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Novel Models for Obtaining the Closest Weak and Strong Efficient Projections in Data Envelopment Analysis

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abstract: Data Envelopment Analysis (DEA) is a nonparametric method for measuring the relative efficiency and performance of Decision Making Units (DMUs). Determining the least distance efficiency measure and thereby identifying the best reference point, is an important issue in recent DEA literature. In this paper, two alternative target setting models based on quadratically constrained programming (QCP), have been developed to allow for the low efficient DMUs to find the easiest way to improve theirs efficiency and reach the efficient boundary. One model seeks the closest weak efficient projection and the other suggests the most appropriate direction towards the strong efficient frontier surface. Both of these models provide the closest projection in one stage. Finally, a proposed problem is empirically checked by using recent data from thirty European airports.

Key Words: Decision Making Units, Data Envelopment Analysis, Efficiency, Closest efficient targets.

Contents

1. Introduction

Analyzing the performance and efficiency of organizations is a crucial challenge of the modern business. Data Envelopment Analysis (DEA) measures the relative efficiency of a number of homogenous Decision Making Units (DMUs) using some inputs to produce some outputs. See [9,19] for more details.

To evaluate the performance of the DMUs in DEA framework, we assume some axioms. DEA considers a set of input-output vectors as the production possibility

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set (PPS) assuming that all the units can convert their inputs into their outputs under these axioms. The efficiency score of DMUs is estimated by investigating how well the outputs are obtained. Therefore, DEA classifies the units of PPS into the efficient and inefficient units. The efficient units lead to the maximum achievable outputs from a given input set (or alternatively the minimum inputs necessary to produce the given outputs). In other words, an efficient unit has no potential improvement, whereas an inefficient unit can reach the efficient boundary by deleting its input excess and/or augmenting the output shortfall, thereby yielding an efficient target. Hence, the input-output vectors on the efficient boundary can play the role of criteria for the inefficient DMUs to apply and indicate keys for improving the performance of an inefficient DMU. Therefore, whenever the projection point on the efficient boundary is closer to an inefficient DMU, it is easier to remove its inefficiency (i.e., less changes in its inputs and/or outputs is required). Additionally, it must be stated that any efficient point can be weak efficient point or strong efficient point. For a weak efficient point, it is impossible to improve all its inputs and outputs in PPS; and for a strong efficient point, it is not possible to improve any of its inputs or outputs in PPS without worsening some other inputs or outputs. Researchers in the field of DEA have been interested in finding efficient targets on the frontier of PPS.

In this regard, by using the weighting scheme provided by the dual prices for calculating a composite point on the frontier, Sherman and Gold [24] determined a projection onto the efficient frontier. However, the closest efficient point was not obtained in their work. Briec [6] and Briec and Lesourd [7] obtained the minimum distance to the efficient frontier and some duality results by using the family of Hölder norms. Although, the models discussed in [6] and [7] are linear, they considered the weak efficient frontier and their models may not obtain the minimum distance of units to the strong efficient frontier. Moreover, these models only just are applicable to the linear norms. In terms of Euclidean norm, Frei and Harker [11] found projection points by minimizing the distance from inefficient DMU to each hyperplane of the efficient frontier. Cherchye and Van Puyenbroeck [8] determined the deviation of the observed input vector and the corresponding reference point in terms of the cosine of the angle, and maximized the cosine to obtain the efficient target. Gonzalez and Alvarez [14] introduced the concept of input-specific contractions and found a relevant benchmark for inefficient units by minimizing this contraction in the inputs required to reach the efficient subset. Nevertheless, it does not guarantee to reach the strong efficient frontier. Lozano and Villa [20] denoted a gradual efficiency improvement strategy that determines a sequence of distanced targets and ends in the efficient frontier. Moreover, targets obtained by [20] is not necessarily the closest projection to assessed unit. Aparicio et al. [5] proposed a single-stage procedure based on solving only one mixed-integer zero-one programming problem to obtain the minimum distance of DMUs from the strong efficient frontier (the set of all strong efficient points) of a PPS. However, Jahanshahloo et al. [16] used a linear bilevel programming problem. Moreover, to obtain the minimum distance of the units from the weak efficient boundary of a convex PPS, Jahanshahloo et al. [15] provided some linear models using l_1 and l_∞ norms and they applied the calculated distances to evaluate the group performance of DMUs. Finally, Amirteimoori and Kordrostami [1] proposed an Euclidean distance based measure of efficiency by stating that it searches the shortest path to the efficient frontier of PPS. However, Aparicio and Pastor [4] showed by a counterexample that Amirteimoori and Kordrostami's method does not necessarily generate the nearest efficient point and even it may fall out of PPS. For more details see [21,22,2,12,23,3].

In this paper, two models, whose all constraints are linear except one, are proposed to calculate the minimum distance from the weak and strong efficient frontiers. Although the presented models are not linear, they have the following advantages:

- Proposed models select the most appropriate efficient target for the evaluated DMUs in one stage directly with the least effort.
- Our models have a few constraints in comparison with some similar models in obtaining the closest efficient targets in one stage.
- Mentioned models can be applied for each arbitrary distance function and efficiency measurements; for instance, $\|.\|_p, p \in [1, \infty]$, the RAM (Range Adjusted Measured) efficiency measure in Cooper et al. [10], the SBM (Slack Based Measure), BRWZ measure by Silva [21] and etc.
- There exist several algorithms for solving quadratically constrained programming (QCP) problems efficiently. In fact, QCP is closely related to other fields in mathematical programming. Hence, the results that are already achieved and will be achieved in the future in these fields of research could be beneficial for obtaining the closest efficient targets.
- The proposed models discuss both weak and strong efficient frontiers.
- Due to the flexibility of the proposed models, monotonicity, translation and unit invariant features can be considered in these models.

