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Simplifying Coefficients in Differential Equations Related to Generating Functions of Reverse Bessel and Partially Degenerate Bell Polynomials

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Abstract: In the paper, by virtue of the Faá di Bruno formula and identities for the Bell polynomials of the second kind, the author simplifies coefficients in a family of ordinary differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials.

Key Words: Ordinary differential equation, Coefficient, Generating function, Simplification, Partially degenerate Bell polynomial, Reverse Bessel polynomial, Bell polynomials of the second kind, Faá di Bruno formula.

Contents

1. Motivation and main results

In [3, Theorem 1], it was established inductively and recursively that the family of differential equations

$$G^{(n)}(t) = G(t) \sum_{i=n}^{2n-1} a_{i-n}(n,x)(1-2t)^{-i/2}, \quad n \in \mathbb{N} \tag{1.1}$$

has the same solution

$$G(t) = e^{x(1-\sqrt{1-2t}\,)}, \tag{1.2}$$

where $a_0(n,x) = x^n$, $a_{n-1}(n,x) = (2n-3)!!x$, and

$$a_j(n,x) = x^{n-j} \sum_{i_j=0}^{n-j-1} \sum_{i_{j-1}=0}^{n-j-1-i_j} \cdots \sum_{i_1=0}^{n-j-1-i_j-\cdots-i_2} \prod_{k=1}^{j} (n - i_j - i_{j-1} - \cdots - i_k - (j-(2k-2))) \tag{1.3}$$

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for $1 \le i \le n-2$. The function $G(t)$ in (1.2) can be used to generate the reverse Bessel polynomials $p_k(x)$ by

$$G(t) = e^{x(1-\sqrt{1-2t}\,)} = \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!}.$$

The expression (1.3) was used in [3, Theorem 2].

In [4, Theorem 2.2], it was also established inductively and recursively that the family of differential equations

$$F_\lambda^{(n)}(t) = F_\lambda(t)\sum_{i=1}^{n} b_i(n,\lambda)x^i(1+\lambda t)^{i/\lambda-n}, \quad n\in\mathbb{N} \tag{1.4}$$

has a solution

$$F_\lambda(t) = e^{x[(1+\lambda t)^{1/\lambda}-1]}, \tag{1.5}$$

where $b_1(n,\lambda) = (1-(n-1)\lambda|\lambda)_{n-1}$,

$$\begin{aligned} b_i(n,\lambda) = \sum_{k_{i-1}=0}^{n-i}\sum_{k_{i-2}=0}^{n-i-k_{i-1}}\cdots\sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2} & \\ \prod_{\ell=2}^{i}\Biggl(\ell-\Biggl(n-\sum_{j=\ell}^{i-1}k_j-i-1+\ell\Biggr)\lambda\Bigg|\lambda\Biggr)_{k_{\ell-1}} & \\ \times\Biggl(1-\Biggl(n-\sum_{j=1}^{i-1}k_j-i\Biggr)\lambda\Bigg|\lambda\Biggr)_{n-\sum_{j=1}^{i-1}k_j-i} & \end{aligned} \tag{1.6}$$

for $2 \le i \le n$,

$$(x|\alpha)_n = \prod_{k=0}^{n-1}(x+k\alpha) = \begin{cases} x(x+\alpha)\cdots[x+(n-1)\alpha], & n\ge 1,\\ 1, & n=0,\end{cases}$$

and the function $F_\lambda(t)$ in (1.5) can be used [42] to generate partially degenerate Bell polynomials $B_{n,\lambda}(x)$ by

$$F_\lambda(t) = e^{x[(1+\lambda t)^{1/\lambda}-1]} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}.$$

For more information about the Bell numbers and polynomials, please refer to [2, 7,9,12,19,20,31,37,39,41] and closely related references.

It is obvious to see that

1. the expression (1.6) is too complicated to be remembered, understood, and computed easily;
2. the original proof of [4, Theorem 2.2] is long and tedious,

3. the generating functions $G(t)$ and $F_\lambda(t)$ are connected by

$$F_2(-t) = \frac{1}{G(t)}.$$

In this paper, we will provide nice and standard proofs for [3, Theorem 1] and [4, Theorem 2.2] and, more importantly, discover simple, meaningful, and significant form for $a_j(n,x)$ and $b_i(n,\lambda)$.

