacuum); Gödel metric. From the above, we can can say that dust as a solution is more applicable.

Let us consider a collection of non-interacting pressure less massive particle (dust) as a space-time (M_4, g). Also we consider that the characteristic vector field ξ to be the unit time like velocity vector of (M_4, g), *i.e.*, $g(\xi, \xi) = -1$. If (M_4, g) is conformally flat and satisfies the Einstein equation without cosmological constant, then

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y), \tag{3.1}$$

where S and r are defined earlier, κ is a gravitational constant and T is the stress energy tensor of the relativistic pressure less fluid, a symmetric tensor of type $(0, 2)$, is given as

$$T(X,Y) = \rho\eta(X)\eta(Y). \tag{3.2}$$

Here ρ is the matter density of the dust and η is the $1-$form associated with the unit time like velocity vector as $\eta(X) = g(X, \xi)$. In view of (3.2), (3.1) becomes

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa\rho\eta(X)\eta(Y) \tag{3.3}$$

which reduces to

$$r = \kappa\rho. \tag{3.4}$$

From (3.4), we can observe that the scalar curvature is constant. We have from (3.3) and (3.4),

$$S(X,Y) = \kappa\rho\{\frac{1}{2}g(X,Y) + \eta(X)\eta(Y)\}, \tag{3.5}$$

which shows that (M_4, g) is a quasi Einstein with

$$a = \frac{1}{2}\kappa\rho \ \ and \ \ b = \kappa\rho. \tag{3.6}$$

In view of (1.1), (1.2), (1.3) and $g(\xi, \xi) = -1$, we get

$$k = \frac{a-b}{3} = -\frac{1}{6}\kappa\rho. \tag{3.7}$$

It is well known that, on a conformally flat manifold the characteristic vector field ξ belongs to $k-$nullity distribution [9]. Thus we can state the following example:

Example 3.1. *A conformally flat space-time (M_4, g) satisfying Einstein equation without cosmological constant is an $N(-\frac{1}{6}\kappa\rho)-$quasi Einstein manifold.*

Again, let us consider a conformally flat space-time (M_4, g) (collection of non-interacting pressure less massive particle (dust)) satisfying the Einstein equation with cosmological constant Λ. Then Einstein equation assumes the form

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \Lambda g(X,Y) = \kappa T(X,Y), \tag{3.8}$$

which becomes

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \Lambda g(X,Y) = \kappa\rho\eta(X)\eta(Y). \tag{3.9}$$

after considering equation (3.2). Contracting (3.9) for X and Y, we find

$$r = 4\Lambda + \kappa\rho. \tag{3.10}$$

In consequence of (3.10), (3.9) gives

$$S(X,Y) = (\Lambda + \frac{1}{2}\kappa\rho)g(X,Y) + \kappa\rho\eta(X)\eta(Y), \tag{3.11}$$

which shows that the space-time (M_4, g) is a quasi Einstein manifold. From (1.1) and (3.11), we have

$$a = \Lambda + \frac{1}{2}\kappa\rho \ \ and \ \ b = \kappa\rho. \tag{3.12}$$

and therefore the equation (3.7) gives

$$k = \frac{a-b}{3} = \{\frac{\Lambda}{3} - \frac{1}{6}\kappa\rho\}.$$

Hence we can state the following example:

Example 3.2. *If the space-time (M_4, g) is conformally flat and satisfies Einstein equation with cosmological constant, then it is an $N(\frac{\Lambda}{3} - \frac{1}{6}\kappa\rho)-$quasi Einstein manifold.*

In 1972, K. Kenmotsu introduced the notion of Kenmotsu manifold and studied its different geometrical properties [2]. Let (M_n, g), (n=2m+1), be an $n-$ dimensional Kenmotsu manifold, where ϕ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a $1-$form and g is a Riemannian metric. It is obvious that the manifold (M_n, g) satisfies the relations

$$\phi^2(X) = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \ \ g(X,\xi) = \eta(X), \ \ and \ \nabla_X\xi = X - \eta(X)\xi, \tag{3.13}$$

for arbitrary vector fields X and Y. Some properties of the Kenmotsu manifold has been noticed in ([30]- [35]). It can be easily see from (3.13) that

$$(L_\xi g)(X,Y) = 2\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.14}$$

where L_ξ denotes the Lie derivative along the characteristic vector field ξ.

