



A Number of Limit Cycle of Sextic Polynomial Differential Systems via the Averaging Theory

Amor Menaceur and Salah Boulaaras*

ABSTRACT: The main purpose of this paper is to study the number of limit cycles of sextic polynomial differential systems (SPDS) via the averaging theory which is an extension to the study of cubic polynomial vector fields in (Nonlinear Analysis 66 (2007), 1707–1721), where we provide an accurate upper bound of the maximum number of limit cycles that SPDS can have bifurcating from the period annulus surrounding the origin of a class of cubic system.

Key Words: Limit cycle, Averaging method, Conic, Sextic polynomial differential systems (SPDS).

Contents

1	Introduction	181
2	The first order averaging method	183
3	Proof of Theorem 1	183
3.1	Proof of statement (a) of Theorem 1 with $(f=(x+a)^2)$	184
3.2	Proof of statement (b) of Theorem 1 with $(f=x-a-y^2)$	189

1. Introduction

The main open problem in the qualitative theory of real planar differential systems is the determination and finding the distribution of their limit cycles, where the famous Hilbert’s sixteenth problem and its weak form (for more detail see [7]) and the limit cycles number which bifurcates from periodic orbits of a nonlinear or linear differential system with a center has been dealt extensively recently using many different methods, the method of inverse integral factor in ([6]), Hopf methods combined with homoclinic bifurcation theory in [12], Lyapunov constants in [4], function of Melnikov in [13] and an averaging theory in [2], [8] and [9]. A previous used method to produce limit cycles is by perturbing a system that has a center, in such a method that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the period annulus of the center of the unperturbed system, see [11]. In the last recent work [5], the authors proved the maximum number of limit cycles which can be bifurcated from the period annulus surrounding the origin of a class of cubic polynomial differential systems using the averaging method. More precisely, they demonstrated that the

* Corresponding author
 2010 *Mathematics Subject Classification:* Primary: 34C23, Secondary: 34C35, 34C25, 34C29.
 Submitted March 04, 2018. Published April 23, 2018

perturbations of the period annulus of the center located at the origin of the cubic polynomial differential system

$$\dot{x} = -yf(x, y), \quad \dot{y} = xf(x, y), \quad (1.1)$$

where $f(x, y) = 0$ is a conic and $f(0, 0) \neq 0$ by using the arbitrary cubic polynomial differential systems provide at least 6 limit cycles bifurcating from the periodic orbits of the period annulus, then, in [10] the authors found a bound for the number of limit cycles which bifurcate from the period annulus of the center, under piecewise smooth cubic polynomial perturbations. Their results explained that the piecewise smooth cubic system can have at least one more limit cycle than the smooth one, Moreover in [1] the author improved the result of the maximum number of limit cycles of quartic and quintic polynomial differential systems which bifurcate from the period annulus surrounding the origin of system (1.1) by using the first order of averaging theory method.

In this paper, we are interested to study the number of limit cycles which can be bifurcated from the period annulus surrounding the origin of a class of cubic polynomial differential system of the form introduced in [5], when it inside the class of all sextic polynomial differential systems having the origin as a singular point is perturbed, i.e., for all ε sufficiently small, we extend the work of Jaume Gine and Jaume Llibre [5] by studying the number of limit cycles of the following system

$$\begin{cases} \dot{x} = -yf(x, y) + \varepsilon \left(\sum_{k=1}^6 P_k(x, y) \right), \\ \dot{y} = xf(x, y) + \varepsilon \left(\sum_{k=1}^6 Q_k(x, y) \right), \end{cases} \quad (1.2)$$

which bifurcate from the period annulus surrounding the origin of system (1.1), where P_k and Q_k homogeneous polynomials of degree k . Thus we can write

$$P_k = \sum_{i+j=k} p_{ij} x^i y^j$$

and

$$Q_k = \sum_{i+j=k} q_{ij} x^i y^j.$$

The main results of this paper is to prove the following theorem:

Theorem 1.1. *For all $|\varepsilon|$ is sufficiently small, the maximum number of limit cycles of the sextic polynomial differential system (1.2) bifurcating from the periodic orbits of the system (1.1) using the first order of averaging theory method and with respect to the conic $f(x, y) = 0$ given by one of the following two cases:*

(a) *Seven limit cycles for a one double invariant real straight line defined as:*

$$f(x, y) = (x + a)^2 = 0, \quad a > 0.$$

(b) *Fourteen limit cycles for Parabola curves defined as:*

$$f(x, y) = x - a - y^2 = 0, \quad a \neq 0.$$

2. The first order averaging method

We introduce in this section the first order of averaging theory method for computing limit cycles up to first order ε which has been extensively given in [3].

Theorem 2.1. Consider the following two initial value problems:

$$\dot{x} = \varepsilon R(t, x) + \varepsilon^2 G(t, x, \varepsilon), \quad x(0) = x_0 \tag{2.1}$$

and

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \tag{2.2}$$

where x, y and $x_0 \in D$ which is an open domain of \mathbb{R} , $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, R and G are periodic functions with their period T with its variable t , and $f^0(y)$ is the average function of $R(t, y)$ with respect to t , i.e.,

$$f^0(y) = \frac{1}{T} \int_0^T F(t, y) dt. \tag{2.3}$$

Assume that:

(i). $R, \frac{\partial R}{\partial x}, \frac{\partial^2 R}{\partial x^2}, G$ and $\frac{\partial G}{\partial x}$ are well defined, continuous and bounded by a constant independent by $\varepsilon \in (0, \varepsilon_0]$ in $[0, \infty) \times D$.

