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Various Contractions in Generalized Metric Space

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abstract: This paper presents a survey that aims to provide a brief study of various contractions and development of fixed point theorems for these contractions in the context of generalized metric space introduced by Branciari [5].

Key Words: Fixed point, Generalized metric space.

Contents

1. Introduction

Fixed point theory is an active area of research with wide range of applications in various directions.Wide application potential of this theory has accelerated the research activities which resulted in an enormous increase in publication.

In 2002, Polish mathematician Stefan Banach [51] formulated and proved a theorem which concerns under appropriate conditions about the existence and uniqueness of a fixed point in complete metric space. His result is known as Banach's contraction principle. Various generalizations and extensions of the Banach Contraction principle have been done mostly by relaxing the contraction condition and sometimes by withdrawing the requirement of completeness [See [4], [8], [10], [28], [53] to [61]]. Recently, a very interesting generalization was obtained by Branciari in 2000, by lessening the structure of a metric space. He generalized the concept of metric space by replacing the triangle inequality of metric space to quadrilateral inequality and named such metric space as generalized metric space which is defined as follows:

Definition 1.1. *[4] Let X be a set and $d : X \times X \to R^+$ (set of positive real numbers) be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$ each of them different from x and y;*

1. $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

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then (X,d) *is a generalized metric space.*

Since Branciari [5] conceived this notion by replacing triangle inequality with a weaker assumption, namely, quadrilateral inequality and hence each metric space is obviously a generalized metric space. However a generalized metric space need not be a metric space. Azam and Arshad [6] presented an example to expose this fact.

Example 1.2. *[6] Let* $X = \{1,2,3,4\}$*. Define* $d : X \times X \to R$ *as follows:*

$$\begin{aligned} &d(1,2) = d(2,1) = 3; \\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1; \\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4. \end{aligned}$$

Then clearly (X,d) *is complete generalized metric space but* (X,d) *is not a metric space because it lacks the triangular property:*

$$3 \quad = \quad d(1,2) > d(1,3) + d(3,2) = 1 + 1 = 2.$$

As in metric space setting, such spaces became topological spaces with neighborhood basis given by

$$B = \{B(x,r) : x \in X, r \in R^{+} - \{0\}\}.$$

Moreover, he introduced the definition of Cauchyness of sequence and completeness of spaces as in metric spaces.

Definition 1.3. *([5]) Let* (X,d) *be a generalized metric space and* $\{x_n\}_{n\in N}$ *be a sequence in* X*, then it is said to be a Cauchy sequence if for all* $\epsilon > 0$*, there exists a natural number* $n_\epsilon \in N$ *such that for all* $n, m \in N, n \geq n_\epsilon$*, one has* $d(x_n, x_{n+m}) < \epsilon$*.*

A generalized metric space (X,d) is called complete if every Cauchy sequence in X is convergent. It is also observed that d is continuous in each of the coordinates. Fixed point theorems are statements containing sufficient conditions that ensure the existence of fixed point. Therefore, one of central concern in fixed point theorem is to find a minimal set of sufficient condition which guarantee a fixed point. Branciari [5] made an effort in the same way and proved a fixed point theorem in the context of gms which is stated as below:

Theorem 1.4. *[5] Let* (X,d) *be a complete generalized metric space,* $c \in [0,1)$ *and* $f : X \to X$ *a mapping such that for each* $x, y \in X$*, one has*

$$d(fx, fy) \leq cd(x,y) \tag{1.1}$$

then

1. *there exists a point* $a \in X$ *such that for each* $x \in X$ *one has* $\lim_{n\to\infty} f^n x = a$*;*

2. $fa = a$ *and for each* $e \in X$ *such that* $fe = e$ *one has* $e = a$.

3. for all $n \in N$,

$$d(f^n x, a) \leq \frac{c^n}{1-c} \max\{d(x, fx), d(x, f^2 x)\}.$$

The intriguing nature of this space has attracted attention and therefore fixed points and common fixed point theorems for various contractions on generalized metric space have been established by many researchers[For more, one can see [5] to [50]].

It is important to note that Banach Contraction is uniformly continuous. Therefore, Banach fixed point theorem fails to ensure the existence of fixed point for those maps which are not continuous. In this direction, Kannan [47] introduced a class of contraction mapping in which the mapping is not to be continuous. Thenafter, these mappings were known as Kannan type mappings.