A triplet (g, V, λ) on a Riemannian manifold is said to be a Ricci soliton if it satisfies the condition

$$\frac{1}{2}L_V g + S = \lambda g, \tag{3.15}$$

where g is the Riemannian metric associated with the smooth vector field V, S is a Ricci tensor and λ is a real constant [3]. A Ricci soliton is a natural generalization of Einstein manifold and it has many applications in physics. Here λ is a real constant, therefore it is characterized in three categories:
(i) $\lambda = 0$, Ricci soliton is steady,
(ii) $\lambda < 0$, Ricci soliton is expanding,
(iii) $\lambda > 0$, Ricci soliton is shrinking.
In first two cases, Ricci soliton to be Einstein but we are interested in non-Einstein manifold and therefore we are going to consider the case of shrinking soliton only. Let us suppose that $V = \xi$ and therefore by equations (3.14) and (3.15), we get

$$S(X,Y) = (\lambda - 1)g(X,Y) + \eta(X)\eta(Y), \tag{3.16}$$

where

$$a = \lambda - 1 \ \ and \ \ b = 1, \tag{3.17}$$

which shows that the Kenmotsu manifold equipped with the Ricci soliton is an $\eta-$Einstein manifold. Now with the help of (2.2) and (3.17), we have $k = \frac{a+b}{n-1} = \frac{\lambda}{n-1}$ and thus we can state the following example as:

Example 3.3. *A conformally flat Kenmotsu manifold equipped with the Ricci soliton is an $N(\frac{\lambda}{n-1})$- quasi Einstein manifold.*

Example 3.4. *Let $(x^1, x^2, \ldots, x^n) \in R_n$, where R_n denotes an $n-$dimensional real number space. We consider a Riemannian metric g on $R_4 = (x^1, x^2, x^3, x^4; x^1 \neq \pi p, \frac{\pi}{2} + p\pi, p \in \mathbb{Z})$, ($\mathbb{Z}$ is the set of integer), by*

$$ds^2 = g_{ij}dx^i dx^j = sin^2(x^1)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2, \tag{3.18}$$

where $(i, j = 1, 2, 3, 4)$. With the help of (3.18), we can see that the non-vanishing components of Riemannian metric are

$$g_{11} = g_{22} = g_{33} = sin^2(x^1), \quad g_{44} = 1 \tag{3.19}$$

and its associated components are

$$g^{11} = g^{22} = g^{33} = cosec^2(x^1), \quad g^{44} = 1. \tag{3.20}$$

In consequence of (3.19) and (3.20), it can be calculated that the non-vanishing components of Christoffel symbols, curvature tensor, Ricci tensor and scalar curvature are given by

$$\begin{aligned} &\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = 2cot(x^1), \ \ \Gamma^1_{22} = \Gamma^1_{33} = -2cot(x^1), \\ &R_{1331} = -cos^2(x^1), \ \ S_{33} = cot^2(x^1), \ \ r = cot^2(x^1)cosec^2(x^1)(\neq 0) \end{aligned} \tag{3.21}$$

and the components obtained by the symmetric properties. Now, we are going to prove the manifold (R_4, g) is an $N(k)-$quasi Einstein manifold. For this purpose we take the associated scalars a and b as follows:

$$a = 0, \quad b = cot^2(x^1)cosec^2(x^1)(\neq 0). \tag{3.22}$$

Now we define the $1-$forms A_i as follows:

$$A_i = \begin{cases} sin(x^1) & , \quad i = 3 \\ 0 & , \quad otherwise \end{cases} \tag{3.23}$$