(ii). T is a constant independent of ε ;

(iii). $y(t)$ belongs to D on the time scale $1/\varepsilon$. Then the following statements hold

(a)- On the time scale $\frac{1}{\varepsilon}$, we have

$$x(t) - y(t) = O(\varepsilon), \text{ as } \varepsilon \rightarrow 0.$$

(b)- If p is an equilibrium point of the averaged system (2.2), such that

$$\left. \frac{\partial f^0}{\partial y} \right|_{y=p} \neq 0, \tag{2.4}$$

then system (2.1) has a T -periodic solution $\phi(t, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

(c)- If (2.4) is a negative. Therefore, the corresponding periodic solution $\phi(t, \varepsilon)$ of equation (2.1) according to (t, x) is asymptotically stable for all ε sufficiently small, if (2.4) is a positive, then it is unstable.

3. Proof of Theorem 1

This proof is based on the first order of averaging theory method, we can write system (1.2) by using the polar coordinates (r, θ) where $x = r \cos \theta, y = r \sin \theta$, it can be easily gotten

$$\begin{cases} \dot{r} = \varepsilon \sum_{k=1}^6 (\cos \theta P_k(\cos \theta, \sin \theta) + \sin \theta Q_k(\cos \theta, \sin \theta)) r^k, \\ \dot{\theta} = f(r \cos \theta, r \sin \theta) \left(1 + \varepsilon \sum_{k=1}^6 \frac{\cos \theta Q_k(\cos \theta, \sin \theta) - \sin \theta P_k(\cos \theta, \sin \theta)}{f(r \cos \theta, r \sin \theta)} r^{k-1} \right), \end{cases}$$

where

$$\begin{aligned} P_k(\cos \theta, \sin \theta) &= \sum_{i+j=k} p_{ij}^i \cos^i \theta \sin^j \theta, \\ Q_k(\cos x, \sin x) &= \sum_{i+j=k} q_{ij} \cos^i \theta \sin^j \theta. \end{aligned}$$

Therefore, we have

$$\frac{dr}{d\theta} = \varepsilon \sum_{k=1}^6 \frac{\cos \theta P_k(\cos \theta, \sin \theta) + \sin \theta Q_k(\cos \theta, \sin \theta)}{f(r \cos \theta, r \sin \theta)} r^k + \varepsilon^2 G(r, \theta, \varepsilon). \quad (3.1)$$

Using the notations introduced in theorem 2, we must calculate

$$f^0(r) = \frac{1}{2\pi} \sum_{k=1}^6 r^k \int_0^{2\pi} \frac{\cos \theta P_k(\cos \theta, \sin \theta) + \sin \theta Q_k(\cos \theta, \sin \theta)}{f(r \cos \theta, r \sin \theta)} d\theta.$$

Therefore, we get

$$f^0(r) = \sum_{k=1}^6 C_k(r) r^k,$$

where

$$C_k(r) = \sum_{i+j=k} (p_{ij} \alpha_{i+1,j}(r) + q_{ij} \alpha_{i,j+1}(r))$$

and

$$\alpha_{p,q} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^p \theta \sin^q \theta}{f(r \cos \theta, r \sin \theta)} d\theta.$$

The function $f^0(r)$ can be written as

$$f^0(r) = \sum_{i=1}^6 \sum_{j=0}^{i+1} (\lambda_{i,j} \alpha_{i+1-j,j}) r^i, \quad (3.2)$$

where

$$\lambda_{i,j} = \begin{cases} p_{i,0} & \text{if } j = 0, \\ p_{i-1,j} + q_{i+1-j,j-1} & \text{if } 1 \leq j \leq i, \\ q_{0,i} & \text{if } j = i + 1. \end{cases}$$

Now we prove the statements of theorem 1, in the first step, we compute the integral $f^0(r)$, and the second step, the number of its simple zeros is studied.

3.1. Proof of statement (a) of Theorem 1 with $(\mathbf{f}=(\mathbf{x}+\mathbf{a})^2)$

For the case of one double invariant real straight line, system (1.1) has a unique period annulus defined as:

$$A = \{(x, y) : 0 < x^2 + y^2 < a^2\}$$

Lemma 3.1. *Under the previous assumptions and notations, we have:*

$$\begin{aligned}
 (i). \alpha_{n+1,0} &= \begin{cases} -\frac{2a}{r}\alpha_{n,0} - \frac{a^2}{r^2}\alpha_{n-1,0} & \text{if } n \text{ is an even number,} \\ \frac{1}{2\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta d\theta - \frac{2a}{r}\alpha_{n,0} - \frac{a^2}{r^2}\alpha_{n-1,0}, & \text{if } n \text{ is an odd number} \end{cases} \\
 (ii). \alpha_{0,n+1} &= \begin{cases} 0 & \text{if } n \text{ is an even number,} \\ \sum_{k=0}^{\frac{n+1}{2}} (-1)^k C_{\frac{n+1}{2}}^k \alpha_{2k,0} & \text{if } n \text{ is an odd number} \end{cases} \\
 (iii). \alpha_{n-2l+1,2l} &= \sum_{k=0}^l (-1)^k C_l^k \alpha_{n-2l+2k+1,0}, \quad l = 1, \dots, \left[\frac{n+1}{2} \right].
 \end{aligned}$$