Finally, a numerical example will be discussed through one of the proposed models. The numerical example has been taken from reference [25] and it will be discussed again in here in order to compare our model with the DFM method in the referenced work.

The paper is organized as follows. DEA is reviewed in Section 2. The main results of the paper are presented in Section 3. A numerical example is provided and the proposed model is compared with the DFM method [25] in Section 4. Finally, conclusions are made in Section 5.

2. Preliminaries

In this section, we first present some preliminaries about DEA. Consider a set of n observed DMUs, $\{\text{DMU}_1,\text{DMU}_2,\ldots,\text{DMU}_n\}$. Assume that each DMU_j $(j =$

$1, 2, \ldots, n$) produces s outputs $\mathbf{y}_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^{\mathrm{t}} \in \mathbb{R}^s_+,\ \mathbf{y}_j \neq \mathbf{0}_s$, using m inputs $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^{\mathrm{t}} \in \mathbb{R}^m_+,\ \mathbf{x}_j \neq \mathbf{0}_m$.
The performance of each DMU is evaluated with respect to the efficient boundary of the so-called production possibility set. The mathematical form of the two most famous PPSs are

$$T_c = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \,|\, \mathbf{x} \geq \sum_{j=1}^{n} \lambda_j \mathbf{x}_j,\ \mathbf{y} \leq \sum_{j=1}^{n} \lambda_j \mathbf{y}_j,\ \lambda_j \geq 0;\ j = 1, 2, \ldots, n\}$$

$$T_v = \left\{ \begin{array}{c} (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \,|\, \mathbf{x} \geq \sum_{j=1}^{n} \lambda_j \mathbf{x}_j,\ \mathbf{y} \leq \sum_{j=1}^{n} \lambda_j \mathbf{y}_j, \\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0;\ j = 1, 2, \ldots, n \end{array} \right\}.$$

which correspond to constant and variable returns to scale (CRS and VRS), respectively. Now, two concepts related to the efficient frontier are defined as follows.

Definition 2.1. $(\mathbf{x}, \mathbf{y}) \in T$ *is called a weak efficient point if there is no other* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in T$ *such that* $(\bar{\mathbf{x}}, -\bar{\mathbf{y}}) < (\mathbf{x}, -\mathbf{y})$.

Definition 2.2. $(\mathbf{x}, \mathbf{y}) \in T$ *is called a strong efficient point if there is no other* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in T$ *such that* $(\bar{\mathbf{x}}, -\bar{\mathbf{y}}) \leq (\mathbf{x}, -\mathbf{y})$ *and strict inequality holds in at least one component.*

Denoting the weak efficient frontier by $\partial^{\mathrm{W}}(T)$ and the strong efficient frontier by $\partial^{\mathrm{S}}(T)$, we have

$$\partial^{\mathrm{S}}(T) \subseteq \partial^{\mathrm{W}}(T).$$

If $(\mathbf{x}, \mathbf{y}) \in T$ and $(\mathbf{x}, \mathbf{y}) \notin \partial^{\mathrm{W}}(T)$, $(\mathbf{x}, \mathbf{y})$ is called an inefficient point.

There are a lot of models in DEA for measuring technical efficiency of DMUs. Let $(\mathbf{x}_o, \mathbf{y}_o) \in T_c$ be an input-output vector under consideration. The input-oriented CCR multiplier model corresponding to $(\mathbf{x}_o, \mathbf{y}_o)$ is as follows:

$$\begin{array}{ll} \max & \mathbf{u}^{\mathrm{t}} \mathbf{y}_o \\ \text{s.t.} & \mathbf{v}^{\mathrm{t}} \mathbf{x}_o = 1 \\ & \mathbf{u}^{\mathrm{t}} \mathbf{y}_j - \mathbf{v}^{\mathrm{t}} \mathbf{x}_j \leq 0; \quad j = 1, 2, \ldots, n \\ & \mathbf{u} \geq \mathbf{0} \\ & \mathbf{v} \geq \mathbf{0}. \end{array} \tag{2.1}$$

If there is a solution $(\mathbf{u}^*, \mathbf{v}^*)$ for Model (2.1) such that $\mathbf{u}^{*\mathrm{t}} \mathbf{y}_o = 1$, then $(\mathbf{x}_o, \mathbf{y}_o)$ is a weak efficient point. Besides the optimality value 1, $(\mathbf{u}^*, \mathbf{v}^*) > (\mathbf{0}, \mathbf{0})$ implies the strong efficiency of $(\mathbf{x}_o, \mathbf{y}_o)$. It is worth to note that one of the observed DMUs is a strong efficient unit. Another model is the well-known additive model which can be formulated under constant returns to scale as follows:

$$\begin{aligned}
\max \quad & \mathbf{1s}^- + \mathbf{1s}^+ \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j \mathbf{x_j} + \mathbf{s}^- = \mathbf{x_o} \\
& \sum_{j=1}^{n} \lambda_j \mathbf{y_j} - \mathbf{s}^+ = \mathbf{y_o}; \\
& \boldsymbol{\lambda} \geq \mathbf{0} \\
& \mathbf{s}^- \geq \mathbf{0} \\
& \mathbf{s}^+ \geq \mathbf{0}.
\end{aligned} \tag{2.2}$$

Note that Model (2.2) maximizes $\|.\|_1$ distance from the $(\mathbf{x}_o, \mathbf{y}_o)$ to the strong efficient frontier.

