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On Graded Quasi-primary Submodules of Graded Modules Over Graded Commutative Rings

Khaldoun Al-Zoubi and Rweili Alkhalaf

ABSTRACT: Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded quasi-primary submodules of graded modules over graded commutative rings. Various properties of graded quasi-primary submodules are considered.

Key Words: Graded primary ideals, Graded quasi-primary submodules, Graded primary submodules.

Contents

1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Graded primary ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [20]. Graded primary submodules of graded modules over graded commutative rings have been introduced and studied in [1,3,11,13,19]. Graded prime submodules of graded modules over graded commutative rings have been introduced and studied in [2,5,6,7,9,10,22,23]. Also graded primary-like submodules of graded modules over graded commutative rings have been introduced and studied by K. Al-Zoubi and M. Al-Dolat in [4]. Here we introduce the concept of graded quasi-primary submodules of graded modules over graded commutative rings and give a number of its properties (see sec. 2).

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [14], [16], [17] and [18] for these basic properties and more information on graded rings and modules. Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also we write, $h(R) = \bigcup_{g\in G} R_g$. Moreover, R_e is a subring of R and $1_R \in R_e$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \cap R_g)$.

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Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded.

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \underset{g \in G}{\oplus} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g \in G}{\cup} M_g$ and the elements of $h(M)$ are called to be *homogeneous*. Let $M = \underset{g \in G}{\oplus} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \underset{g \in G}{\oplus} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N.

Let R be a G-graded ring and M a graded R-module.
A proper graded ideal I of R is said to be *a graded maximal ideal* of R if J is a graded ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$ (see [21].)

A proper graded ideal p of R is said to be *a graded prime ideal* if whenever $rs \in p$, we have $r \in p$ or $s \in p$, where $r, s \in h(R)$ (see [21].) We denote the set of all graded prime ideals of R containing p by $V_R^g(p)$.

The *graded radical* of I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [21].)

A proper graded ideal p of R is said to be *a graded primary ideal* if whenever $r, s \in h(R)$ with $rs \in p$, then either $r \in p$ or $s \in Gr(p)$ (see [20].)

A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M) = \{r \in R : rM \subseteq P\}$ or $m \in P$ (see [7].) In this case, P is called graded p-prime, where $p = (P :_R M)$.

A proper graded submodule P of a graded R-module M is said to be *a graded primary submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $m \in P$ or $r \in Gr((P :_R M))$ (see [11].)

The *graded radical* of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [11].)

A proper graded submodule N of M is said to be *a graded primary-like submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M)$ or $m \in Gr_M(N)$ (see [4].)

We say that a graded submodule N of M satisfies the gr-primeful property if for each graded prime ideal p of R with $(N :_R M) \subseteq p$, there exists a graded prime submodule P of M containing N such that $(P :_R M) = p$ (see [4].)

2. Results

Definition 2.1. *Let R be a G-graded ring and M a graded R-module. A proper graded submodule Q of M is said to be a graded quasi-primary if whenever $r \in h(R)$*

and $m \in h(M)$ *with* $rm \in Q$*, then either* $r \in Gr((Q :_R M))$ *or* $m \in Gr_M(Q)$*.*

The following Lemma is known (see [15, Lemma 1.2 and Lemma 2.7), we write be it her for the sake of references.

Lemma 2.2. *Let* R *be a* G*-graded ring and* M *a graded* R*-module. Then the the following hold:*

1. *If* N *is a graded submodule of* M*, then* $(N :_R M) = \{r \in R : rM \subseteq N\}$ *is a graded ideal of* R*.*
2. *If* N *is a graded submodule of* M*,* $r \in h(R)$*,* $x \in h(M)$ *and* I *is a graded ideal of* R*, then* Rx*,* IN *and* rN *are graded submodules of* M*.*
3. *If* N *and* K *are graded submodules of* M*, then* $N + K$ *and* $N \cap K$ *are also graded submodules of* M*.*
4. *If* $\{N_i\}_{i\in I}$ *is a collection of graded submodules of* M*, then* $N = \bigcap_{i\in I} N_i$ *is a graded submodule of* M*.*