Proof. Putting $x = r \cos \theta$, $y = r \sin \theta$, we get for (i)

$$\begin{aligned}
 \frac{\cos^{n+1} \theta}{f(r \cos \theta, r \sin \theta)} &= \frac{(r \cos \theta + a)^2 \cos^{n-1} \theta - 2ar \cos^n \theta - a^2 \cos^{n-1} \theta}{r^2 f(r \cos \theta, r \sin \theta)} \\
 &= \frac{1}{r^2} \cos^{n-1} \theta - \frac{2a}{r} \frac{\cos^n \theta}{f(r \cos \theta, r \sin \theta)} - \frac{a^2}{r^2} \frac{\cos^{n-1} \theta}{f(r \cos \theta, r \sin \theta)},
 \end{aligned}$$

thus

$$\begin{aligned}
 \alpha_{n+1,0} &= \frac{1}{2\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta d\theta - \frac{2a}{2\pi r} \int_0^{2\pi} \frac{\cos^n \theta}{f(r \cos \theta, r \sin \theta)} d\theta \\
 &\quad - \frac{a^2}{2\pi r^2} \int_0^{2\pi} \frac{\cos^{n-1} \theta}{f(r \cos \theta, r \sin \theta)} d\theta.
 \end{aligned}$$

Therefore,

$$\alpha_{n+1,0} = \begin{cases} -\frac{2a}{r}\alpha_{n,0} - \frac{a^2}{r^2}\alpha_{n-1,0} & \text{if } n \text{ is an even number,} \\ \frac{1}{2\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta d\theta - \frac{2a}{r}\alpha_{n,0} - \frac{a^2}{r^2}\alpha_{n-1,0} & \text{if } n \text{ an odd number.} \end{cases}$$

(ii)

$$\frac{\sin^{n+1} \theta}{f(r \cos \theta, r \sin \theta)} = \begin{cases} 0 & \text{if } n \text{ is an even number,} \\ \frac{(1-\cos^2 \theta)^{\frac{n+1}{2}}}{f(r \cos \theta, r \sin \theta)} = \sum_{k=0}^{\frac{n+1}{2}} (-1)^k C_{\frac{n+1}{2}}^k \frac{\cos^{2k} \theta}{f(r \cos \theta, r \sin \theta)}, & \text{if } n \text{ an odd number,} \end{cases}$$

then

$$\alpha_{0,n+1} = \begin{cases} 0 & \text{if } n \text{ is an even number,} \\ \sum_{k=0}^{\frac{n+1}{2}} (-1)^k C_{\frac{n+1}{2}}^k \alpha_{2k,0} & \text{if } n \text{ an odd number.} \end{cases}$$

(iii)

$$\begin{aligned}
 \frac{\cos^{n-2l+1} \theta \sin^{2l} \theta}{f(r \cos \theta, r \sin \theta)} &= \frac{\cos^{n-2l+1} \theta (1 - \cos^2 \theta)^l}{f(r \cos \theta, r \sin \theta)} \\
 &= \sum_{k=0}^l (-1)^k C_l^k \frac{\cos^{n-2l+2k+1} \theta}{f(r \cos \theta, r \sin \theta)}, \quad l = 1, \dots, \left[\frac{n+1}{2} \right].
 \end{aligned}$$

Thus,

$$\alpha_{n-2l+1,2l} = \sum_{k=0}^l (-1)^k C_l^k \alpha_{n-2l+2k+1,0}, \quad l = 1, \dots, \left[\frac{n+1}{2} \right].$$

This completes the proof. \square

Remark 3.2. We note for any p, k with $p + 2k \leq 6$ that

$$\alpha_{p,2k+1} = 0.$$

Lemma 3.3. Under the previous notations, we have

$$\alpha_{0,0} = \frac{a}{g^3} \text{ and } \alpha_{1,0} = -\frac{r}{g^3}, \text{ such that } g = \sqrt{a^2 - r^2}.$$

Proof. Assume that

$$z = e^{i\theta} \text{ and } C = \{z : |z| = 1\},$$

we get

$$\int_0^{2\pi} \frac{d\theta}{f(r \cos \theta, r \sin \theta)} \equiv \frac{4}{i} \int_C \frac{z}{(rz^2 + 2az + r)^2} dz$$

and

$$\int_0^{2\pi} \frac{\cos \theta d\theta}{f(r \cos \theta, r \sin \theta)} \equiv \frac{2}{i} \int_C \frac{z^2 + 1}{(rz^2 + 2az + r)^2} dz,$$

which whose poles are

$$z_1 = \frac{-a + \sqrt{a^2 - r^2}}{r} \text{ and } z_2 = \frac{-a - \sqrt{a^2 - r^2}}{r},$$

Since $a > 0$ implies that $|z_1| < 1$ and $|z_2| > 1$ and by using the residue theorem, we get

$$\int_0^{2\pi} \frac{d\theta}{f(r \cos \theta, r \sin \theta)} = \frac{2\pi a}{(\sqrt{a^2 - r^2})^3}, \quad \int_0^{2\pi} \frac{\cos \theta d\theta}{f(r \cos \theta, r \sin \theta)} = -\frac{2\pi r}{(\sqrt{a^2 - r^2})^3}.$$

Therefore, we have

$$\alpha_{0,0} = \frac{a}{(\sqrt{a^2 - r^2})^3}, \quad \alpha_{1,0} = -\frac{r}{(\sqrt{a^2 - r^2})^3}$$