Our main results can be stated as the following theorem.

Theorem 1.1. *For $n \ge 0$, the function $F_\lambda(t)$ defined by (1.5) satisfies*

$$F_\lambda^{(n)}(t) = \frac{F_\lambda(t)}{(1+\lambda t)^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left[\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda)\right] \left[x(1+\lambda t)^{1/\lambda}\right]^k \tag{1.7}$$

and the function $G(t)$ defined by (1.2) satisfies

$$G^{(n)}(t) = \frac{(-1)^n G(t)}{(1-2t)^n} \sum_{k=0}^{n} \frac{1}{k!} \left[\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m)\right] \bigl(x\sqrt{1-2t}\,\bigr)^k \tag{1.8}$$

and

$$G^{(n)}(t) = \frac{G(t)}{(1-2t)^n} \sum_{k=0}^{n} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \bigl(x\sqrt{1-2t}\,\bigr)^k, \tag{1.9}$$

where the empty product means 1 as usual.

2. Proof of Theorem 1.1

The famous Faà di Bruno formula reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) \,\mathrm{B}_{n,k}\bigl(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\bigr) \tag{2.1}$$

for $n \ge 0$, where the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ are defined [1, p. 134, Theorem A] and [1, p. 139, Theorem C] by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Applying $u = h(t) = (1+\lambda t)^{1/\lambda}$ and $f(u) = e^{x(u-1)}$ to (2.1) gives

$$F_\lambda^{(n)}(t) = \sum_{k=0}^{n} \frac{\mathrm{d}^k e^{x(u-1)}}{\mathrm{d} u^k} \mathrm{B}_{n,k}\Biggl((1+\lambda t)^{1/\lambda-1}, (1+\lambda t)^{1/\lambda-2}(1-\lambda),$$
$$(\lambda t+1)^{1/\lambda-3}(1-\lambda)(1-2\lambda), \dots, (1+\lambda t)^{1/\lambda-(n-k+1)} \prod_{\ell=1}^{n-k}(1-\ell\lambda)\Biggr)$$
$$= \sum_{k=0}^{n} \frac{x^k e^{x(u-1)}}{(1+\lambda t)^{n-k/\lambda}} \mathrm{B}_{n,k}\Biggl(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\Biggr)$$
$$= F_\lambda(t) \sum_{k=0}^{n} \frac{x^k}{(1+\lambda t)^{n-k/\lambda}} \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - q\lambda),$$

where we used the identities

$$\mathrm{B}_{n,k}\bigl(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\bigr) = a^k b^n \,\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.2}$$

and

$$\mathrm{B}_{n,k}\Biggl(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\Biggr)$$
$$= \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - q\lambda), \quad \lambda \in \mathbb{C}, \tag{2.3}$$

which is equivalent to

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle\alpha\ell\rangle_n, \quad \alpha \in \mathbb{C}, \tag{2.4}$$

in [1, p. 135], [21, First proof of Theorem 2], [22, Lemma 2.2], [25, Remark 6.1], [26, Lemma 4], [29, Remark 1], [37, Lemma 2.6], and [38, Theorems 2.1 and 4.1]. The formula (1.7) is thus proved.

Similarly, applying $u = h(t) = \sqrt{1-2t}$ and $f(u) = e^{x(1-u)}$ to (2.1) yields

$$G^{(n)}(t) = \sum_{k=0}^{n} \frac{\mathrm{d}^k e^{x(1-u)}}{\mathrm{d} u^k} \mathrm{B}_{n,k}\biggl(-\frac{1}{(1-2t)^{1/2}}, -\frac{1}{(1-2t)^{3/2}},$$
$$-\frac{3}{(1-2t)^{5/2}}, \dots, -\frac{[2(n-k+1)-3]!!}{(1-2t)^{[2(n-k+1)-1]/2}}\biggr)$$
$$= \sum_{k=0}^{n} (-x)^k e^{x(1-u)} \frac{(-1)^k}{(1-2t)^{n-k/2}} \mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!)$$
$$= G(t) \sum_{k=0}^{n} \frac{x^k}{(1-2t)^{n-k/2}} \frac{(-1)^n}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - 2q),$$

where we used the identity

$$\mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) = \frac{(-1)^n}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - 2q) \quad (2.5)$$

in [25,27,29,46,42]. The formula (2.5) can also be derived from taking $\lambda = 2$ in (2.3) or taking $\alpha = \frac{1}{2}$ in (2.4) and utilizing the identity (2.2). The formula (1.8) is thus proved.