Now, we have to prove the following:

$$S_{ij} = ag_{ij} + bA_iA_j \tag{3.24}$$

for $i, j = 1, 2, 3, 4$. For instant, we have to show that

$$S_{33} = ag_{33} + bA_3A_3. \tag{3.25}$$

Left hand side of (3.25)=S_{33}=$cot^2(x^1)$ (from (3.21)). In view of (3.19), (3.22) and (3.23), right hand side=$ag_{33} + bA_3A_3$=$cot^2(x^1)$. In the similar way, we can verify for other components of S_{ij}. We have from (1.1), (1.2), (1.3) and (3.22),

$$k = \frac{a+b}{n-1} = \frac{cot^2(x^1)cosec^2(x^1)}{3}$$

and $r = 4a+b$ hold on (R_4, g). Therefore (R_4, g) is an $N\left(\frac{cot^2(x^1)cosec^2(x^1)}{3}\right)-$quasi Einstein manifold.

Example 3.5. *We consider a $3-$dimensional manifold $M_3 = \{(x, y, z) \in R_3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R_3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M_3 given by*

$$E_1 = (\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial z}), \quad E_2 = \frac{\partial}{\partial y}, \ E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in TM$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -2yE_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

Taking $E_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1}E_3 = yE_2, \qquad \nabla_{E_3}E_3 = 0, \qquad \nabla_{E_2}E_3 = -yE_1,$$

$$\nabla_{E_2}E_2 = 0, \qquad \nabla_{E_1}E_2 = -yE_3, \qquad \nabla_{E_2}E_1 = yE_3,$$
$$\nabla_{E_1}E_1 = 0, \qquad \nabla_{E_3}E_2 = -yE_1, \qquad \nabla_{E_3}E_1 = yE_2.$$

From the above it can be easily see that (ϕ, ξ, η, g) is a trans-Sasakian structure on M_3. Consequently $M_3(\phi, \xi, \eta, g)$ is an $N(k)$-manifold with $k = y^2 \neq 0$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows :

$$\begin{aligned} &R(E_1, E_2)E_2 = E_3 - 3y^2E_1, \qquad R(E_1, E_2)E_1 = E_3 + 3y^2E_2,\\ &R(E_2, E_3)E_3 = y^2E_2, \qquad R(E_1, E_3)E_3 = y^2E_1 \end{aligned} \tag{3.26}$$

and the components which can be obtained from these by the symmetry properties. Using the components of the curvature tensor, we can easily calculate the non-vanishing components of the Ricci tensor S as follows :

$$S(E_1, E_1) = -2y^2 = S(E_2, E_2), \qquad S(E_3, E_3) = 2y^2$$

and the scalar curvature r is given by $r = -2y^2$. Clearly, $M_3(\phi, \xi, \eta, g)$ is an $N(k)-$quasi Einstein manifold with a=$-2y^2$ and b=$4y^2$.

Example 3.6. *[12] A conformally flat perfect fluid space time (M_4, g) satisfying Einstein's equation with cosmological constant λ is an $N(\frac{\lambda}{3} + \frac{\kappa(3\sigma+p)}{6})-$quasi Einstein manifold.*

Example 3.7. *[12] A conformally flat perfect fluid space time (M_4, g) satisfying Einstein's equation without cosmological constant is an $N(\frac{\kappa(3\sigma+p)}{6})-$quasi Einstein manifold.*

Example 3.8. *[14] A special para-Sasakian manifold with vanishing $D-$concircular curvature tensor is an $N(k)-$quasi Einstein manifold.*

Example 3.9. *[14] A perfect fluid pseudo Ricci symmetric space time is an $N(\frac{2r}{9})-$quasi Einstein manifold.*

4. $N(k)$-quasi Einstein manifolds satisfying $R(\xi, X).W^* = 0$

From (1.1), (1.2), (2.1), (2.8) and (2.9) it is obvious that

$$\eta(W^*(X, Y)Z) = \lambda\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{4.1}$$

where

$$\lambda = k + aa^* + (a + b)b^* - \frac{c^*r}{n}\left(\frac{1}{n-1} + a^* + b^*\right). \tag{4.2}$$