Definition 1.5. *[47] A mapping* $T : X \to X$*, where* (X, d) *is a metric space, is called a Kannan type mapping for all* $x, y \in X$*, the following inequality holds:*

$$d(Tx, Ty) \leq \frac{\lambda}{2}[d(x, Tx) + d(y, Ty)],$$

where $\lambda \in [0, 1)$.

In 2002, Das [47], Azam, in 2008 [6] and Mihet [15] introduced a fixed point theorem for mappings of Kannan's type in generalized metric space:

Theorem 1.6. *[47] Let* (X, d) *be generalized metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$$

where $0 \prec \beta \prec \frac{1}{2}$*. If* X *is* T*-orbitally complete, then* T *has a unique fixed point in* X.

Now, we present the examples to show the comparison between Banach contraction and Kannan contraction.

Example 1.7. *Let* $X = [0, \infty)$ *and let* d *be the usual metric on* X*. Define* $T : X \to X$ *as*

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{x}{4} & \text{if } \frac{1}{3} \leq x \leq 1 \\ \frac{1}{12} & \text{if } x \geq 1 \end{cases}$$

Then T *is Kannan contraction.* T *is not Banach contraction because it is clear that the condition of Banach contraction implies the continuity of the map in the whole space but* T *is not continuous at* $x = \frac{1}{3}$.

Example 1.8. *Let $E = [0, 1]$, $Tx = \frac{x}{2}$ for $x \in E$ and the distance function is the ordinary usual distance. Clearly, Banach contraction is satisfied but at $x = \frac{1}{3}$ and $y = 0$ the condition of Kannan is not satisfied.*

Both the examples show that Banach contraction and Kannan contraction are independent to each other. In 2009, Fora et al. [2] studied Das's theorem and introduced the class of all non-decreasing upper semi-continuous functions $\phi : R^+ \to R^+$ such that $\sum_{n=1}^{\infty} \phi^n(t) \prec \infty$ for all $t \succ 0$.

Theorem 1.9. *Let (X, d) be a generalized metric space, let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \phi(max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\})$$

where $\phi \in \Phi$, and if there exists $x \in X$ such that $O(x)$ is orbitally complete. Then T has a unique fixed point in X.

Also, Das [47] introduced the notion of Cauchy sequence in the following way:

Let (X, d) be a generalized metric space. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for any $\epsilon \succ 0$, there exists $n_\epsilon \in N$ such that for all $m, n \in N, n \leq n_\epsilon$, one has $d(x_n, x_{n+m}) \prec \epsilon$.

This is different from the classical concept of Cauchy sequence. In 2003, Akram et al. [30] introduced the notion of A-contraction mapping as a generalization of Kannan's maps, where A is collection of all functions $\alpha : R^3_+ \to R_+$ satisfying

1. α is continuous on the set R^3_+ of all triplets of non-negative reals;
2. $a \leq Kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

Definition 1.10. *[30] A self-map T on a gms X is said to be a A- contraction if it satisfies the condition that $d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$ for all $x, y \in X$ and for some $\alpha \in A$.*

Theorem 1.11. *Let T be an A-contraction on orbitally complete generalized metric space. Then T has a unique fixed point in X.*

Definition 1.12. *[30] Let $T : X \to X$ be a mapping where (X, d) is gms. (X, d) is said to be $T-$ orbitally complete if and only if every Cauchy sequence which is contained in $\{x, Tx, T^2x, T^3x....\}$ for some $x \in X$ converges in X.*

Remark 1.13. *A complete metric space is orbitally complete with respect to any self mapping of X, but a T-orbitally complete gms may not be complete [7]. It can be observed that a non complete gms X may be T-orbitally complete.*

In 2007, Das et al. [48] proved that for uniformly locally contractive mappings, there is unique fixed point under certain condition in gms. For this, firstly $\epsilon-$ chainable space is defined:

Definition 1.14. *[48] A gms X is said to be ϵ-chainable if for any two points $a, b \in X$ there exists a finite set of points $a = x_0, x_1, x_2, ...x_{n-1}, x_n = b$ such that*

$$d(x_{i-1}, x_i) \preceq \epsilon,$$

for $i = 1, 2, 3...,$ *where* $\epsilon \succ 0$.