Lemma 2.3. *Let* R *be a* G*-graded ring and* M *a graded* R*-module. If* Q *is a graded quasi-primary submodule of* M *satisfying the gr-primeful property, then* $p = Gr((Q :_R M))$ *is a graded prime ideal of* R *and we say that* Q *is a graded* p*-quasi-primary submodule of* M*.*

Proof: Suppose that $ab \in Gr((Q :_R M))$ and $b \notin Gr((Q :_R M))$ for some $a, b \in h(R)$. We show that $a \in (Gr_M(Q) :_R M)$. Let $m = \sum_{g\in G} m_g \in M$. Hence $\sum_{g\in G}(ab)^n m_g \in Q$ for some $n \in \mathbb{Z}$. Since Q is a graded quasi-primary submodule of M, $a^n b^n m_g \in Q$ and $b^n \notin (Q :_R M)$ for all $g \in G$, we get $a^n m_g \in Gr_M(Q)$ for all $g \in G$. Thus $a^n m \in Gr_M(Q)$. This shows that $a^n \in (Gr_M(Q) : M)$. Since Q satisfies the gr-primeful property, by [4, Theorem 3.5], we have $a^n \in (Gr_M(Q) :_R M) = Gr((Q :_R M))$ and so $a \in Gr((Q :_R M))$. Thus $Gr((Q :_R M))$ is a graded prime ideal of R. □

Recall that a graded R-module M over G-graded ring R is said to be *a graded free* R-module if it has an R-basis consisting of homogeneous elements (see [18].)

Lemma 2.4. *Let* R *be a* G*-graded ring,* M *a graded free* R*-module and* I *a graded ideal of* R*. Then* $Gr_M(IM) = Gr(I)M$*.*

Proof: Let $r \in Gr(I)\cap h(R)$. Then $r^n \in I$ for some $n \in \mathbb{Z}$. Let P be a graded prime submodule of M such that $IM \subseteq P$ and $m \in M$. Then $r^n m \in IM \subseteq P$. Since P is a graded prime submodule of M, we conclude that $rm \in P$. Thus $rM \subseteq P$ this implies that $rM \subseteq Gr_M(IM)$. Thus $Gr(I)M \subseteq Gr_M(IM)$. Conversely, note first that if $I = R$, then $Gr_M(IM) = Gr(I)M = M$. Assume that $I \neq R$. Note that $Gr(I)M = (\cap_{Q\in\Omega} Q)M = \cap_{Q\in\Omega}(QM)$ where Ω is the collection of graded prime ideals of R such that $I \subseteq Q$. By [11, Proposition 2.6(ii)], QM is a graded prime submodule of M and $IM \subseteq QM$. Hence $Gr_M(IM) \subseteq QM$ for all $Q \in \Omega$. Thus $Gr_M(IM) \subseteq \cap_{Q\in\Omega}(QM) = Gr(I)M$. Therefore $Gr_M(IM) = Gr(I)M$. □

A proper graded ideal I of a G-graded ring R is said to be a graded quasi-primary ideal if $ab \in I$ for $a, b \in h(R)$ implies $a \in Gr(I)$ or $b \in Gr(I)$.

Theorem 2.5. *Let R be a G-graded ring, M a graded free R-module and I a graded ideal of R. Then IM is a graded quasi-primary submodule of M if and only if I is a graded quasi-primary ideal of R.*

Proof: ($\Rightarrow$) Let IM be a graded quasi-primary submodule of M. Since $(IM :_R M) = I$, I is a graded proper ideal of R. Assume that $ab \in I$ and $b \notin Gr(I)$ for some $a, b \in h(R)$. By Lemma 2.3, $Gr_M(IM) = Gr(I)M$. This yields that $b \notin (Gr_M(IM) :_R M)$. Since IM is a graded quasi-primary submodule of M, $abM \subseteq IM$ and $b \notin (Gr_M(IM) :_R M)$, we conclude that $a \in Gr((IM :_R M)) = Gr(I)$. Therefore I is a graded quasi-primary ideal of R.