Now, the target setting for an inefficient DMU is discussed. Let DMU under evaluation be inefficient. It is possible to determine a projection on the efficient frontier through decreasing its inputs or/and augmenting its outputs. The coordinates of this projection on the efficient frontier will be the targets for the mentioned DMU. With the oriented framework, for inefficient DMU, increasing the outputs without requiring any decrease in the inputs is called output-oriented inefficiency. Alternatively, decreasing the inputs without requiring any increase in the outputs is called input-oriented inefficiency. In most applications, in practice, technical efficiency measure includes both input-saving and output-expanding components which is called non-oriented. In DEA models the distance between DMUs and the efficient frontier of the PPS is sometimes approximated to evaluate the performance of DMUs, thereby determining the efficient targets. Sometimes the furthest projection is obtained from the traditional DEA models such as Model (2.2) for inefficient unit under evaluation due to the maximizing slack variables. However, recently some authors argue that the distance should be minimized instead of maximized in order to find targets as similar as possible to the inefficient DMU under evaluation. The idea behind this viewpoint is that the closer the efficient projection to the DMU under evaluation, the easier it is to reach the efficient frontier with less variation in its inputs and outputs. In this regard, on one hand, some researchers focus their works on finding all defining hyperplanes to obtain projection points which is NP-hard from computational point of view [17,18]. On the other hand, some researchers attempt to present a mathematical programming problem [5,16]. Similarly, this alternative is again associated with a NP-hard problem. Therefore, target settings have not been satisfactorily found from the viewpoint of computational complexity and further effort is required to develop new models in order to solve the problem. In the next section, two models will be proposed which can be solved easier for finding the closest weak and strong targets. All of the constraints in these models are linear except one.

3. Models based on the least distance to the efficient boundary of PPS

Having stated, we are interested in identifying of the targets on the weak and strong efficient frontier for each inefficient decision making unit.

3.1. Model for determining the closest weak efficient projection

Here, a new model is proposed to obtain the targets on the weak efficient frontier based on arbitrary norm. Consider the following pair of primal and dual programming problems corresponding to $(\mathbf{x}, \mathbf{y})$.

Primal

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i - t, \quad i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r + t, \quad r = 1, 2, \ldots, s \\ & \lambda_j \geq 0, \qquad\qquad\qquad j = 1, 2, \ldots, n \\ & t \geq 0. \end{aligned} \tag{3.1}$$

Dual

$$\begin{aligned} \min \quad & -(\sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i) \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, 2, \ldots, n \\ & \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i \geq 1 \\ & u_r \geq 0, \qquad\qquad r = 1, 2, \ldots, s \\ & v_i \geq 0, \qquad\qquad i = 1, 2, \ldots, m. \end{aligned} \tag{3.2}$$

Consider $(\boldsymbol{\lambda}^*, t^*)$ and $(\mathbf{u}^*, \mathbf{v}^*)$ as the optimal solutions of Models (3.1) and (3.2), respectively. The following lemma and theorem are provided to validate these models.

Lemma 3.1. $(\mathbf{x}, \mathbf{y}) \in \partial^W(T_c)$ *if and only if* $t^* = 0$.

Proof: To prove necessary condition, let $(\mathbf{x}, \mathbf{y}) \in \partial^{\mathrm{W}}(T_c)$ and by contradiction suppose that $t^* \neq 0$. Since $(\boldsymbol{\lambda}^*, t^*)$ is an optimal solution of Model (3.1), $(\mathbf{x} - \mathbf{1}_m t^*, \mathbf{y} + \mathbf{1}_s t^*) \in T_c$ and dominates $(\mathbf{x}, \mathbf{y})$ which contradict the assumption of the weak efficiency of $(\mathbf{x}, \mathbf{y})$.
For the sufficient condition, let $t^* = 0$ and by contradiction assume that $(\mathbf{x}, \mathbf{y}) \notin \partial^{\mathrm{W}}(T_c)$. Thus, there exists $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in T_c$ such that $\bar{\mathbf{x}} < \mathbf{x}$ and $\bar{\mathbf{y}} > \mathbf{y}$.
Assume that $\bar{\mathbf{x}} + \mathbf{1}_m \bar{t} \leq \mathbf{x}$ and $\bar{\mathbf{y}} - \mathbf{1}_s \bar{t} \geq \mathbf{y}$ where $\bar{t} \neq 0$. Since $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in T_c$, there exists $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^n$ such that

$$\begin{cases} \bar{\boldsymbol{\lambda}}\mathbf{X} \leq \bar{\mathbf{x}} \\ \bar{\boldsymbol{\lambda}}\mathbf{Y} \geq \bar{\mathbf{y}} \end{cases}$$

where $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$ and $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s]$. It is clear that $(\bar{\boldsymbol{\lambda}}, \bar{t})$ is a feasible solution of Model (3.1) and this is contradictory with $t^* = 0$. □