Definition 1.15. *[48] A mapping* $T : X \to X$ *is called locally contractive if for every* $x \in X$ *there exists an* $\epsilon_x \succ 0$ *and* $\lambda_x \in [0, 1)$ *such that for all* $p, q \in \{y : d(x, y) \leq \epsilon_x\}$ *the relation* $d(T(p), T(q)) \leq \lambda_x d(p, q)$ *holds.*

Definition 1.16. *[48] A mapping* $T : X \to X$ *is called* (ϵ, λ)*uniformly locally contractive if it is locally contractive at all points* $x \in X$ *and* ϵ, λ *do not depend on* x *i.e.*

$$d(x, y) \prec \epsilon \Longrightarrow d(Tx, Ty) \prec \lambda d(x, y),$$

for all $x, y \in X$.

Remark 1.17. *It can be easily observed that uniformly locally contractive mapping is continuous.*

Theorem 1.18. *[48] If* T *is an* (ϵ, λ) *uniformly locally contractive mapping defined on* T*-orbitally complete,* $\frac{\epsilon}{2}-$*chainable gms* X *satisfying the following condition for all* $x, y, z \in X$,

$$d(x, y) \prec \tfrac{\epsilon}{2} \text{ and } d(y, z) \prec \tfrac{\epsilon}{2} \text{ implies } d(x, z) \prec \epsilon$$

Then T *has a unique fixed point in* X.

Till now, all the work on gms have been done by considering the fact of Branciari [5] that gms d is continuous and a topology can be generated in a gms (X, d) with the help of neighborhood basis given by

$$B = \{B(x, r) : x \in X, r \in R^+ - \{0\}\},$$

where $B(x, r) = \{y \in X : d(x, y) \prec r\}$ is the open ball with center x and radius r.

Then in 2009, a revolutionary fact came across in the paper of Das et al. [49]. They deeply studied the research of Branciari [5] and pointed out that the family $\{B(x, r) : x \in x, r \succ 0\}$ is not a neighborhood basis for any topology on X and present an example to show that d is not continuous. All these facts are clarified by the following example:

Example 1.19. *Define* $X = \{\frac{1}{n} : n = 1, 2, 3...\} \cup \{0\}$ *and* $d : X \times X \to R^+$ *such that*

$$d(x, y) = \begin{cases} 0 & x = y; \\ \frac{1}{n} & \text{if } \{x, y\} = \{0, \frac{1}{n}\}, x \neq y; \\ 1 & \text{if } x \neq y, x, y \in X - 0. \end{cases}$$

Clearly, (X, d) *satisfies axioms of generalized metric space, i.e. for all* $x, y \in X$.

It can be observe that $B(\frac{1}{3}, \frac{1}{2}) \cap B(\frac{1}{4}, \frac{1}{2}) = 0$, *hence there are no* $r \succ 0$ *with* $B(0, r) \subset B(\frac{1}{3}, \frac{1}{2}) \cap B(\frac{1}{4}, \frac{1}{2})$. *Therefore the family* $\{B(x, r) : x \in X, r \succ 0\}$ *is not a neighborhood basis for any topology on* X.

Also it is observed that $\lim_{n\to\infty} d(\frac{1}{2},\frac{1}{n}) = 1$ whereas $d(\frac{1}{2},0) = \frac{1}{2} \neq 1$ which shows that d is not continuous.

In 2010, Samet [9] exposed the incorrect property of generalized metric space with the help of an example:

Example 1.20. *Let $\{x_n\}$ be a sequence in Q and $a,b \in R - Q, a \neq b$. we put the set $X = \{x_1, x_2, x_3, ...x_n, ...\} \cup \{a,b\}$ and we consider $d : X \times X \to R$ defined by*

$$\begin{cases} d(x,x) = 0 & \text{for all } x \in X, \\ d(x,y) = d(y,x) & \text{for all } x,y \in X, \\ d(x_n,x_m) = 1 & \text{for all } n,m \in N^*, n \neq m, \\ d(x_n,b) = \frac{1}{n} & \text{for all } n \in N^*, \\ d(x_n,a) = \frac{1}{n} & \text{for all } n \in N^*, \\ d(a,b) = 1. \end{cases}$$

It is clear that (X,d) is gms. Here, we have $\lim_{n\to\infty} x_n = a$ because $d(x_n,a) = \frac{1}{n} \to 0$ as $n \to \infty$. But $|d(x_n,b) - d(a,b)| = 1 - \frac{1}{n} \to 1$ as $n \to \infty$.