This completes the proof. \square

Based on Lemmas 1 and 2, we can deduce to calculate the following amounts

$$\begin{aligned}\alpha_{2,0} &= \frac{1}{2\pi r^2} \int_0^{2\pi} d\theta - \frac{2a}{r} \alpha_{1,0} - \frac{a^2}{r^2} \alpha_{0,0} \\ &= \frac{1}{r^2} - \frac{2a}{r} \left(-\frac{r}{g^3} \right) - \frac{a^2}{r^2} \frac{a}{g^3} \\ &= \frac{g^3 + 2ar^2 - a^3}{g^3 r^2},\end{aligned}$$

then

$$\begin{aligned}\alpha_{0,2} &= \alpha_{0,0} - \alpha_{2,0} \\ &= \frac{a}{g^3} - \frac{(g^3 + 2ar^2 - a^3)}{g^3 r^2} \\ &= \frac{-g^3 - ar^2 + a^3}{g^3 r^2}\end{aligned}$$

and we also have

$$\begin{aligned}\alpha_{3,0} &= \frac{-2ag^3 - (3a^2r^2 - 2a^4)}{g^3 r^3} \\ \alpha_{1,2} &= \frac{2ag^3 - (r^4 - 3a^2r^2 + 2a^4)}{g^3 r^3}, \\ \alpha_{4,0} &= \frac{(r^2 + 6a^2)g^3 + (8a^3r^2 - 6a^5)}{2g^3 r^4}, \\ \alpha_{2,2} &= \frac{(r^2 - 6a^2)g^3 + (4ar^4 - 10a^3r^2 + 6a^5)}{2g^3 r^4}\end{aligned}$$

and

$$\begin{aligned}\alpha_{0,4} &= \frac{-(3r^2 - 6a^2)g^3 - (6ar^4 - 12a^3r^2 + 6a^5)}{2g^3 r^4}, \\ \alpha_{5,0} &= \frac{-(ar^2 + 4a^3)g^3 - (5a^4r^2 - 4a^6)}{g^3 r^5}, \\ \alpha_{3,2} &= \frac{-(ar^2 - 4a^3)g^3 - (3a^2r^4 - 7a^4r^2 + 4a^6)}{g^3 r^5}, \\ \alpha_{1,4} &= \frac{(3ar^2 - 4a^3)g^3 - (r^6 - 6a^2r^4 + 9a^4r^2 - 4a^6)}{g^3 r^5}, \\ \alpha_{6,0} &= \frac{(3r^4 + 12a^2r^2 + 40a^4)g^3 + (48a^5r^2 - 40a^7)}{8g^3 r^6}, \\ \alpha_{4,2} &= \frac{(r^4 + 12a^2r^2 - 40a^4)g^3 + (32a^3r^4 - 72a^5r^2 + 40a^7)}{8g^3 r^6}.\end{aligned}$$

In addition, we have

$$\begin{aligned}
\alpha_{2,4} &= \frac{(3r^4 - 36a^2r^2 + 40a^4)g^3 + (16ar^6 - 72a^3r^4 + 96a^5r^2 - 40a^7)}{8g^3r^6}, \\
\alpha_{0,6} &= \frac{-(15r^4 - 60a^2r^2 + 40a^4)g^3 - (40ar^6 - 120a^3r^4 + 120a^5r^2 - 40a^7)}{8g^3r^6}, \\
\alpha_{7,0} &= \frac{(-3ar^4 - 8a^3r^2 - 24a^5)g^3 + (-28a^6r^2 + 24a^8)}{4g^3r^7}, \\
\alpha_{5,2} &= \frac{(-ar^4 - 8a^3r^2 + 24a^5)g^3 + (-20a^4r^4 + 44a^6r^2 - 24a^8)}{4g^3r^7}, \\
\alpha_{3,4} &= \frac{(-3ar^4 + 24a^3r^2 - 24a^5)g^3 + (-12a^2r^6 + 48a^4r^4 - 60a^6r^2 + 24a^8)}{4g^3r^7}, \\
\alpha_{1,6} &= \frac{(15ar^4 - 40a^3r^2 + 24a^5)g^3}{4g^3r^7} \\
&\quad + \frac{(-4r^8 + 37a^2r^6 - 84a^4r^4 + 76a^6r^2 - 24a^8)}{4g^3r^7}.
\end{aligned}$$

Using (3.2), we get

$$f^0(r) = \frac{(a_4r^4 + a_2r^2 + a_0)g^3 + (b_8r^8 + b_6r^6 + b_4r^4 + ab_2r^2 - a^3a_0)}{8g^3r}$$

where

$$\begin{aligned}
a_0 &= 48(-\lambda_{6,0} + \lambda_{6,2} - \lambda_{6,4} + \lambda_{6,6})a^5 + 40(\lambda_{5,0} - \lambda_{5,2} + \lambda_{5,4} - \lambda_{5,6})a^4 \\
&\quad + 32(-\lambda_{4,0} + \lambda_{4,2} - \lambda_{4,4})a^3 + 24(\lambda_{3,0} - \lambda_{3,2} + \lambda_{3,4})a^2 + \\
&\quad + 16(\lambda_{2,2} - \lambda_{2,0})a + 8(\lambda_{1,0} - \lambda_{1,2}),
\end{aligned}$$