($\Leftarrow$)Assume that I is a graded quasi-primary ideal of R. It follows that $Gr(I)$ is a graded prime ideal of R. Since $(IM :_R M) = I$, IM is a proper graded submodule of M. Now, Let $rm \in IM \subseteq Gr(I)M$ and $r \notin Gr((IM :_R M)) = Gr(I)$ for some $r \in h(R)$ and $m \in h(M)$. Since $Gr(I)$ is a graded prime ideal of R by [11, Proposition 2.6(ii)], we conclude that $Gr(I)M$ is a graded prime submodule of M. Since $Gr(I)M$ is a graded prime submodule of M, $rm \in Gr(I)M$ and $r \notin Gr(I) = (Gr(I)M :_R M)$, we have $m \in Gr(I)M$ and hence $m \in Gr_M(IM)$ by Lemma 2.3. Therefore IM is a graded quasi-primary submodule of M. □

Let R be a G-graded ring and M a graded R-module. Let p be a graded prime ideal of R and N a graded submodule of M. Then $N(p) = \{m \in M : rm \in N$ for some $r \in h(R)\setminus p\}$ is a graded submodule of M containing N with the property that for each graded ideal q of R with $q \cap (h(R)\setminus p) \neq \emptyset, (qN)(p) = N(p)$ (see [24, p. 694)].)

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module and Q a graded submodule of M satisfying the gr-primeful property. Then Q is a graded p-quasi-primary submodule of M if and only if $Gr((Q :_R M)) = p$ is a graded prime ideal of R and $Q(p) \subseteq Gr_M(Q)$.*

Proof: ($\Rightarrow$) Assume that Q is a graded p-quasi-primary submodule of M. By Lemma 2.2., $Gr((Q :_R M)) = p$ is a graded prime ideal of R. Let $m \in Q(p)$. Then $rm \in Q$ for some $r \in h(R)\setminus p$. Since Q is a graded p-quasi-primary submodule, $rm \in Q$ and $r \notin p = Gr((Q :_R M))$, we have $r \in Gr_M(Q)$. Thus $Q(p) \subseteq Gr_M(Q)$.

($\Leftarrow$) Assume that $Gr((Q :_R M)) = p$ is a graded prime ideal of R and $Q(p) \subseteq Gr_M(Q)$. Let $rm \in Q$ and $m \notin Gr_M(Q)$ for some $r \in h(R)$ and $m \in h(M)$. Then $m \notin Q(p)$. Since $rm \in Q$ and $m \notin Q(p)$, $r \in p = Gr((Q :_R M)$. Therefore Q is a graded p-quasi-primary submodule of M. □

Lemma 2.7. *Let R be a G-graded ring, M a graded R-module and Q a graded quasi-primary submodule of M. Then $Q(p) \subseteq Gr_M(Q)$ for every $p \in V_R^g((Q :_R M))$. In particular, if $Q(p)$ is a graded prime submodule of M for some $p \in V_R^g((Q :_R M))$, then $Q(p) = Gr_M(Q)$.*

Proof: Let $p \in V_R^g((Q :_R M))$. Since p is a graded prime ideal containing $(Q :_R M)$, $Gr((Q :_R M)) \subseteq p$. Let $m \in Q(p) \cap h(M)$. Then $rm \in Q$ for some $r \in h(R) \setminus p$. Since Q a graded quasi-primary submodule of $M$and $r \notin Gr((Q :_R M))$, we have $m \in Gr_M(Q)$. Now, assume that $Q(p)$ is a graded prime submodule of M for some $p \in V_R^g((Q :_R M))$. Since $Q(p)$ is a graded prime submodule of M containing $Q, Gr_M(Q) \subseteq Q(p)$. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module and Q a graded p-quasi-primary submodule of M satisfying the gr-primeful property. Then $Q(p)$ is a graded p-prime submodule of M if and only if $Q(p) = Gr_M(Q)$.*

Proof: (⇒)Assume that $Q(p)$ is a graded p-prime submodule of M. Since $Q \subseteq Q(p)$, we have $(Q :_R M) \subseteq (Q(p) :_R M) = p$. By Lemma 2.4, $Q(p) = Gr_M(Q)$.