Boyd and Wong [14] proved fixed point theorem in the setup of metric space. In 2009, Das and Dey [49], work in generalized metric space of Branciari [5] and proved the existence of fixed point for more general type of contractive mapping of Boyd and Wong:

Theorem 1.21. *Let (X,d) be a complete gms and let $T : X \to X$ satisfy*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

where $\psi : \bar{P} \to [0,\infty)$ is upper semi-continuous from right on $\bar{P}$(the closure of range of d) and satisfies $\psi(t) \prec t$ for all $t \in \bar{P} - \{0\}$. Then T has a unique fixed point x_0 and $T^n x \to x_0$ for each $x \in X$.

In 2009, Samet [8] introduced a Lebesgue integrable mapping ϕ and proved fixed point results for a contractive condition of integral type as follows:

Theorem 1.22. *Let (X,d) be a complete gms, $c \in (0,1)$, and let $f : X \to X$ be a mapping such that for each $x,y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)} \phi(t)dt,$$

where $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$, non-negative and such that for all $\epsilon > 0, \int_0^\epsilon \phi(t)dt >$ 0.

Then f has a unique fixed point $a \in X$ such that for each $x \in X, \lim_{n\to\infty} f^n x = a$.

In all the theorems discussed above, the following three conditions were taken for granted.

1. $\{B_r(x) : r \succ 0, x \in X\}$ is a basis for a topology on X;

2. d is continuous in each of the coordinates and

3. a gms is a Hausdorff space.

But Das et al. [49] and Sarma et al. [21] mentioned that the above propositions are not true in generalized metric space. Several examples were given to support this. Thenafter, researchers assume the space X to be Hausdorff to ensure the existence and uniqueness of fixed point in gms. The following is the extension of Banach fixed point theorems to gms with the assumption of Hausdorffness of the space.

Theorem 1.23. *let (X, d) be a Hausdorff and complete gms and let $f : X \to X$ be a mapping and $0 \prec \lambda \prec 1$ satisfying*

$$d(fx, fy) \leq \lambda d(x, y);$$

for all $x, y \in X$. Then there is a unique point $x \in X$ satisfying $fx = x$.

In 2013, Kirk et al. [59] gave condition to prove the distance function to be continuous in another form.

Proposition 1.24. *If (X, d) is a generalized metric space which satisfies for each pair of distinct points $a, b \in X$, there is a number $r_{a,b} \prec 0$ such that for every $c \in X$,*

$$r_{a,b} \leq d(a, c) + d(c, b),$$

then the distance function is continuous.

The following preposition shows that the quadrilateral inequality implies a weaker but useful form of distance continuity.

Proposition 1.25. *Let $\{q_n\}$ be a Cauchy sequence in a generalized metric space (X, d) and suppose $\lim_{n\to\infty} d(q_n, q) = 0$. Then $\lim_{n\to\infty} d(p, q_n) = d(p, q)$ for all $p \in X$. In particular, $\{q_n\}$ does not converge to p if $p \neq q$.*

In 2010, Al-Bsoul et al. [1] studied the properties of generalized metric space and gave necessary and sufficient conditions for the generalized metric spaces to be a metric space.

Proposition 1.26. *Let (X, d) be a gms. Let $x_i \in X, 0 \leq i \leq n, n \in N, x_0 = x, x_n = y, x \neq x_i$. Then, we have either*

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x, y),$$

or

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x, x_1) + d(x_1, y).$$

Theorem 1.27. *Let (X, d) be a gms, then the following are necessary and sufficient conditions that d is metric without an isolated point:*

1. *Every convergent sequence is a Cauchy sequence;*

2. *for all $x \in X$, there exists $x_n \in X - \{x\}$, such that $\{x_n\}$ converges to x.*

In 2012, Bari and Vetro [12] extended the work in generalized metric space from one maps to two maps and find common fixed point theorems by using a $(\psi - \phi)-$weakly contractive condition, where $\psi \in \Psi, \phi \in \Phi$;

Ψ is the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

1. ψ is continuous and non-decreasing;

2. $\psi(t) = 0$ if and only if $t = 0$.

and Φ is the set of functions $\phi : [0, \infty) \to [o, \infty)$ satisfying the following conditions:

1. $\phi(t)$ is lower semi-continuous;

2. $\phi(t) = 0$ iff $t = 0$.

Bari and Vetro [12] obtained the following lemma and utilized it to prove the existence of common fixed point under some assumptions.