$$\begin{aligned}
a_2 &= 16(-\lambda_{6,0} - \lambda_{6,2} + 3\lambda_{6,4} - 5\lambda_{6,6})a^3 \\
&\quad + 12(\lambda_{5,0} + \lambda_{5,2} - 3\lambda_{5,4} + 5\lambda_{5,6})a^2 \\
&\quad + 8(-\lambda_{4,0} - \lambda_{4,2} + 3\lambda_{4,4})a + 4(\lambda_{3,0} + \lambda_{3,2} - 3\lambda_{3,4}),
\end{aligned}$$

$$a_4 = 2(-3\lambda_{6,0} - \lambda_{6,2} - 3\lambda_{6,4} + 15\lambda_{6,6})a + (3\lambda_{5,0} + \lambda_{5,2} + 3\lambda_{5,4} - 15\lambda_{5,6}),$$

$$\begin{aligned}
b_2 &= 8(-7\lambda_{6,0} + 11\lambda_{6,2} - 15\lambda_{6,4} + 19\lambda_{6,6})a^6 \\
&\quad + 8(6\lambda_{5,0} - 9\lambda_{5,2} + 12\lambda_{5,4} - 15\lambda_{5,6})a^4 \\
&\quad + 8(-5\lambda_{4,0} + 7\lambda_{4,2} - 9\lambda_{4,4})a^3 + 8(4\lambda_{3,0} - 5\lambda_{3,2} + 6\lambda_{3,4})a^2 \\
&\quad + 24(-\lambda_{2,0} + \lambda_{2,2})a + 8(2\lambda_{1,0} - \lambda_{1,2}),
\end{aligned}$$

$$\begin{aligned}
b_4 &= 8(-5\lambda_{6,2} + 12\lambda_{6,4} - 21\lambda_{6,6})a^4 + 8(4\lambda_{5,2} - 9\lambda_{5,4} + 15\lambda_{5,6})a^3 \\
&\quad + 24(-\lambda_{4,2} + 2\lambda_{4,4})a^2 + 8(2\lambda_{3,2} - 3\lambda_{3,4})a - 8\lambda_{2,2},
\end{aligned}$$

$$b_6 = 24(-\lambda_{6,4} + 3\lambda_{6,6})a^2 + 8(2\lambda_{5,4} - 5\lambda_{5,6})a - 8\lambda_{4,4},$$

$$b_8 = -8\lambda_{6,6}$$

and

$$g = \sqrt{a^2 - r^2}.$$

In fact, there are only seven independent parameters between the a_i and b_i with respect to p_{ij}, q_{ij} and a . In order to bound the number of zeros of numerator of f^0 , it is sufficient to bound the number of zeros of

$$k(r) = (a_4r^4 + a_2r^2 + a_0)^2 (g^3)^2 - (b_8r^8 + b_6r^6 + b_4r^4 + ab_2r^2 - a^3a_0)^2$$

we can obtain

$$\begin{aligned} k(r) = & r^2 [(-b_8^2)r^{14} + (-a_4^2 - 2b_6b_8)r^{12} + (3a_4^2a^2 - 2a_2a_4 - b_6^2 - 2b_4b_8)r^{10} \\ & + (-3a_4^2a^4 + 6a_2a_4a^2 - 2b_2b_8a - (a_2^2 + 2a_0a_4 + 2b_4b_6))r^8 \\ & + (a_4^2a^6 - 6a_2a_4a^4 + 2a_0b_8a^3)r^6 \\ & + ((3a_2^2 + 6a_0a_4)a^2 - 2b_2b_6a - (b_4^2 + 2a_0a_2))r^6 \\ & + (2a_2a_4a^6 + (-3a_2^2 - 6a_0a_4)a^4 + 2a_0b_6a^3 + 6a_0a_2a^2 - 2b_2b_4a - a_0^2)r^4 \\ & + ((a_2^2 + 2a_0a_4)a^6 - 6a_0a_2a^4 + 2a_0b_4a^3 + (3a_0^2 - b_2^2)a^2)r^2 \\ & + (2a_0a_2a^6 + (2a_0b_2 - 3a_0^2)a^4)] \end{aligned}$$

Hence, $f^0(r)$ has at most seven simple zeros. Therefore, statement (a) in Theorem 1 holds.

3.2. Proof of statement (b) of Theorem 1 with (f=x-a-y²)

For the case of the parabola, system (1.1) has a unique period annulus as

$$A = \{(x, y) : 0 < x^2 + y^2 < r_1^2\}$$

where

$$\begin{aligned} r_1 &= \min \left\{ \sqrt{x^2 + y^2} : x - a - y^2 = 0 \right\} \\ &= \begin{cases} |a| & \text{if } a > -1/2, \\ \sqrt{-4a - 1}/2 & \text{if } a \leq -1/2. \end{cases} \end{aligned}$$

Lemma 3.4. *We have*

$$\alpha_{n+1,0} = \begin{cases} -\frac{1}{r}\alpha_{n,0} + \frac{r^2 + a}{r^2}\alpha_{n-1,0} & \text{if } n \text{ is an even numbers,} \\ \frac{1}{2\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta d\theta - \frac{1}{r}\alpha_{n,0} + \frac{r^2 + a}{r^2}\alpha_{n-1,0} & \text{if } n \text{ is an odd number.} \end{cases}$$