In [42, Theorem 1.2] and [40, Eq. (1.10)], it was derived that

$$\mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) = \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!!. \quad (2.6)$$

Substituting (2.6) into (1.8) concludes (1.9). The proof of Theorem 1.1 is complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 3.1. *The equation (1.1) can be reformulated as*

$$G^{(n)}(t) = G(t) \sum_{i=n}^{2n-1} \frac{a_{i-n}(n,x)}{(1-2t)^{i/2}}, \quad n \in \mathbb{N}.$$

This means that the function $\frac{G^{(n)}(t)}{G(t)}$ *is a linear combination of the base*

$$\left(\frac{1}{\sqrt{1-2t}}\right)^n, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{n+1}, \quad \dots, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{2n-1}.$$

The equation (1.8) shows that the function $\frac{G^{(n)}(t)}{G(t)}$ *is a linear combination of the base*

$$\left(\frac{1}{\sqrt{1-2t}}\right)^{2n}, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{2n-1}, \quad \dots, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{n+1}, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{n}.$$

These two bases are not equivalent to each other. Therefore, we surely disclose that the equation (1.1) is wrong and, consequently, main results in [3] are all wrong.

Remark 3.2. *Comparing (1.4) with (1.7) reveals that*

$$b_k(n,\lambda) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda).$$

This form for $b_k(n,\lambda)$ *is apparently simpler, more meaningful, and more significant than the expression (1.6).*

From (2.5) and (2.6), it follows that

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-2q)=(-1)^{n}k!\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!. \tag{3.1}$$

Consequently, we obtain

$$b_k(n,2)=(-1)^{n-k}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!.$$

Remark 3.3. *The equation (1.7) can be rewritten as*

$$\begin{aligned}F_{\lambda}^{(n)}(t)=\frac{F_{\lambda}(t)}{(1+\lambda t)^{n}}\sum_{\ell=0}^{n}\frac{[x(1+\lambda t)^{1/\lambda}]^{\ell}}{\ell!}&\\ \times\left[\prod_{q=0}^{n-1}(\ell-q\lambda)\right]\sum_{m=0}^{n-\ell}(-1)^{m}\frac{[x(1+\lambda t)^{1/\lambda}]^{m}}{m!}&\\ =\frac{e^{-x}}{(1+\lambda t)^{n}}\sum_{\ell=0}^{n}\frac{[x(1+\lambda t)^{1/\lambda}]^{\ell}}{\ell!(n-\ell)!}\left[\prod_{q=0}^{n-1}(\ell-q\lambda)\right]\Gamma\bigl(n-\ell+1,-x(1+\lambda t)^{1/\lambda}\bigr),&\end{aligned}$$

where $\Gamma(a,x)=\int_{x}^{\infty}t^{a-1}e^{-t}\,\mathrm{d}\,t$ denotes the incomplete gamma function which has been investigated in [10,30,34] and closely related references.

Remark 3.4. *Applying (3.1) to (1.7) results in*

$$F_{2}^{(n)}(t)=\frac{F_{2}(t)}{(1+2t)^{n}}\sum_{k=0}^{n}(-1)^{n-k}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!\bigl[x(1+2t)^{1/2}\bigr]^{k}.$$

Remark 3.5. *We leave a question to readers: what are the inversion formulas of the equations from (1.7) to (1.9)?*

Remark 3.6. *The motivations in the papers [5,6,8,11,13,14,15,16,17,23,24,28,29,31,32,33,35,36,37,46,43,44,45,47,48,49] are same as the one in this paper.*

Remark 3.7. *This paper is a slightly modified version of the preprint [18].*

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