Lemma 1.28. *[12] Let $\{S_n\}$ be a sequence of non-negative real numbers. If*

$$\psi(s_{n+1}) \leq \psi(s_n) - \phi(S_n)$$

for all $n \in N$ where $\psi \in \Psi$ and $\phi \in \Phi$, then the following hold:

1. *$s_{n+1} \leq s_n$ for all $n \in N$,*

2. *$s_{n+1} \prec s_n$ if $s_n \succ 0$,*

3. *$s_n \to 0$ as $n \to \infty$.*

Theorem 1.29. *Let (X, d) be a Hausdorff gms and let T and f be self mappings on X such that $TX \subset fX$. Assume that (fX, d) is a complete gms and that the following conditions hold:*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \phi(d(fx, fy))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover if T and f are weakly compatible, then T and f have a unique common fixed point.

In 1986, Jungck [17] introduced the concept of compatible mappings. Chi-Ming Chen [11] defined $\wp$-function and $\hbar$- function and ensure the existence of common fixed point for the pairs of compatible maps in the context of generalized metric space.

Definition 1.30. *$\phi : R^+ \to R^+$ be a $\wp$ function if the function ϕ satisfies the following conditions:*

1. *$\phi(t) \prec t$ for all $t \succ 0$ and $\phi(0) = 0$,*

2. *$\lim_{t_n \to t} \inf \phi(t_n) \prec t$ for all $t \succ 0$.*

Definition 1.31. *$\phi : R^{+3} \to R^+$ is said to be a $\hbar$ function if the function ϕ satisfies the following conditions:*

1. *ϕ is strictly increasing, continuous function in each coordinates;*

2. *for all $t \succ 0$, $\phi(t,t,t) \prec t$, $\phi(t,0,0) \prec t$, $\phi(0,t,0) \prec t$ and $\phi(0,0,t) \prec t$.*

Theorem 1.32. *Let (X, d) be a Hausdorff and complete gms, and let $\phi : R^+ \to R^+$ be a $\wp$ function. Let S, T, F and $G : X \to X$ be four single valued functions such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \phi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\}).$$

Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$, and the pairs $\{S, F\}$ and $\{T, G\}$ are compatible. If F or G is continuous, then S, T, F and G have a unique common fixed point in X.

Theorem 1.33. *Let (X, d) be a Hausdorff and complete gms and let $\phi : R^{+3} \to R^+$ be a $\hbar$-function. Let S, T, f and $G : X \to X$ be four single valued functions such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \phi(d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)).$$

Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$, and the pairs $\{S, F\}$ and $\{T, G\}$ are compatible. If F or G is continuous, then S, T, F and G have a unique common fixed point in X.

In 1969, Boyd and Wong [14] defined a class of contractive mapping by introducing a function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) \succ 0$ for all $t \succ 0$. In order to enlarge the domain of theory of fixed point, Lakzian and Samet [20] introduced a new function namely altering distance function, ψ, such that $\psi : [0, \infty) \to [0, \infty)$ be non-decreasing and $\psi(t) = 0$ iff $t = 0$ and obtained the fixed point by using altering distance function.

Thenafter, Erhan et al. [22] provided fixed points for a general class of $(\psi - \phi)$ contractive mappings in complete gms in the form of following:

Theorem 1.34. *[22] Let (X,d) be a a Hausdorff and complete gms and $T : X \to X$ be a self map satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) + Lm(x,y)$$

for all $x,y \in X$ and $\psi,\phi \in \Psi$, where $L \succ 0$, the function ψ is non-decreasing and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$
$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

Then T has a unique fixed point in X.

Also Erhan et al. [22] gave existence and uniqueness theorem of fixed point under the conditions which involves rational expressions:

Theorem 1.35. *Let (X,d) be a Hausdorff and complete gms and Let $T : X \to X$ be a self-map satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y));$$

for all $x,y \in X$ and $\psi,\phi \in \Psi$, where $M(x,y) = \max\{d(x,y), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}\}$.

Then T has a unique fixed point in X.

Ahmad, Arshad and Vetro [23] provide a method to construct a new generalized metric space from a family of given gms.The next example depicts that from the family of generalized metric space, how one can derive a new generalized metric space:

Example 1.36. *[23] Let $\{(X_n,d_n) : n \in J \subset N\}$ be a family of disjoint generalized metric spaces and let $X = \cup\{X_n : n \in J\}$. Define $d : X \times X \to [0,\infty)$ by $d(x,y) = d(y,x)$ with*

$$d(x,y) = \begin{cases} d_n(x,y) & \{if x,y \in X_n, n \in J\} \\ 1, & \{if x \in X_n, y \in X_m, m,n \in J, m \neq n\}. \end{cases}$$

Then (X,d) is a gms.