(⇐) Assume that $Q(p) = Gr_M(Q)$. Let $rm \in Q(p)$ and $m \notin Q(p)$ for some $r \in h(R)$ and $m \in h(M)$. It follows that $r \in p = Gr((Q :_R M))$. By [4, Theorem 3.5], $r \in p = Gr((Q :_R M)) = (Gr_M(Q) :_R M) = (Q(p) :_R M)$. Therefore $Q(p)$ is a graded p-prime submodule of M. □

Recall that a graded R-module M over G-graded ring R is said to be *a graded multiplication module (gr-multiplication module)* if for every graded submodule N of M there exists a graded ideal I of R such that $N = IM$. It is clear that M is gr-multiplication R- module if and only if $N = (N :_R M)M$ for every graded submodule N of M (see [12].)

Lemma 2.9. *Let R be a G-graded ring, M a graded multiplication R-module and I a graded ideal of R. If IM satisfies the gr-primeful property, then so dose $Gr(I)M$. In this case $Gr((IM :_R M)) = Gr((Gr(I)M :_R M))$.*

Proof: Let p be a graded prime ideal of R containing $(Gr(I)M :_R M)$. Since IM satisfies the gr-primeful property and $(IM :_R M) \subseteq (Gr(I)M :_R M) \subseteq p$, there exists a graded prime submodule P of M containing IM such that $(P :_R M) = p$. By [19, Corollary 3], $P = qM$ for some graded prime ideal q of R containing $ann(M)$. Since $IM \subseteq qM$, by [8, Lemma 3.9], $I \subseteq q$. Hence $Gr(I)M \subseteq qM = P$ by [20, Proposition 1.2]. Therefore $Gr(I)M$ satisfies the gr-primeful property. Also the similar argument follows that $Gr_M(IM) = Gr_M(Gr(I)M)$ and so we have the second part. □

Theorem 2.10. *Let R be a G-graded ring, M a graded multiplication R-module and Q a graded submodule of M satisfies the gr-primeful property. Then the following statements are equivalent:*

1. *Q is a graded quasi-primary submodule of M;*
2. *$(Q :_R M)$ is a graded quasi-primary ideal of R;*
3. *$Gr_M(Q)$ is a graded prime submodule of M;*
4. *$Q = qM$ for some graded quasi-primary ideal q of R with $ann(M) \subseteq q$.*

Proof: $(i) \Rightarrow (ii)$ Let $rs \in (Q :_R M)$ and $s \notin Gr((Q :_R M))$ for some $r, s \in h(R)$. By [4, Theorem 3.5], $s \notin Gr((Q :_R M)) = (Gr(Q) :_R M)$. Then there exists $m \in M$ such that $sm \notin Gr_M(Q)$ and $rsm \in Q$. Since Q is a graded quasi-primary submodule of M, we have $r \in Gr((Q :_R M))$. Therefore $(Q :_R M)$ is a graded quasi-primary ideal of R.

$(ii) \Rightarrow (iii)$ It is easy to check that $Gr((Q :_R M))$ is a graded prime ideal of R, since $(Q :_R M)$ is a graded quasi-primary ideal of R. By [19, Theorem 9], $Gr_M(Q) = Gr((Q :_R M))M$. It follows that $Gr_M(Q)$ is a graded prime submodule of M by [19, Corollary 3].

$(iii) \Rightarrow (i)$ Let $rm \in Q$ and $r \notin Gr((Q :_R M))$ for some $r \in h(R)$ and $m \in h(M)$. By [4, Theorem 3.5], $r \notin (Gr_M(Q) :_R M) = Gr((Q :_R M)$. Since $Gr_M(Q)$ is a graded prime submodule, $rm \in Q \subseteq Gr_M(Q)$ and $r \notin (Gr_M(Q) :_R M)$, we have $m \in Gr_M(Q)$. Therefore Q is a graded quasi-primary submodule of M.

$(ii) \Rightarrow (iv)$ If we let $(Q :_R M) = q$, it is clear.