Proof. Putting $x = r \cos \theta$, $y = r \sin \theta$, we get

$$\begin{aligned} \frac{\cos^{n+1} \theta}{f(r \cos \theta, r \sin \theta)} &= \frac{\cos^{n+1} \theta}{(r \cos \theta + \frac{1}{2})^2 - (r^2 + a + \frac{1}{4})} \\ &= \frac{((r \cos \theta + \frac{1}{2})^2 - (r^2 + a + \frac{1}{4})) \cos^{n-1} \theta - r \cos^n \theta + (r^2 + a) \cos^{n-1} \theta}{r^2 f(r \cos \theta, r \sin \theta)} \\ &= \frac{1}{r^2} \cos^{n-1} \theta - \frac{1}{r} \frac{\cos^n \theta}{f(r \cos \theta, r \sin \theta)} + \frac{r^2 + a}{r^2} \frac{\cos^{n-1} \theta}{f(r \cos \theta, r \sin \theta)} \end{aligned}$$

Hence

$$\alpha_{n+1,0} = \begin{cases} -\frac{1}{r} \alpha_{n,0} + \frac{r^2+a}{r^2} \alpha_{n-1,0}, & \text{if } n \text{ is even} \\ \frac{1}{2\pi r^2} \int_0^{2\pi} \cos^{n-1} \theta d\theta - \frac{1}{r} \alpha_{n,0} + \frac{r^2+a}{r^2} \alpha_{n-1,0} & \text{if } n \text{ is odd} \end{cases}$$

□

Lemma 3.5. *Under the previous assumptions and notations, we have*

$$\alpha_{0,0} = -\frac{G}{2H} \quad \text{and} \quad \alpha_{1,0} = \frac{G+K}{4rH},$$

where

$$H = hg \quad \text{and} \quad G = \begin{cases} (g_1 + g_2) & \text{if } a > \frac{-1}{2}, \\ (g_1 - g_2) & \text{if } a \leq \frac{-1}{2} \end{cases},$$

with

$$K = \begin{cases} (g_1 - g_2) h, & \text{if } a > -1/2 \\ (g_1 + g_2) h, & \text{if } a \leq -1/2 \end{cases},$$

such that

$$\begin{aligned} g_1 &= \sqrt{4a + 2 - 2h}, \quad g_2 = \sqrt{4a + 2 + 2h} \\ h &= \sqrt{4r^2 + 4a + 1} \neq 0 \quad \text{and} \quad g = \sqrt{a^2 - r^2} \neq 0. \end{aligned}$$

Proof. Letting $z = e^{i\theta}$ and $C = \{z : |z| = 1\}$, we can obtain

$$\int_0^{2\pi} \frac{d\theta}{f(r \cos \theta, r \sin \theta)} \equiv \frac{4}{i} \int_C \frac{z}{(rz^2 + z + r)^2 - (4r^2 + 4a + 1)z^2} dz$$

and

$$\int_0^{2\pi} \frac{\cos \theta d\theta}{f(r \cos \theta, r \sin \theta)} \equiv \frac{2}{i} \int_C \frac{z^2 + 1}{(rz^2 + z + r)^2 - (4r^2 + 4a + 1)z^2} dz,$$

with whose poles are

$$z_{1,2} = \frac{-1 + h \pm g_1}{r} \quad \text{and} \quad z_{3,4} = \frac{-1 - h \pm g_2}{r},$$

where

$$g_1 = \sqrt{4a + 2 - 2h} \text{ and } g_2 = \sqrt{4a + 2 + 2h}.$$

Using the residue theorem $|z_2| < 1, |z_3| < 1, |z_1| > 1$ and $|z_4| > 1$ for all $a > \frac{-1}{2}$, $|z_1| < 1, |z_4| < 1, |z_2| > 1$ and $|z_3| > 1$ for $a \leq \frac{-1}{2}$, we obtain

$$\int_0^{2\pi} \frac{d\theta}{f(r \cos \theta, r \sin \theta)} = \begin{cases} -\frac{\pi}{hg}(g_1 + g_2) & \text{if } a > \frac{-1}{2} \\ -\frac{\pi}{hg}(g_1 - g_2) & \text{if } a \leq \frac{-1}{2} \end{cases},$$

and

$$\int_0^{2\pi} \frac{\cos \theta d\theta}{f(r \cos \theta, r \sin \theta)} = \begin{cases} \frac{\pi}{2rhg}[(g_1 + g_2) + (g_1 - g_2)h] & \text{if } a > \frac{-1}{2}, \\ \frac{\pi}{2rhg}[(g_1 - g_2) + (g_1 + g_2)h] & \text{if } a \leq \frac{-1}{2}. \end{cases}$$

Thus, we have

$$\alpha_{0,0} = -\frac{G}{2H} \text{ and } \alpha_{1,0} = \frac{G + K}{4rH}$$

This completes the proof. □

Based on Lemmas 3 and 4, we can get

$$\begin{aligned} \alpha_{2,0} &= \frac{1}{2\pi r^2} \int_0^{2\pi} d\theta - \frac{1}{r} \alpha_{1,0} + \frac{r^2+a}{r^2} \alpha_{0,0} \\ &= \frac{1}{r^2} - \frac{1}{r} \left(\frac{G + K}{4rH} \right) + \frac{r^2+a}{r^2} \left(-\frac{G}{2H} \right) \\ &= -\frac{(2r^2 + 2a + 1)G + K - 4H}{4Hr^2}, \end{aligned}$$