By inspiring Khan [28], the following new version of the theorem of Khan [28] was obtained in the generalized metric space.

Theorem 1.37. *[23] Let (X,d) be a complete gms and let $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty) \leq \begin{cases} \lambda d(x,y) + \mu\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)} & \{d(x,Ty)+d(y,Tx) \neq 0\} \\ 0 & \{d(x,Ty)+d(y,Tx) = 0\}; \end{cases}$$

for all $x,y \in X$ and $x \neq y$, and for some $\lambda,\mu \in [0,1)$ with $\lambda+\mu \prec 1$. Then T has a unique fixed point in X.

The following example support this theorem.

Example 1.38. *Let $X = \{1, -1, i, -i\}$ and define the generalized metric $d : X \times X \to [0, \infty)$ by $d(x, y) = d(y, x)$ with*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \in \{-1, 1\}, y = i, \\ 3 & \text{if } x, y \in \{-1.1\}, x \neq y, \\ 0 & \text{otherwise}, \end{cases}$$

Then clearly d is complete gms and not a metric space and define $T : X \to X$ by

$$Tx = \begin{cases} i & x \neq i \\ 1 & x = -i, \end{cases}$$

then obviously T satisfies the contractive condition of above theorem with $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{4}$. This implies $x = i$ is unique fixed point of T.

It is to be noticed that, a sequence in gms may have two limits. In 2014, Kadelburg and Radenovic [61] proved that a Cauchy sequence may converge to unique limit point under certain conditions. This is clarified in the following lemma:

Lemma 1.39. *[61] Let (X, d) be a gms and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to atmost one point.*

Next lemma is a modification of a result in generalized metric space which is well known in metric spaces:

Lemma 1.40. *[61] Let (X, d) be a gms and let $\{y_n\}$ be sequence in X with distinct elements ($y_n \neq y_m$ for $n \neq m$). Suppose that $d(y_n, y_{n+1})$ and $d(y_n, y_{n+2})$tend to 0 as $n \to \infty$ and that y_n is not a Cauchy sequence. Then there exist $\epsilon \succ 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k \succ m_k \succ k$ and the following four sequences tends to ϵ as $k \to \infty$*

$$d(y_{m_k}, y_{n_k}), d(y_{m_k}, y_{n_k+1}), d(y_{m_k-1}, y_{m_k}), d(y_{m_k-1}, y_{m_k+1}).$$

In 1973, Geraghty [35] introduced a contractive condition in which he defined a class of real functions $\beta : [0, \infty) \to (0, 1)$ such that

$$\beta(t_n) \to 1 \text{ as } n \to \infty \text{ implies } t_n \to 0 \text{ as } n \to \infty.$$

Here we highlight the difference between fixed point theorem in generalized metric space proved in 2000 and in the recent time, In starting all the theorems in gms was proved by considering either both or one of the condition that d is continuous and relative topology is Hausdorff explicitly. But in 2014, Kadelburg and Radenovic [61] proved common fixed point theorem by neither assuming space is Hausdorff nor d is continuous.

Theorem 1.41. *Let (X, d) be a gms and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If for some Geraghty function β*

$$d(fx, fy) \leq \beta(d(gx, gy))d(gx, gy)$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence $y*$. If, moreover, f, g are weakly compatible, then they have a unique common fixed point .

In 2014, Kadelburg and Radenovic [61] proved fixed point theorem for $\alpha - \psi$ contractive mapping introduced by Samet et al. [10].

Theorem 1.42. *Let (X, d) be a complete gms and $f : X \to X$ be an $\alpha - \psi$ contractive mapping satisfying the following conditions:*

1. *f is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(x_0, f^2x_0) \geq 1$;*
3. *f is continuous; or*
4. *if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then f has a fixed point. where $\psi \in \Psi$, Ψ denote the family of non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t)\infty$ for each $t \succ 0$, where ψ^n is the n^{th} iterate of ψ. Also $\psi(t) \prec t$ for $\psi \in \Psi$ and $t \succ 0$.

In 2014, Asadi et al. [39] exercise on the concept of Geraghty [35] and Samet et al. [10] and they developed a new class of contractive mapping namely $\alpha - \psi$ Geraghty contraction. Further they investigated the existence and uniqueness of fixed point theorem in gms.

Definition 1.43. *Let (X, d) be a generalized metric space and let $\alpha : X \times X \to R$ be a function. A map $T : X \to X$ is called $\alpha - \psi$ -Geraghty contraction if there exists Geraghty function β such that for all $x, y \in X$*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)),$$

where $\psi \in \Psi$.