From (2.12) we have

$$\begin{aligned} (R(\xi, X) \cdot W^*)(Y, Z)U &= R(\xi, X)W^*(Y, Z)U - R(W^*(\xi, X)Y, Z)U\\ &\quad -R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U. \end{aligned} \tag{4.3}$$

With the help of (1.2) and (2.5), (4.3) turns into the form

$$\begin{aligned}(R(\xi, X) \cdot W^*)(Y, Z)U &= k[{}'W^*(Y, Z, U, X)\xi - \eta(W^*(Y, Z)U)X \\ &+\eta(Y)W^*(X, Z)U - g(X, Z)W^*(Y, \xi)U \\ &+\eta(Z)W^*(Y, X)U + g(X, U)W^*(Y, Z)\xi \\ &-\eta(U)W^*(Y, Z)X] - g(X, Y)W^*(\xi, Z)U,\end{aligned}$$

where ${}'W^*(Y, Z, U, X) = g(W^*(Y, Z)U, X)$. Let us suppose that $R(\xi, X).W^* = 0$ and then with the help of (1.2), (4.4) gives

$$\begin{aligned}&k[{}'W^*(Y, Z, U, X) - \eta(W^*(Y, Z)U)\eta(X) - g(X, Y)\eta(W^*(\xi, Z)U) \\ &+\eta(Y)\eta(W^*(X, Z)U) - g(X, Z)\eta(W^*(Y, \xi)U) + \eta(Z)\eta(W^*(Y, X)U) \\ &+g(X, U)\eta(W^*(Y, Z)\xi) - \eta(U)\eta(W^*(Y, Z)X)] = 0,\end{aligned} \tag{4.4}$$

which shows that either $k = 0$ or

$$\begin{aligned}&{}'W^*(Y, Z, U, X) - \eta(W^*(Y, Z)U)\eta(X) - g(X, Y)\eta(W^*(\xi, Z)U) \\ &+\eta(Y)\eta(W^*(X, Z)U) - g(X, Z)\eta(W^*(Y, \xi)U) + \eta(Z)\eta(W^*(Y, X)U) \\ &+g(X, U)\eta(W^*(Y, Z)\xi) - \eta(U)\eta(W^*(Y, Z)X) = 0.\end{aligned} \tag{4.5}$$

If we consider that $k \neq 0$, then in view of (1.1), (1.2), (2.9), (4.1) and (4.4), we obtain

$$\begin{aligned}{}'R(Y, Z, U, X) &= \alpha\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\} \\ &+\beta\{\eta(Y)\eta(U)g(X, Z) - \eta(U)\eta(Z)g(X, Y) \\ &+\eta(X)\eta(Y)g(Z, U) - \eta(Z)\eta(X)g(Y, U)\},\end{aligned} \tag{4.6}$$

where $\alpha = (\lambda + \frac{a}{n-1})$ and $\beta = (2\lambda - \frac{b}{2(n-1)})$.

A Riemannian manifold (M_n, g) is said to be quasi-constant curvature if the curvature tensor R is not identically zero and satisfies the relation

$$\begin{aligned}{}'R(X, Y, Z, U) &= \alpha\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\} \\ &+\beta\{\eta(Y)\eta(Z)g(X, U) - \eta(X)\eta(Z)g(Y, U) \\ &+\eta(X)\eta(U)g(Y, Z) - \eta(Y)\eta(U)g(X, Z)\}\end{aligned} \tag{4.7}$$

for arbitrary vector fields X, Y, Z and U [29]. Here α and β are smooth functions on M_n. If $\beta = 0$, then (M_n, g) reduces to a space of constant curvature α. In consequence of (4.6) and (4.7), we can state the following theorem:

Theorem 4.1. *Let (M_n, g) be an $n-$dimensional $N(k)$-quasi Einstein manifold, then it is a space of quasi constant curvature for each of curvature restriction and the condition*