Theorem 3.2. *$(\mathbf{x}, \mathbf{y}) \in \partial^W(T_c)$ if and only if $(\mathbf{x}, \mathbf{y})$ satisfies in the following equations.*

$$\begin{array}{ll}
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, & i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, & r = 1, 2, \ldots, s \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, & j = 1, 2, \ldots, n \\
\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i \geq 1, & \\
\sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i = 0, & \\
u_r \geq 0, & r = 1, 2, \ldots, s \\
v_i \geq 0, & i = 1, 2, \ldots, m \\
\lambda_j \geq 0, & j = 1, 2, \ldots, n \\
y_r \geq 0, & r = 1, 2, \ldots, s.
\end{array} \tag{3.3}$$

Proof: Considering the duality strong theorem for both of the above primal (3.1) and dual (3.2) Models and $t = 0$ in the optimality, the proof is trivial. □

Now, consider the following model.

$$\begin{array}{llll}
\min & \|(\mathbf{x}_o, \mathbf{y}_o) - (\mathbf{x}, \mathbf{y})\| & & (3.4.1) \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, & i = 1, 2, \ldots, m & (3.4.2) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, & r = 1, 2, \ldots, s & (3.4.3) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, & j = 1, 2, \ldots, n & (3.4.4) \\
& \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i \geq 1, & & (3.4.5) \\
& \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i = 0, & & (3.4.6) \\
& v_i \geq 0, & i = 1, 2, \ldots, m & (3.4.7) \\
& u_r \geq 0, & r = 1, 2, \ldots, s & (3.4.8) \\
& \lambda_j \geq 0 & j = 1, 2, \ldots, n & (3.4.9) \\
& x_i \geq 0, & i = 1, 2, \ldots, m & (3.4.10) \\
& y_r \geq 0, & r = 1, 2, \ldots, s. & (3.4.11)
\end{array} \tag{3.4}$$

Suppose that $(\mathbf{v}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \mathbf{x}^*, \mathbf{y}^*)$ be the optimal solution to Model (3.4). The optimal value and solution of this model provide the minimum distance and the closest weak efficient target of $(\mathbf{x}_o, \mathbf{y}_o)$ by an arbitrary measure $\|.\|$, respectively. Utilizing a linear norm such as $\|.\|_1$ and $\|.\|_\infty$ of p-norms, the objective function can be converted to a linear function.

3.2. Model for determining the closest strong efficient projection

In this section, we propose a new model to identify a strong efficient projection. This approach is based on minimizing the distance of a point to the strong efficient frontier by $\|.\|$. It is a Quadratically Constrained Programming problem which identifies a new efficiency improvement projection to a given inefficient unit within the CRS technology of DEA. The presented approach leads to the closest strong efficient targets by means of a single stage procedure which is easy to perform and directly yields the closest strong projection unit.

Let $(\mathbf{x}_o, \mathbf{y}_o) \in T_c$ be an inefficient point under evaluation and let $(\mathbf{v}^*, \mathbf{u}^*, \lambda^*, \mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution of Model (3.5). Then $(\mathbf{x}^*, \mathbf{y}^*)$ is the closest strong efficient projection to $(\mathbf{x}_o, \mathbf{y}_o)$:

$$
\begin{array}{lllr}
\min & \|(\mathbf{x}_o, \mathbf{y}_o) - (\mathbf{x}, \mathbf{y})\| & & (3.5.1) \\
\text{s.t.} & \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i = 0 & & (3.5.2) \\
 & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, & j = 1, 2, \ldots, n & (3.5.3) \\
 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, & i = 1, 2, \ldots, m & (3.5.4) \\
 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, & r = 1, 2, \ldots, s & (3.5.5) \\
 & u_r \geq 1, & r = 1, 2, \ldots, s & (3.5.6) \\
 & v_i \geq 1, & i = 1, 2, \ldots, m & (3.5.7) \\
 & \lambda_j \geq 0, & j = 1, 2, \ldots, n & (3.5.8) \\
 & x_i \geq 0, & i = 1, 2, \ldots, m & (3.5.9) \\
 & y_r \geq 0, & r = 1, 2, \ldots, s. & (3.5.10)
\end{array}
\quad (3.5)
$$

In fact, if $(\mathbf{x}^*, \mathbf{y}^*, \lambda^*, \mathbf{u}^*, \mathbf{v}^*)$ is an optimal solution of Model (3.5), then $(\frac{\mathbf{u}^*}{\mathbf{v}^{*t}\mathbf{x}^*}, \frac{\mathbf{v}^*}{\mathbf{v}^{*t}\mathbf{x}^*})$ is an optimal solution of the input-oriented CCR multiplier model corresponding to $(\mathbf{x}^*, \mathbf{y}^*)$ with the optimal value 1.