Theorem 1.44. *Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

1. *T is an $\alpha - \psi$-Geraghty contraction mapping;*
2. *T is triangular α-admissible;*

3. *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$;

4. T *is continuous.*

Then T *has a fixed point* x^* *in* X *and* $\{T^n x_0\}$ *converges to* x^*.

Asadi et al. [39] also introduced the notion of generalized $\alpha - \psi$ Geraghty contraction, which is stronger version of $\alpha - \psi$ Geraghty contraction:

Definition 1.45. *Let* (X, d) *be a generalized metric space, and let* $\alpha : X \times X \to R$ *be a function.A map* $T : X \to X$ *is called generalized* $\alpha - \psi$*-Geraghty contraction mapping if there exists Geraghty function* β *such that for all* $x, y \in X$,

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ *and* $\psi \in \Psi$.

Theorem 1.46. *Let* (X, d) *be a complete generalized metric space,* $\alpha : X \times X \to R$ *be a function and let* $T : X \to X$ *be a map. Suppose that the following conditions are satisfied:*

1. T *is a generalized* $\alpha - \psi$*-Geraghty contraction mapping;*

2. T *is triangular* $\alpha -$ *admissible;*

3. *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$;

4. T *is continuous.*

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_0\}$ *converges to* x^*.

To ensure the uniqueness of a fixed point for $\alpha - \psi$-Geraghty contraction as well as generalized $\alpha - \psi$ Geraghty contraction mapping, the condition for all $x, y \in F(T)$, $\alpha(x, y) \geq 1$, where $F(T)$ denotes the fixed point of T, is compulsory.

After the introduction of $\alpha - \psi$ contraction condition Aydi et al. [19] established fixed point results for mapping involving generalized $(\alpha - \psi)$-contractive mappings.

Definition 1.47. *Let* (X, d) *be a generalized metric space and* $T : X \to X$ *be a given mapping. We say that* T *is generalized* $(\alpha - \psi)$*-contractive mapping of type-I if there exist two functions* $\alpha : X \times X \to [0, \infty)$ *and* $\psi \in \Psi$ *such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in X$ *and* $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Definition 1.48. *Let (X, d) be a generalized metric space and $T : X \to X$ be a given mapping. We say that T is a generalized $(\alpha - \psi)$-contractive mapping of type-II if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(N(x, y)),$$

for all $x, y \in X$ and $N(x, y) = \max\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}\}$

Remark 1.49. *From above two definitions, we observe that $N(x, y) \leq M(x, y)$.*

Theorem 1.50. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a generalized $\alpha - \psi$ contractive mapping of type-I. Suppose that*

1. *T is α-admissible;*

2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*

3. *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

One can prove the above theorem for generalized $\alpha - \psi$ contractive mapping of type-II also.

Jleli et al. [43] established the generalization of Banach fixed point theorem in the context of gms by introducing a new family of function Θ, the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

1. θ is non-decreasing;

2. for each sequence $t_n \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$,

3. there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Theorem 1.51. *[43] Let (X, d) be a complete gms and $T : X \to X$ be a given mapping. Suppose that there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that for all $x, y \in X, d(Tx, Ty) \neq 0$ implies*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k$$

then T has unique fixed point.

Theorem 1.52. *[43] Let (X, d) be a complete gms and $T : X \to X$ be a given mapping. Suppose that there exists $\theta \in \Theta$ that is continuous and $k \in (0, 1)$ such that for all $x, y \in X, d(Tx, Ty) \neq 0$ implies*

$$\theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^k,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$, *Then* T *has a unique fixed point.*

In 2015, Kikina and Kikina [24] proved fixed point theorems in gms for self maps in a class of almost contractions defined by an implicit relation. Firstly they proved a lemma in which they provided the conditions how to prove a sequence a Cauchy sequence?

Lemma 1.53. *Let* (X,d) *be a generalized metric space, let* $\{x_n\}$ *be a sequence of distinct points* ($x_n \neq x_m$ *for all* $n \neq m$) *in* X *and* $l \geq 0$. *If*

1. $d(x_n, x_{n+1}) \leq \delta^n l, 0 \leq \delta \prec 1$, *for all* $n \in N$ *and*

2. $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$,

then $\{x_n\}$ *is a Cauchy sequence.*

Definition 1.54. *[58] Let* (X,d) *be a metric space. A map* $T : X \to X$ *is called weak(almost) contraction if there exists a constant* $\delta \in (0,1)$ *and some* $L \geq 0$ *such that*

$$d(Tx,Ty) \leq \delta d(x,y) + L.d(y,Tx),$$

for all $x, y \in X$.