Curvature restriction	condition
$(R(\xi,X)\cdot R)(Y,Z)U=0$ $(a^*=b^*=c^*=0)$	$b\neq 0,$
$(R(\xi,X)\cdot E)(Y,Z)U=0$ $(a^*=b^*=0,\ c^*=1)$	$4na\neq -(n+4)b$
$(R(\xi,X)\cdot C)(Y,Z)U=0$ $(a^*=b^*=-\frac{1}{n-2}, c^*=1)$	$4a(n-2)\neq -(3n-2)b$
$(R(\xi,X)\cdot \hat{C})(Y,Z)U=0$ $(a^*=b^*=-\frac{1}{n-2}, c^*=0)$	$4a(2n-1)\neq -(5n-4)b$
$(R(\xi,X)\cdot P)(Y,Z)U=0$ $(a^*=-\frac{1}{n-1}, b^*=c^*=0)$	$4a\neq -b$
$(R(\xi,X)\cdot H)(Y,Z)U=0$ $(a^*=b^*=-\frac{1}{2(n-1)},\ c^*=0)$	$4a\neq -3b$

Let $\{e_i, i=1,2,...,n\}$, be an orthonormal basis of the tangent space at any point of the manifold (M_n,g). Then putting $X=Y=e_i$ in (4.6) and taking summation over i, $1\leq i\leq n$, we get

$$S(Z,U)=a_1g(Z,U)+b_1\eta(Z)\eta(U), \tag{4.8}$$

where $a_1=a+(n-1)\lambda+\frac{b}{2(n-1)}$ and $b_1=-\frac{bn}{2(n-1)}$. Since, on a quasi-Einstein manifold the smooth functions a and b are unique, as if $S=a_1g+b_1\eta\otimes\eta$, then $(a-a_1)g+(b-b_1)\eta\otimes\eta$=0 and thus g is of rank ≤ 1, a contradiction and therefore $k=0$. Conversely, if we consider $k=0$, then from (4.4) we have $R(\xi,X).W^*=0$. Hence we can state the following theorem:

Theorem 4.2. *An n-dimensional $N(k)$-quasi Einstein manifold (M_n,g) satisfies the condition $R(\xi,X)\cdot W^*=0$ if and only if $k=0$.*

Let us suppose that $k=0 \implies b=-a\neq 0$ and thus (1.2) becomes

$$S(Y,Z)=a\{g(Y,Z)-\eta(Y)\eta(Z)\}. \tag{4.9}$$

With the help of (4.9), we find

$$(\nabla_XS)(Y,Z)=da(X)\{g(Y,Z)-\eta(Y)\eta(Z)\}-a\{(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)\}. \tag{4.10}$$

Taking cyclic sum of (4.10), we have

$$\begin{aligned}(\nabla_XS)(Y,Z)+(\nabla_YS)(Z,X)+(\nabla_ZS)(X,Y)&=da(Z)\{g(X,Y)-\eta(X)\eta(Y)\}\\ &+da(X)\{g(Y,Z)-\eta(Y)\eta(Z)\}-a\{(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)\}\\ &+da(Y)\{g(Z,X)-\eta(Z)\eta(X)\}-a\{(\nabla_Y\eta)(Z)\eta(X)+(\nabla_Y\eta)(X)\eta(Z)\}\\ &-a\{(\nabla_Z\eta)(X)\eta(Y)+(\nabla_Z\eta)(Y)\eta(X)\}.\end{aligned} \tag{4.11}$$

If we suppose that the manifold (M_n,g) is cyclic parallel, then from (4.11) we have

$$\begin{aligned}0 \;=\;& da(X)\{g(Y,Z)-\eta(Y)\eta(Z)\}\\ &-a\{(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)\}\\ &+da(Y)\{g(Z,X)-\eta(Z)\eta(X)\}\\ &-a\{(\nabla_Y\eta)(Z)\eta(X)+(\nabla_Y\eta)(X)\eta(Z)\}\\ &+da(Z)\{g(X,Y)-\eta(X)\eta(Y)\}\\ &-a\{(\nabla_Z\eta)(X)\eta(Y)+(\nabla_Z\eta)(Y)\eta(X)\}.\end{aligned} \tag{4.12}$$