$(iv) \Rightarrow (iii)$ Assume that $Q = qM$ for some graded quasi-primary ideal q of R with $ann(M) \subseteq q$. By [19, Theorem 9] and Lemma 2.5, we have $Gr_M(Q) = Gr((Q :_R M))M = Gr((qM :_R M)) = Gr((Gr(q)M :_R M)) = Gr(Gr(q))M = Gr(q)M$. Hence by [19, Corollary 3], $Gr_M(Q)$ is a graded prime submodule of M. □

Theorem 2.11. *Let R be a G-graded ring, M a graded multiplication R-module and U a graded submodule of M. Let Q be a graded quasi-primary submodule of M satisfying the gr-primeful property such that $p = Gr((U :_R M)) = Gr((Q :_R M))$ and $U \subseteq N \subseteq Q$. Then N is a graded p-quasi-primary submodule of M.*

Proof: It is easy to see $Gr((U :_R M)) = Gr((N :_R M)) = Gr((Q :_R M)) = p$. This yields that $(N :_R M)$ is a graded p-quasi-primary ideal of R. Now we show that N satisfies the gr-primeful property. Let p be a graded prime ideal of R containing $(N :_R M)$. Then $(Q :_R M) \subseteq G((Q :_R M)) \subseteq G((N :_R M)) \subseteq p$ by [20, Proposition 1.2]. Since Q satisfies the gr-primeful property, there exists a graded prime submodule P of M containing Q and so N such that $(P :_R M) = p$. Therefore N satisfies the gr-primeful property. By Theorem 2.4, N is a graded p-quasi-primary submodule of M. □

Lemma 2.12. *Let R be a G-graded ring and M a graded R-module. If Q is a graded p-quasi-primary submodule of M satisfying the gr-primeful property, then $Gr_M(Q) = Gr_M(Q + pM)$.*

Proof: Clearly $Gr_M(Q) \subseteq Gr_M(Q+pM)$. Let P be a graded q-prime submodule such that $Q \subseteq P$. Then $(Gr_M(Q) :_R M) \subseteq (P :_R M) = q$. By [4, Theorem 3.5], we have $p = Gr(Q :_R M) = (Gr_M(Q) :_R M) \subseteq (P :_R M) = q$. So $Q + pM \subseteq P + qM \subseteq P$. Hence $Q + pM \subseteq Gr_M(N)$ and so $Gr_M(Q+pM) \subseteq Gr_M(Q)$ by [23, Theorem 2.12]. Therefore $Gr_M(Q) = Gr_M(Q+pM)$. □

Theorem 2.13. *Let R be a G-graded ring, M a graded R-module and q a graded maximal ideal of R. If Q is a graded q-quasi-primary submodule of M satisfying the gr-primeful property, then $Gr_M(Q) = Gr_M(Q+qM) = Q+qM$.*

Proof: By [4, Theorem 3.5], we have $(Gr_M(Q) :_R M) = Gr((Q :_R M)) = q$. Since $(Gr_M(Q) :_R M) = q$ is a graded maximal ideal of R, by [9, Proposition 2.4(i)], we have $Gr_M(Q)$ is a graded q-prime submodule of M. Thus $Q + qM \subseteq Gr_M(Q+qM) = Gr_M(Q)$ by Lemma 2.6 . It follows that $q \subseteq (Q+qM :_R M) \subseteq (Gr_M(Q) :_R M) = q$. This yields that $(Q+qM :_R M) = q$. By [9, Proposition 2.4(i)], we conclude that $Q+qM$ is a graded prime submodule of M containing Q. Hence $Gr_M(Q) \subseteq Q+qM$. Therefore $Gr_M(Q) = Gr_M(Q+qM) = Q+qM$. □

Theorem 2.14. *Let R be a G-graded ring and M a graded R-module. If Q is a graded quasi-primary submodule of M and K a graded submodule of M such that $Gr_M(Q) \cap Gr_M(K) = Gr_M(Q \cap K)$, then $K \subseteq Q$ or $Q \cap K$ is a graded quasi-primary submodule of K.*

Proof: Assume that $K \nsubseteq Q$. Then $Q \cap K$ is a proper graded submodule of K. Let $rk \in Q \cap K$ and $r \notin Gr((Q \cap K :_R K))$ for some $r \in h(R)$ and $k \in h(K)$. It follows that $rk \in Q$ and $r \notin Gr((Q :_R M))$. Since Q is a graded quasi-primary submodule of M, $rk \in Q$ and $r \notin Gr((Q :_R M))$, we conclude that $k \in Gr_M(Q)$. Hence $k \in Gr_M(Q) \cap Gr_M(K) = Gr_M(Q \cap K)$. Therefore $Q \cap K$ is a graded quasi-primary submodule of K. □