$$\begin{aligned} \alpha_{0,2} &= \alpha_{0,0} - \alpha_{2,0} \\ &= \frac{-G}{2H} - \left(-\frac{(2r^2 + 2a + 1)G + K - 4H}{4Hr^2} \right) \\ &= \frac{(2a + 1)G + K - 4H}{4Hr^2}, \end{aligned}$$

$$\begin{aligned} \alpha_{3,0} &= -\frac{1}{r} \alpha_{2,0} + \frac{r^2 + a}{r^2} \alpha_{1,0} \\ &= -\frac{1}{r} \left(-\frac{(2r^2 + 2a + 1)G + K - 4H}{4Hr^2} \right) \\ &\quad + \frac{r^2 + a}{r^2} \left(\frac{G + K}{4rH} \right) \\ &= \frac{(3r^2 + 3a + 1)G + (r^2 + a + 1)K - 4H}{4Hr^3}, \end{aligned}$$

$$\begin{aligned}
\alpha_{1,2} &= \alpha_{1,0} - \alpha_{3,0} \\
&= \frac{G + K}{4rH} - \frac{(3r^2 + 3a + 1)G + (r^2 + a + 1)K - 4H}{4Hr^3} \\
&= -\frac{(2r^2 + 3a + 1)G + (a + 1)K - 4H}{4Hr^3},
\end{aligned}$$

In addition, we can find

$$\begin{aligned}
\alpha_{2,2} &= \frac{((2a + 3)r^2 + (2a^2 + 4a + 1))G}{4Hr^4} \\
&\quad + \frac{(r^2 + (2a + 1))K - (2r^2 + (4a + 4))H}{4Hr^4} \\
\alpha_{0,4} &= -\frac{(2r^2 + (2a^2 + 4a + 1))G}{4Hr^4} \\
&\quad + \frac{(2a + 1)K + (2r^2 - 4a - 4)H}{4Hr^4}, \\
\alpha_{5,0} &= \frac{(5r^4 + (10a + 5)r^2 + (5a^2 + 5a + 1))G}{4Hr^5} \\
&\quad + \frac{(r^4 + (2a + 3)r^2 + (a^2 + 3a + 1))K - (10r^2 + (8a + 4))H}{4Hr^5} \\
\alpha_{3,2} &= -\frac{(2r^4 + (7a + 4)r^2 + (5a^2 + 5a + 1))G}{4Hr^5} \\
&\quad + \frac{((a + 2)r^2 + (a^2 + 3a + 1))K - (6r^2 + 8a + 4)H}{4Hr^5}, \\
\alpha_{1,4} &= \frac{((4a + 3)r^2 + (5a^2 + 5a + 1))G}{4Hr^5} \\
&\quad + \frac{(r^2 + (a^2 + 3a + 1))K - (2r^2 + 8a + 4)H}{4Hr^5}, \\
\alpha_{6,0} &= -\frac{1}{8Hr^6} [(4r^6 + (12a + 18)r^4 + (12a^2 + 36a + 12))G \\
&\quad + (4a^3 + 18a^2 + 12a + 2)G \\
&\quad + (6r^4 + (12a + 8)r^2 + (6a^2 + 8a + 2))K \\
&\quad - (15r^4 + (20a + 28)r^2 + (8a^2 + 24a + 8))H],
\end{aligned}$$

$$\begin{aligned}\alpha_{4,2} &= \frac{1}{8Hr^6} [(4a+10)r^4 + (8a^2+28a+10)r^2) G \\ &\quad + (4a^3+18a^2+12a+2) G \\ &\quad + (2r^4 + (8a+6)r^2 + (6a^2+8a+2)) K \\ &\quad - (3r^4 + (12a+20)r^2 + (8a^2+24a+8)) H],\end{aligned}$$

$$\begin{aligned}\alpha_{0,6} &= -\frac{1}{8Hr^6} [(4a+14)r^4 + (12a^2+36a+12)r^2) G \\ &\quad + (4a^3+18a^2+12a+2) G \\ &\quad + (2r^4 + (12a+8)r^2 + (6a^2+8a+2)) K \\ &\quad + (r^4 + (-20a-28)r^2 - (8a^2+24a+8)) H],\end{aligned}$$

$$\begin{aligned}\alpha_{7,0} &= \frac{1}{8Hr^7} [(14r^6 + (42a+28)r^4 + (42a^2+56a+14)r^2) G \\ &\quad + (14a^3+28a^2+14a+2) G \\ &\quad + (2r^6 + (6a+12)r^4 + (6a^2+24a+10)r^2) K \\ &\quad + (2a^3+12a^2+10a+2) K \\ &\quad - (35r^4 + (56a+36)r^2 + (24a^2+32a+8)) H],\end{aligned}$$

$$\begin{aligned}\alpha_{5,2} &= -\frac{1}{8Hr^7} [(4r^6 + (22a+18)r^4 + (32a^2+46a+12)r^2) G \\ &\quad + (14a^3+28a^2+14a+2) G \\ &\quad + ((2a+6)r^4 + (4a^2+18a+8)r^2) K \\ &\quad + (2a^3+12a^2+10a+2) K \\ &\quad - (15r^4 + (40a+28)r^2 + (24a^2+32a+8)) H],\end{aligned}$$

$$\begin{aligned}\alpha_{3,4} &= \frac{1}{8Hr^7} [(8a+10)r^4 + (22a^2+36a+10)r^2) G \\ &\quad + (14a^3+28a^2+14a+2) G \\ &\quad + (2r^4 + (2a^2+12a+6)r^2 + (2a^3+12a^2+10a+2)) K \\ &\quad - (3r^4 + (24a+20)r^2 + (24a^2+32a+8)) H],\end{aligned}$$

$$\begin{aligned}\alpha_{1,6} &= -\frac{1}{8Hr^7} [(4r^4 + (12a^2+26a+8)r^2) G \\ &\quad + (14a^3+28a^2+14a+2) G \\ &\quad + ((6a+4)r^2 + (2a^3+12a^2+10a+2)) K \\ &\quad + (r^4 + (-8a-12)r^2 - (24a^2+32a+8)) H]\end{aligned}$$

and for any p, k with $p + 2k \leq 6$: $\alpha_{p,2k+1} = 0$.