Let $\{e_i, i = 1, 2, ..., n\}$, be an orthonormal basis of the tangent space at any point of the manifold (M_n, g). Then putting $Y = Z = e_i$ in (4.12) and taking summation over i, $1 \leq i \leq n$, we get

$$(n+1)da(X) - 2da(\xi)\eta(X) - 2a\{(\nabla_\xi \eta)(X) + \sum_{n=1}^{n}(\nabla_{e_i}\eta)(e_i)\eta(X)\} = 0. \quad (4.13)$$

Again putting $Y = Z = \xi$ in (4.12) and then using (1.2), we find

$$(\nabla_\xi \eta)(X) = 0. \quad (4.14)$$

In light of (4.14), (4.13) becomes

$$(n+1)da(X) - 2da(\xi)\eta(X) = 2a\sum_{n=1}^{n}(\nabla_{e_i}\eta)(e_i)\eta(X). \quad (4.15)$$

Setting $X = \xi$ in (4.15) and then using (1.2), we get

$$(n+1)da(\xi) - 2da(\xi) = 2a\sum_{n=1}^{n}(\nabla_{e_i}\eta)(e_i). \quad (4.16)$$

In view of (4.16), (4.15) gives

$$da(X) = \eta(X)da(\xi). \quad (4.17)$$

Using (4.17) in (4.12), we obtain

$$\begin{aligned} & da(\xi)\{\eta(X)g(Y,Z) + \eta(Y)g(Z,X) + \eta(Z)g(X,Y) - 3\eta(X)\eta(Y)\eta(Z)\} \\ & -a\{(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y) + (\nabla_Y\eta)(Z)\eta(X) + (\nabla_Y\eta)(X)\eta(Z) \\ & +(\nabla_Z\eta)(X)\eta(Y) + (\nabla_Z\eta)(Y)\eta(X)\} = 0. \end{aligned} \quad (4.18)$$

Replacing the vector field Z with the characteristic vector field ξ in equation (4.18) and then using (1.2) and (4.14), we get

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0 \quad (4.19)$$

which is equivalent to

$$(\nabla_X\eta)(X) = 0. \quad (4.20)$$

Thus we can state the following theorem:

Theorem 4.3. *If an $n-$dimensional $N(k)-$quasi Einstein manifold (M_n, g) equipped with the cyclic parallel Ricci tensor satisfies $R(\xi, X) \cdot W^* = 0$, then its generator ξ is a Killing vector field.*

On a compact Riemannian manifold, the following inequality holds for a vector field X,

$$\int_M \langle S(X,X) - |\nabla X|^2 - (divX)^2\rangle dM \leq 0. \quad (4.21)$$

Equality holds if and only if X is a Killing vector field [1]. Let $X = \xi$ and by the use of $a + b = 0$, (1.1) and Theorem 4.3, equation (4.21) turns into

$$\int_M \langle |\nabla\xi|^2 - (div\xi)^2 \rangle dM = 0. \tag{4.22}$$

Since ξ is a Killing vector field, therefore $div\xi = 0$ and equation (4.22) becomes

$$\int_M |\nabla\xi|^2 dM = 0, \tag{4.23}$$

which is equivalent to

$$\nabla\xi = 0. \tag{4.24}$$

Hence we can state the following theorem:

Theorem 4.4. *Let (M_n, g) be an $n-$dimensional $N(k)-$quasi Einstein manifold equipped with the cyclic parallel Ricci tensor satisfying $R(\xi, X) \cdot W^* = 0$. Then the characteristic vector field on (M_n, g) is a parallel unit vector field.*

5. $\mathcal{Z}$-pseudosymmetric $N(k)$-quasi Einstein manifolds

Recently Mallick et al. [28] considered an $N(k)$-quasi Einstein manifold satisfies the condition $R(\xi, X) \cdot \mathcal{Z} = 0$ and proved that the smooth functions a and b are in opposite direction. In this section, we generalize the result proved by Mallick et al. [28].