Lemma 3.3. *Problem (3.5) is feasible.*

Proof: It is known that one of the observed DMUs such as DMU_k is strong efficient. So, there exist an optimal solution $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) > 0$ to Model (2.1) corresponding to DMU_k. Thus, $\bar{\mathbf{u}}^t \mathbf{y}_k - \bar{\mathbf{v}}^t \mathbf{x}_k = 0$ and the constraints (3.5.3), (3.5.6) and (3.5.7) are satisfied at $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ (Note that $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ can be multiplied by a large number to satisfy (3.5.6) and (3.5.7) if it is necessary). On the other hand, since

$DMU_k \in T_c$, there exists $\bar{\lambda} \in \mathbb{R}^n_+$ such that $\mathbf{X}\bar{\lambda} \leq \mathbf{x}_k$, $\mathbf{Y}\bar{\lambda} \geq \mathbf{y}_k$. Therefore, $(\mathbf{u}, \mathbf{v}, \boldsymbol{\lambda}, \mathbf{x}, \mathbf{y}) = (\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\lambda}, \mathbf{x}_k, \mathbf{y}_k)$ is a feasible solution of Model (3.5). □

Theorem 3.4. *For the optimal solution of Model (3.5), $(\mathbf{u}^*, \mathbf{v}^*, \lambda^*, \mathbf{x}^*, \mathbf{y}^*)$, $(\mathbf{x}^*, \mathbf{y}^*)$ is the closest strong efficient projection to the unit under evaluation $(\mathbf{x}_o, \mathbf{y}_o)$.*

Proof: It is obvious that Problem (3.5) is minimizing the distance between the given point $(\mathbf{x}_o, \mathbf{y}_o)$ and the point $(\mathbf{x}, \mathbf{y})$ by the $\|.\|$ norm. Constraints (3.5.4), (3.5.5) and (3.5.8) along with Constraints (3.5.6) and (3.5.7) implies that in the optimality, $(\mathbf{x}, \mathbf{y})$ is a strong efficient unit. □

3.3. Minimum distance measure p-norm and monotonicity

Monotonicity of efficiency measure is an indispensable property for any technical efficiency measure. In other words, an axiomatic approach for finding closest targets needs the efficiency measure satisfies weak or strong monotonicity. In detail, the measure should not provide a better evaluation score to an inferior input-output vector in the PPS than to a superior one. However, as far as authors know, regarding the practical least distance p-norm ($p \in [1, \infty]$) measure in DEA literature, satisfying these norms both the weak and strong monotonicity on the weak efficient, there was no approach that meet strong monotonicity on the strongly efficient frontier. To overcome this, several suggested solutions have been presented. Ando et al. [2] gave weakly monotonic least distance measure with the incorporation of a free disposable set and showed that it satisfies weak monotonicity over the strongly efficient frontier. Using extended efficient faces and based on full dimensional effcient facets instead of standard PPS, Aparicio and Pastor [3] developed an output-oriented strongly monotonic least distance measure. Fukuyama et al. [12] by extending the free disposable set defined in [2] and introducing the so-called tradeoff set, developed a least distance p-norm inefficiency (efficiency) measure satisfying strong monotonicity over the strongly efficient portion of the boundary. For our model, first, we apply the free disposable set in [2] which guarantees the weak monotonicity of the p-norm ($p \in [1, \infty]$) measure on the strong efficient frontier. Therefore, Model (3.5) can be revised as follows:

$$\min \|(\bar{\mathbf{x}}, \bar{\mathbf{y}}) - (\mathbf{x}, \mathbf{y})\| \qquad (3.6.1) \qquad (3.6)$$

$$\text{s.t.} \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i = 0 \qquad (3.6.2) \qquad (3.7)$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, 2, \ldots, n \ (3.6.3) \qquad (3.8)$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \qquad i = 1, 2, \ldots, m \ (3.6.4) \qquad (3.9)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \qquad r = 1, 2, \ldots, s \ (3.6.5) \qquad (3.10)$$

$$\bar{x}_i \geq x_{io}, \quad i = 1, 2, \ldots, m \ (3.6.6) \qquad (3.11)$$
$$\bar{y}_r \leq y_{ro}, \quad r = 1, 2, \ldots, s \ (3.6.7) \qquad (3.12)$$
$$u_r \geq 1, \quad r = 1, 2, \ldots, s \ (3.6.8) \qquad (3.13)$$
$$v_i \geq 1, \quad i = 1, 2, \ldots, m \ (3.6.9) \qquad (3.14)$$
$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n \ (3.6.10) \qquad (3.15)$$
$$x_i \geq 0, \quad i = 1, 2, \ldots, m \ (3.6.11) \qquad (3.16)$$
$$y_r \geq 0, \quad r = 1, 2, \ldots, s. \ (3.6.12) \qquad (3.17)$$

Second, for the sake of strong monotonicity on the strong efficient frontier, we are going to utilize the tradeoff set which is introduced by Fukuyama et al [12]. They extended free disposable set in [2] with the incorporation of the coefficient ε. For more details about choice of the positive value ε, see [13].