References

1. Al-Zoubi, K., *The graded primary radical of a graded submodules*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 1, 395-402, (2016).
2. Al-Zoubi, K., Abu-Dawwas, R. and Al-Ayyoub, I., *Graded semiprime submodules and graded semi-radical of graded submodules in graded modules*, Ricerche mat., 66(2), 449-455, (2017).
3. Al-Zoubi, K. and Al-Dolat, M., *On graded classical primary submodules*, Adv. Pure Appl. Math., 7 (2), 93-96, (2016).
4. Al-Zoubi, K. and Al-Dolat, M., *On graded primary-like submodules of graded modules over graded commutative rings*, submitted.
5. Al-Zoubi, K., Jaradat, M. and Abu-Dawwas, R., *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc., 41 (1), 217-225, (2015).
6. Al-Zoubi, K. and Qarqaz, F., *An intersection condition for graded prime submodules in Gr-multiplication modules*, Math. Reports, 20 (3), 329–336, (2018).
7. Atani, S. E., *On graded prime submodules*, Chiang Mai J. Sci., 33 (1), 3-7, (2006).

8. Atani S. E., and Atani, R. E., *Graded multiplication modules and the graded ideal $\theta_g(M)$*, Turk. J. Math., 33, 1-9, (2009).

9. Atani, S. E., and Farzalipour, F., *Notes on the graded prime submodules*, Int. Math. Forum 1, 38, 1871-1880, (2006).

10. Atani, S. E. and Saraei, F. E. K.,*Graded Modules which Satisfy the Gr-Radical Formula*, Thai J. Math., 8 (1), 161-170, (2010).

11. Atani, S. E. and Farzalipour, F., *On graded secondary modules*, Turk. J. Math., 31, 371-378, (2007).

12. Escoriza, J. and Torrecillas, B., *Multiplication Objects in Commutative Grothendieck Categories*, Comm. in Algebra, 26 (6), 1867-1883, (1998).

13. Ghiasvand, P. and Farzalipour, F., *On Graded Primary Submodules of Graded Multiplication Modules*, Int. J. Alg., 4 (9), 429-434, (2010).

14. Hazrat, R., *Graded Rings and Graded Grothendieck Groups*, Cambridge University Press, Cambridge, (2016).

15. Lee, S.C. and Varmazyar, R., *Semiprime submodules of Graded multiplication modules*, J. Korean Math. Soc., 49 (2), 435-447, (2012).

16. C. Nastasescu and V.F. Oystaeyen, Graded and filtered rings and modules. Lecture notes in mathematics 758, Berlin-New York: Springer-Verlag, (1982).

17. Nastasescu, C. and Van Oystaeyen, F.,*Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, (1982).

18. Nastasescu, C. and Van Oystaeyen, F., *Methods of Graded Rings*, LNM 1836. Berlin-Heidelberg: Springer-Verlag, (2004).

19. Oral, K. H. Tekir, U. and Agargun, A. G., *On Graded prime and primary submodules*, Turk. J. Math., 35, 159-167, (2011).

20. Refai, M. and Al-Zoubi, K.,*On graded primary ideals*, Turk. J. Math., 28, 217-229, (2004).

21. Refai, M., Hailat, M. and Obiedat, S.,*Graded radicals on graded prime spectra*, Far East J. of Math. Sci., part I, 59-73, (2000).

22. Tavallaee, H. A. and Zolfaghari, M.,*Graded weakly semiprime submodules of graded multiplication modules*, Lobachevskii J. Math., 34 (1), 61-67, (2013).

23. Varmazyar, R.,*Graded coprime submodules*, Math. Sci., 6(1), Art. 70, 4 pp., (2012).

24. Zamani, N., *Finitely generated graded multiplication modules*, Glasgow Math. J., 53(3), 693-705, (2011).

Khaldoun Al-Zoubi,
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
Jordan.
E-mail address: kfzoubi@just.edu.jo

and

Rweili Alkhalaf,
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
Jordan.
E-mail address: rmalkhalaf16@sci.just.edu.jo