An n-dimensional Riemannian or pseudo Riemannian manifold (M_n, g) is said to be $\mathcal{Z}-$ pseudosymmetric if the tensors $R \cdot \mathcal{Z}$ and $Q(g, \mathcal{Z})$ defined by

$$(R(X, Y) \cdot \mathcal{Z})(Z, W) = -\mathcal{Z}(R(X, Y)Z, W) - \mathcal{Z}(Z, R(X, Y)W) \tag{5.1}$$

and

$$Q(g, \mathcal{Z})(Z, W; X, Y) = -\mathcal{Z}((X \wedge_{\mathcal{Z}} Y)Z, W) - \mathcal{Z}(Z, (X \wedge_{\mathcal{Z}} Y)W) \tag{5.2}$$

are linearly dependent, *i.e.*

$$(R(X, Y) \cdot \mathcal{Z})(Z, W) = L_{\mathcal{Z}} Q(g, \mathcal{Z})(Z, W; X, Y), \tag{5.3}$$

for arbitrary vector fields X, Y, Z and W. Here $X \wedge_{\mathcal{Z}} Y$ denotes the endomorphism defined as

$$(X \wedge_{\mathcal{Z}} Y)Z = g(Y, Z)X - g(X, Z)Y \tag{5.4}$$

and $L_{\mathcal{Z}}$ is a smooth function holds on

$$U_{\mathcal{Z}} = \{x \in M : \mathcal{Z} \neq \frac{r}{n} g \ \ at \ \ x\}.$$

If we consider that $L_{\mathcal{Z}} = 0$, then the manifold (M_n, g) becomes $\mathcal{Z}-$semisymmetric.

In consequence of (5.1), (5.2), (5.3) and (5.4), we have

$$\mathcal{Z}(R(X,Y)Z,W) + \mathcal{Z}(Z,R(X,Y)W) = L_{\mathcal{Z}}\{g(Y,Z)\mathcal{Z}(X,W) \\ -g(X,Z)\mathcal{Z}(Y,W) + g(Y,W)\mathcal{Z}(Z,X) - g(X,W)\mathcal{Z}(Z,Y)\}. \tag{5.5}$$

In view of (1.2), (1.3) and (5.5), we find that

$$(k - L_{\mathcal{Z}})\{g(Y,Z)\mathcal{Z}(X,W) - g(X,Z)\mathcal{Z}(Y,W) \tag{5.6}$$
$$+g(Y,W)\mathcal{Z}(Z,X) - g(X,W)\mathcal{Z}(Z,Y)\} = 0, \tag{5.7}$$

which shows that either $k = L_{\mathcal{Z}}$ or

$$g(Y,Z)\mathcal{Z}(X,W) - g(X,Z)\mathcal{Z}(Y,W) \tag{5.8}$$
$$+g(Y,W)\mathcal{Z}(Z,X) - g(X,W)\mathcal{Z}(Z,Y) = 0. \tag{5.9}$$

With the help of (1.2) and (2.11), (5.8) becomes

$$b\{\eta(X)\eta(W)g(Y,Z) - \eta(Y)\eta(W)g(X,Z) \\ +\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)\} = 0.$$

Since $b \neq 0$, therefore above equation converts into

$$\eta(W)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} + \eta(Z)\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\} = 0.$$

Let $\{e_i, i = 1,2,...,n\}$, be an orthonormal basis of the tangent space at any point of the manifold (M_n, g). Then putting $X = W = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$g(Y,Z) = n\eta(Y)\eta(Z). \tag{5.10}$$

Also putting $Y = Z = \xi$ in (5.10) and using (1.2), we obtain $n = 1$, which is a contradiction. Hence we can state:

Theorem 5.1. *In an $n(>1)-$dimensional $\mathcal{Z}-$pseudosymmetric $N(k)-$quasi Einstein manifold $L_{\mathcal{Z}} = k$.*

If we suppose that the manifold (M_n, g) is $\mathcal{Z}-$semisymmetric, then $L_{\mathcal{Z}} = 0$. Thus we can state the following corollary:

Corollary 5.2. *The necessary and sufficient condition for an $n-$dimensional $N(k)$-quasi Einstein manifold (M_n, g) to be $\mathcal{Z}$-semisymmetric is that $k = 0$.*

From Theorem (4.2) and Corollary (5.2), we can conclude the following:

Corollary 5.3. *On an $n-$dimensional $N(k)-$quasi Einstein manifold, the following results are equivalent:*
(i) (M_n, g) is $\mathcal{Z}$-semisymmetric,
(ii) (M_n, g) satisfies $R(\xi, X) \cdot W^* = 0$,
(iii) the smooth functions a and b are in opposite direction.