$$\begin{array}{lll}
\min & \|(\bar{\mathbf{x}}, \bar{\mathbf{y}}) - (\mathbf{x}, \mathbf{y})\| & (3.18.1) \\
\text{s.t.} & \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_i = 0 & (3.18.2) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, 2, \ldots, n & (3.18.3) \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \quad i = 1, 2, \ldots, m & (3.18.4) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \quad r = 1, 2, \ldots, s & (3.18.5) \\
& \bar{x}_i = x_{io} + d_i^x, \quad i = 1, 2, \ldots, m & (3.18.6) \\
& \bar{y}_r = y_{ro} - d_r^y, \quad r = 1, 2, \ldots, s & (3.18.7) \\
& d_k^x + \varepsilon(\sum_{i=1, i\neq k}^{m} d_i^x) \geq 0 \quad k = 1, 2, \ldots, m & (3.18.8) \\
& d_k^y + \varepsilon(\sum_{r=1, r\neq k}^{s} d_r^y) \geq 0 \quad k = 1, 2, \ldots, s & (3.18.9) \\
& u_r \geq 1, \quad r = 1, 2, \ldots, s & (3.18.10) \\
& v_i \geq 1, \quad i = 1, 2, \ldots, m & (3.18.11) \\
& \lambda_j \geq 0, \quad j = 1, 2, \ldots, n & (3.18.12) \\
& x_i \geq 0, \quad i = 1, 2, \ldots, m & (3.18.13) \\
& y_r \geq 0, \quad r = 1, 2, \ldots, s. & (3.18.14)
\end{array} \qquad (3.18)$$

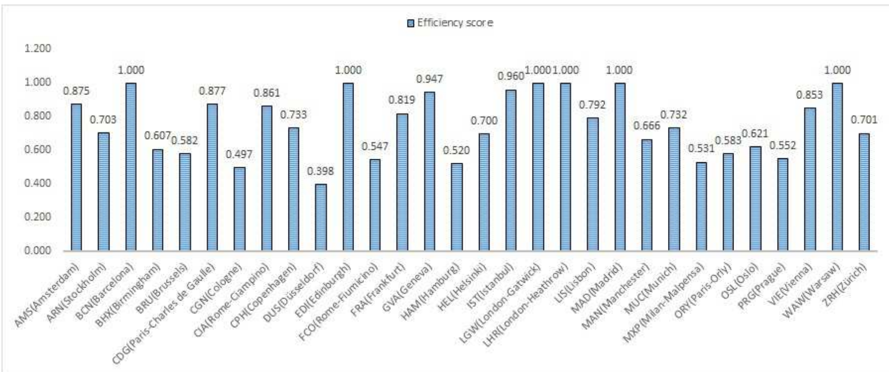
Finally, it is notable that all previous results also hold in the case of the variable returns to scale technology.

4. Empirical illustration

In the previous section two new models were proposed for determining the weak and strong efficient targets. However, the strong efficient projections is more

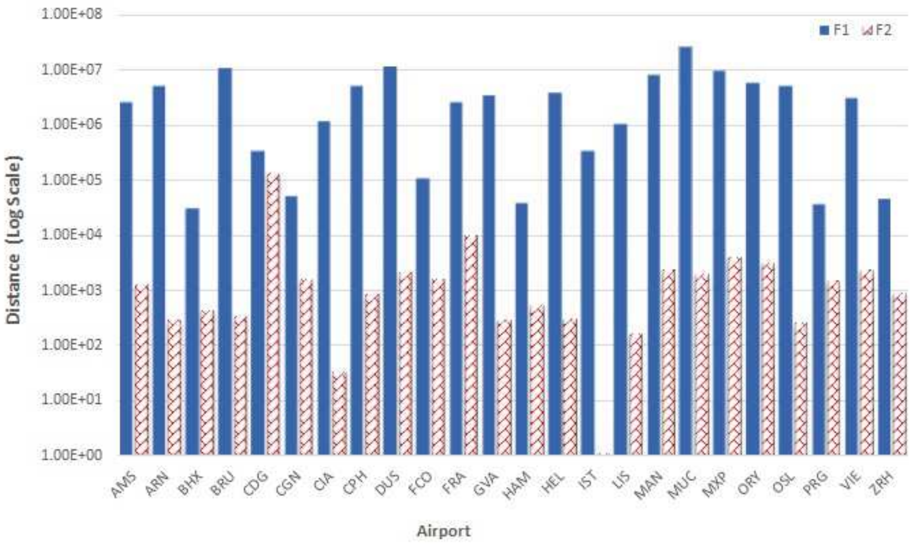
applicable and we are rather interested in finding the strong efficient target. Therefore, to verify the proposed model, we introduce an empirical illustration based on a real data set of 30 selected European airports (which was taken from Suzuki et al.'s paper [25]. Suzuki et al.'s paper propose a multi-stage Distance Friction Minimization (DFM) approach to generate an appropriate efficiency improving projection from an inefficient DMU to the strongly efficient frontier. Additionally, projection function for efficiency improvement is given by a Multiple Objective Quadratic Programming (MOQP) model. Therefore, our proposed model and Suzuki et al.'s method (DFM) are compared in terms of the role of the model. The data consists of four inputs and two outputs as follows:
[*Input* 1] RN: Number of runways.
[*Input* 2] TS: Terminal space (m^2).
[*Input* 3] GN: Number of gates.
[*Input* 4] EN: Number of employees.
[*Output* 1] PN: Number of passengers.
[*Output* 2] AM: Aircraft movements.
Figure 1 demonstrates the efficiency evaluation results of these airports through CCR-I model.
Six of these airports are determined efficient and for the rest of airports, we apply

Figure 1: Efficiency score of 30 airports.



both Model (3.5) and DFM method Suzuki et al. [25] to identify a target on the strong efficient frontier. The results are depicted in Table 1 and Figure 2. Note that $F1$ and $F2$ in Table 1 refer to the Euclidean distance of the observed DMUs to the projections produced by the DFM method and proposed model, respectively. we have also reported the percentages of improvements which is required for inefficient airport to reach the efficiency. Therefore, we can observe a significant difference between the two DFM and proposed approaches. The other thing that needs to be highlighted from this comparison is that DFM method is based on the Euclidean distance. So, we used $\|.\|_2$ in the objection function of Model (3.5). However, one can use linear norm such norms, $\|.\|_1$ or $\|.\|_\infty$. Finally, for the sake of making the comparison easier between targets, Figure 2 is depicted which shows that the proposed model gives a closer strong efficient projection than the DFM method.