Definition 1.55. *[24] The set of real functions* $\phi : R_+^6 \to R$, *which are upper semi-continuous in each coordinate variable and satisfy atleast one of the following conditions:*

1. *if* $\phi(u,v,v,u,u,0) \leq$ *for all* $u, v \geq 0$, *then there exists a real constant* $h \in [0,1)$ *such that* $u \leq hv$,

2. *if* $\phi(u,v_1,v_2,v_3,0,v_4) \leq 0$ *for all* $u, v_1, v_2, v_3, v_4 \geq 0$, *then there exists a real constant* $\delta \in [0,1)$ *and some* $L \geq 0$ *such that* $u \leq \delta \max\{v_1,v_2,v_3,v_4\} + Lv_4$,

3. $\phi(u,u,0,0,u,u) \leq 0$ *implies* $u = 0$, *called a* ϕ_6- *function*

Example 1.56. *Let* $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2$, *where* $0 \leq a \prec 1$, *then* ϕ *is a* ϕ_6 *with* $h = \delta = a$ *and* $L = 0$.

Definition 1.57. *Let* (X,d) *be a generalized metric space and* $\phi \in \phi_6$. *A map* $T : X \to X$ *is called an almost* ϕ*- contraction if*

$$\phi[d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(y,T^2x), d(y,Tx)] \leq 0,$$

for all $x, y \in X$.

Theorem 1.58. *Let* (X,d) *be a gms and* $\phi \in \phi_6$ *and let* $T : X \to X$ *be an almost* ϕ*-contraction. If* ϕ *satisfies the first and second conditions of* ϕ_6 *and* (X,d) *is* $T-$ *orbitally complete, then*

1. *$Fix(T) = \{x \in X : Tx = x\} \neq \phi$,*

2. *for any $x_0 \in X$, the picard iteration $\{x_n\}$ defined by $x_n = Tx_{n-1}, n = 1, 2, 3...$ converges to some $\alpha \in Fix(T)$.*

Theorem 1.59. *Let (X, d) be a gms, $\phi \in \phi_6$ and let $T : X \to X$ be an almost ϕ-contraction. If (X, d) is T -orbitally complete and ϕ satisfies the first, second and third condition of ϕ_6 function, then*

1. *T has a unique fixed point $\alpha \in X$.*

2. *for any $x_0 \in X$, the picard iteration $x_n = Tx_{n-1}, n \in N$ converges to α.*

Since in gms, every convergent sequence is not Cauchy so Abtahi [29] naming convergence in two sense. He said

1. A sequence is said to converge in (X, d) if $d(x, x_n) \to 0$ as $n \to \infty$.

2. A sequence $\{x_n\}$ is said to converge to x in strong sense if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x.

Abtahi [29], in 2015, found the conditions to prove a sequence $\{x_n\}$ a $\nu-$ Cauchy sequence.

Lemma 1.60. *Let $\{x_n\}$ be a sequence in a ν- generalized metric space X such that $x_n (n \in N)$ are all different. Suppose for every $\epsilon \succ 0$, for any two sub-sequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \leq \epsilon$, then , for some N,*

$$d(x_{p_i+1}, x_{q_i+1}) \leq \epsilon, (i \geq N).$$

If $d(x_n, x_{n+1}) \to 0$, then $\{x_n\}$ is a ν- Cauchy.

Definition 1.61. *Let (X, d) be a ν- generalized metric space. A mapping $T : X \to X$ is said to be a Ciric Matkowski contraction if $d(Tx, Ty) \prec d(x, y)$ for every $x, y \in X$, with $x \neq y$, and, for any $\epsilon \succ 0$, there exists $\delta \succ 0$ such that for all $x, y \in X, d(x, y) \prec \delta + \epsilon \longrightarrow d(Tx, Ty) \leq \epsilon$.*

Lemma 1.62. *Let $T : X \to X$ be a mapping. Suppose $d(T^n x, T^{n+1} x) \to 0$, for some $x \in X$. Then, for some $k \in N$, either the Picard iterates $T^n x (n \geq k)$ are all different or they are all the same.*

Competing Interests

The authors declare that they have no competing interests.

Authors Contribution

Both authors contributed equally to the writing of this paper.

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