6. $N(k)$-quasi Einstein manifolds satisfying $W^*(\xi, X) \cdot R = 0$

From (1.1), (1.2), (1.3), (2.2), (2.5) and (2.9) it follows that

$$W^*(\xi, Z)U = b[b^* g(Z,U)\xi - a^* \eta(U)Z + (a^* - b^*)\eta(U)\eta(Z)\xi] + \left[k + a(a^* + b^*) - \frac{c^* r}{n}\left(\frac{1}{n-1} + a^* + b^*\right)\right][g(U,Z)\xi - \eta(U)Z]. \quad (6.1)$$

It is obvious from (2.13) that

$$(W^*(\xi, X) \cdot R)(Y,Z)U = W^*(\xi, X)R(Y,Z)U - R(W^*(\xi, X)Y, Z)U - R(Y, W^*(\xi, X)Z)U - R(Y,Z)W^*(\xi, X)U. \quad (6.2)$$

Let us consider that the $N(k)$-quasi Einstein manifold satisfies $W^*(\xi, X) \cdot R = 0$, then (6.2) gives

$$0 = \eta(W^*(\xi, X)R(Y,Z)U) - \eta(R(W^*(\xi, X)Y, Z)U) - \eta(R(Y, W^*(\xi, X)Z)U) - \eta(R(Y,Z)W^*(\xi, X)U). \quad (6.3)$$

which yields after a straight forward calculation

$$0 = \left[k + b^* b + a(a^* + b^*) - \frac{c^* r}{n}\left(\frac{1}{n-1} + a^* + b^*\right)\right][R(Y,Z,U,X) + kg(Y,U)g(X,Z) - kg(X,Y)g(Z,U) + kg(X,Y)\eta(U)\eta(Z) - kg(X,Z)\eta(U)\eta(Y)] + \left[k + a^* b + a(a^* + b^*) - \frac{c^* r}{n}\left(\frac{1}{n-1} + a^* + b^*\right)\right] \times [kg(X,Z)\eta(U)\eta(Y) - kg(X,Y)\eta(Z)\eta(U)] \quad (6.4)$$

From (6.4), one can easily bring out for $a^* = b^*$ that

$$'R(Y,Z,U,X) = k\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}, \quad (6.5)$$

which informs that the manifold (M_n, g) is an Einstein space. Thus we state the following theorem:

Theorem 6.1. *There does not exit an $N(k)$-quasi Einstein manifold satisfying each of the curvature restriction $R(\xi, X) \cdot R = 0$, $E(\xi, X) \cdot R = 0$, $C(\xi, X) \cdot R = 0$, $\hat{C}(\xi, X) \cdot R = 0$ and $H(\xi, X) \cdot R = 0$*

Again, for $a^* \neq b^*$ that is for $P(\xi, X) \cdot R = 0$, we have

$$R(Y,Z,U,X) = k[g(X,Y)g(Z,U) - g(Y,U)g(X,Z)] + k\left[\frac{(n-1)k - b}{(n-1)k - a}\right][g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(U)\eta(Y)]. \quad (6.6)$$

Let $\{e_i, i = 1, 2, ..., n\}$, be an orthonormal basis of the tangent space at any point of the manifold (M_n, g). Then putting $X = Y = e_i$ in (6.6) and taking summation over i, $1 \le i \le n$, we get

$$S(Z,U) = (a+b)g(Z,U),$$

which shows that the manifold is class of Einstein manifold, a contradiction. Thus we can say that there does not exist an $N(k)-$quasi Einstein manifold with $P(\xi, X)\cdot R = 0$.

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