Figure 2: Comparison of the DFM method and proposed model.



5. Conclusion

In the context of data envelopment analysis, several models have been proposed to improve the efficiency. However, their drawbacks deemed researchers seek recent better models. In this paper, we developed two new models based on the weak and strong efficient frontier to locate the coordinates of efficient projection points for inefficient DMU which is under evaluation. They involve solving only one programming problem that directly yields the closest efficient targets and guarantee that the closest efficient target is reached. Also, some discussions are made to satisfy both weak and strong monotonicity which this is done by revising the presented model. Finally, an empirical illustration for 30 airports was applied and the results of the proposed Quadratically Constrained Quadratic Programming model (for $\|.\|_2$) and DFM method were compared to show that the proposed model in this paper is more applicable than the DFM one.

Table 1: The projections produced by the DFM method and proposed model and their distances from the DMU under evaluation.

I/O	Data	DFM method	F1	proposed model	F2	I/O	Data	DFM method	F1	proposed model	F2
DMU: AMS						DMU: ARN					
(I) RN	5	0.0%	2.7×10^6	-4.2%	1.3×10^3	(I) RN	3	-27.9%	5.3×10^6	-0.2%	2.9×10^2
(I) TS	370000	-63.4%		0.0%		(I) TS	108700	-28.2%		0.0%	
(I) GN	89	-8.8%		-3.5%		(I) GN	61	-26.9%		0.0%	

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Table 1: The projections produced by the DFM method and proposed model and their distances from the DMU under evaluation.

I/O	Data	DFM method	F1	proposed model	F2	I/O	Data	DFM method	F1	proposed model	F2
(I) EN	2231	0.0%		-59.6%		(I) EN	700	0.0%		-41.5%	
(O) PN	39960400	6.7%		0.0%		(O) PN	15100000	35.4%		0.0 %	
(O) AM	392997	43.0%		0.0%		(O) AM	228000	17.4%		0.0%	
DMU: BHX						DMU: BRU					
(I) RN	2	0.0%	3.1×10^4	-0.9%	4.5×10^2	(I) RN	3	-29.4%	1.1×10^7	-0.2%	3.5×10^2
(I) TS	66488	0.0%		0.0%		(I) TS	190804	-35.2%		0.0%	
(I) GN	31	-34.5%		0.0%		(I) GN	109	-41.6%		0.0%	
(I) EN	685	-10.7%		-66.2%		(I) EN	786	0.0%		-44.1%	
(O) PN	9079172	0.0%		0.0%		(O) PN	15192952	75.2%		0.0%	
(O) AM	128740	24.5%		0.0%		(O) AM	244633	26.4%		0.0 %	
DMU: CDG						DMU: CGN					
(I) RN	4	0.0%	3.4×10^5	0.0	1.4×10^5	(I) RN	3	0.0%	5.1×10^4	0.0%	1.6×10^3
(I) TS	542300	-62.3%		-25.8%		(I) TS	204000	0.0%		0.0%	
(I) GN	124	-12.3%		-16.5%		(I) GN	40	-35.2%		0.0%	
(I) EN	4071	-57.2%		-71.9%		(I) EN	1890	0.0%		-85.4%	
(O) PN	48122038	0.0%		0.0%		(O) PN	7758000	0.0%		0.0%	
(O) AM	515025	7.7%		0.1%		(O) AM	153372	33.6%		0.0%	
DMU: CIA						DMU: CPH					
(I) RN	1	-66.3%	1.2×10^6	-0.1%	3.3×10^1	(I) RN	3	-20.8%	5.3×10^6	-36.0%	8.5×10^2
(I) TS	10320	0.0%		0.0%		(I) TS	90300	0.0%		0.0%	
(I) GN	7	-11.5%		0.0%		(I) GN	106	-52.0%		0.0%	
(I) EN	136	-13.3%		-24.8%		(I) EN	1375	-45.6%		-62.0%	
(O) PN	1794285	65.7%		0.0%		(O) PN	17714007	29.9%		0.0%	
(O) AM	37130	7.4%		0.0%		(O) AM	259002	15.4%		0.0%	
DMU: DUS						DMU: FCO					
(I) RN	3	-45.4%	1.2×10^7	-0.2%	2.1×10^3	(I) RN	4	-29.7%	1.1×10^5	0.0%	1.6×10^3
(I) TS	231000	-46.4%		0.0%		(I) TS	285000	0.0%		0.0 %	
(I) GN	104	-40.6%		0.0%		(I) GN	107	-28.8%		0.0%	
(I) EN	2394	-53.7%		-86.0%		(I) EN	2200	0.0%		-73.4%	
(O) PN	14276045	81.9%		0.0%		(O) PN	26284759	0.0%		0.0%	
(O) AM	186159	43.1%		0.0%		(O) AM	300831	33.8%		0.0%	
DMU: FRA						DMU: GVA					
(I) RN	3	0.0%	2.6×10^6	0.0%	1.0×10^4	(I) RN	1	-4.3%	3.5×10^6	-4.4%	3.0×10^2
(I) TS	800000	-69.5%		0.0%		(I) TS	53000	0.0%		0.0%	
(I) GN	147	-17.6%		-29.5%		(I) GN	30	-8.2%		0.0%	
(I) EN	13006	-82.0%		-79.6%		(I) EN	540	-43.0%		-55.6%	
(O) PN	48359320	5.3%		0.0%		(O) PN	8048698	44.1%		0.0%	
(O) AM	458865	10.0%		0.0%		(O) AM	133312	2.7%		0.0%	
DMU: HAM						DMU: HEL					
(I) RN	2	-43.9%	4.0×10^4	-54.3%	5.5×10^2	(I) RN	3	-45.7%	4.0×10^6	0.0%	3.1×10^2
(I) TS	68300	0.0%		0.0%		(I) TS	110000	-59.3%		0.0%	
(I) GN	50	0.0%		0.0%		(I) GN	38	-26.8%		0.0%	
(I) EN	777	0.0%		-70.7%		(I) EN	594	0.0%		-51.7%	

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Table 1: The projections produced by the DFM method and proposed model and their distances from the DMU under evaluation.

I/O	Data	DFM method	F1	proposed model	F2	I/O	Data	DFM method	F1	proposed model	F2
(O) PN	9529924	0.0%		0.0%		(O) PN	9710920	41.8%		0.0%	
(O) AM	126878	31.6%		0.0%		(O) AM	159520	17.6%		0.0%	
DMU: IST						DMU: LIS					
(I) RN	2	-5.5%	3.5×10^5	-7.4%	1.1×10^0	(I) RN	2	-43.0%	1.1×10^6	0.0 %	1.7×10^2
(I) TS	236250	-84.0%		0.0%		(I) TS	140775	-55.3%		0.0%	
(I) GN	27	-2.0%		-4.0%		(I) GN	25	-13.4%		0.0%	
(I) EN	975	-22.5%		0.0%		(I) EN	389	0.0%		-44.9%	
(O) PN	14030000	2.0%		0.0%		(O) PN	9636400	11.6%		0.0%	
(O) AM	161827	25.7%		0.0%		(O) AM	112500	18.1%		0.0%	
DMU: MAN						DMU: MUC					
(I) RN	2	-30.6%	8.5×10^6	-12.6%	2.4×10^3	(I) RN	2	-15.5%	2.6×10^7	-26.8%	2.0×10^3
(I) TS	136400	-5.8%		0.0%		(I) TS	458000	-44.6%		0.0	
(I) GN	103	-38.0%		0.0%		(I) GN	210	-42.8%		0.0%	
(I) EN	2852	-50.7%		-84.6%		(I) EN	4891	-30.9%		-40.1%	
(O) PN	19699256	43.5%		0.0%		(O) PN	24193304	110.1%		0.0%	
(O) AM	207118	20.1%		0.0%		(O) AM	343027	15.5%		0.0%	
DMU: MXP						DMU: ORY					
(I) RN	2	-12.8%	9.9×10^6	0.0%	4.1×10^3	(I) RN	3	0.0%	5.9×10^6	-33.7%	3.2×10^3
(I) TS	329000	-60.0%		0.0%		(I) TS	371500	-76.2%		0.0%	
(I) GN	114	-42.3%		0.0%		(I) GN	78	-29.8%		0.0%	
(I) EN	4500	-73.8%		-91.3%		(I) EN	3710	-57.9%		-86.6%	
(O) PN	17630000	56.6%		0.0%		(O) PN	22390000	26.3%		0.0%	
(O) AM	216910	30.6%		0.0%		(O) AM	206767	69.7%		0.0%	
DMU: OSL						DMU: PRG					
(I) RN	2	-26.3%	5.2×10^6	-36.9%	2.7×10^2	(I) RN	3	0.0%	3.7×10^4	-0.4%	1.5×10^3
(I) TS	144000	-38.9%		0.0%		(I) TS	78048	0.0%		0.0%	
(I) GN	86	-47.5%		0.0%		(I) GN	27	-28.9%		0.0%	
(I) EN	583	0.0%		-45.8%		(I) EN	1702	0.0%		-87.8%	
(O) PN	13646890	38.5%		0.0%		(O) PN	7463120	0.0%		0.0%	
(O) AM	175878	23.4%		0.0%		(O) AM	115765	32.5%		0.0%	
DMU: VIE						DMU: ZRH					
(I) RN	2	-10.5%	3.1×10^6	-19.4%	2.4×10^3	(I) RN	3	-28.2%	4.7×10^4	0.0%	9.4×10^2
(I) TS	55700	0.0%		0.0%		(I) TS	138614	0.0%		0.0%	
(I) GN	57	-41.7%		-0.1%		(I) GN	67	-3.2%		0.0%	
(I) EN	2918	-78.7%		-81.7%		(I) EN	1425	0.0%		-66.0%	
(O) PN	12784504	24.3%		0.0%		(O) PN	17024937	0.0%		0.0%	
(O) AM	197089	8.0%		0.0%		(O) AM	269392	17.6%		0.0%	

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