Using (3.2), we get

$$\begin{aligned} f^0(r) &= \frac{1}{8rH} (A_r H + B_r G + C_r K) \\ &= \begin{cases} \frac{1}{8rhg} (A_r hg + B_r (g_1 + g_2) + C_r (g_1 - g_2) h) & \text{if } a > -\frac{1}{2}, \\ \frac{1}{8rhg} (A_r hg + B_r (g_1 - g_2) + C_r (g_1 + g_2) h) & \text{if } a \leq -\frac{1}{2} \end{cases} \end{aligned}$$

where

$$\begin{aligned} A_r &= a_4 r^4 + a_2 r^2 + a_0, \\ B_r &= b_6 r^6 + b_4 r^4 + b_2 r^2 + b_0, \\ C_r &= c_6 r^6 + c_4 r^4 + c_2 r^2 + \frac{a}{2} a_0 + b_0 \end{aligned}$$

with

$$a_4 = 15\lambda_{5,0} + 3\lambda_{5,2} - \lambda_{5,4} - \lambda_{5,6} - 35\lambda_{6,0} + 15\lambda_{6,2} - 3\lambda_{6,4} - \lambda_{6,6},$$

$$\begin{aligned} a_2 &= (20\lambda_{5,0} + 12\lambda_{5,2} + 4\lambda_{5,4} + 20\lambda_{5,6} - 56\lambda_{6,0} + 40\lambda_{6,2} - 24\lambda_{6,4} + 8\lambda_{6,6}) a \\ &\quad + (12\lambda_{3,0} - 4\lambda_{3,2} - 4\lambda_{3,4} - 20\lambda_{4,0} + 12\lambda_{4,2} - 4\lambda_{4,4} + 28\lambda_{5,0} + 20\lambda_{5,2} \\ &\quad + 12\lambda_{5,4} + 28\lambda_{5,6} - 36\lambda_{6,0} + 28\lambda_{6,2} - 20\lambda_{6,4} + 12\lambda_{6,6}), \end{aligned}$$

$$\begin{aligned} a_0 &= (8\lambda_{5,0} + 8\lambda_{5,2} + 8\lambda_{5,4} + 8\lambda_{5,6} - 24\lambda_{6,0} + 24\lambda_{6,2} - 24\lambda_{6,4} + 24\lambda_{6,6}) a^2 \\ &\quad + (8\lambda_{3,0} - 8\lambda_{3,2} + 8\lambda_{3,4} - 16\lambda_{4,0} + 16\lambda_{4,2} - 16\lambda_{4,4} + 24\lambda_{5,0} + 24\lambda_{5,2} \\ &\quad + 24\lambda_{5,4} + 24\lambda_{5,6} - 32\lambda_{6,0} + 32\lambda_{6,2} - 32\lambda_{6,4} + 32\lambda_{6,6}) a \\ &\quad + (8\lambda_{1,0} - 8\lambda_{1,2} - 8\lambda_{2,0} + 8\lambda_{2,2} + 8\lambda_{3,0} - 8\lambda_{3,2} + 8\lambda_{3,4} - 8\lambda_{4,0} + 8\lambda_{4,2} \\ &\quad - 8\lambda_{4,4} + 8\lambda_{5,0} + 8\lambda_{5,2} + 8\lambda_{5,4} + 8\lambda_{5,6} - 8\lambda_{6,0} + 8\lambda_{6,2} - 8\lambda_{6,4} + 8\lambda_{6,6}), \end{aligned}$$

$$b_6 = -4\lambda_{5,0} + 14\lambda_{6,0} - 4\lambda_{6,2},$$

$$\begin{aligned} b_4 &= (-12\lambda_{5,0} - 4\lambda_{5,2} - 4\lambda_{5,6} + 42\lambda_{6,0} - 22\lambda_{6,2} + 8\lambda_{6,4}) a \\ &\quad + (-4\lambda_{3,0} + 10\lambda_{4,0} - 4\lambda_{4,2} - 18\lambda_{5,0} - 10\lambda_{5,2} - 4\lambda_{5,4} - 14\lambda_{5,6} + 28\lambda_{6,0} \\ &\quad - 18\lambda_{6,2} + 10\lambda_{6,4} - 4\lambda_{6,6}), \end{aligned}$$

$$\begin{aligned}
 b_2 = & (-12\lambda_{5,0} - 8\lambda_{5,2} - 4\lambda_{5,4} - 12\lambda_{5,6} - 32\lambda_{6,2} + 42\lambda_{6,0} + 22\lambda_{6,4} - 12\lambda_{6,6}) a^2 \\
 & + (-8\lambda_{3,0} + 4\lambda_{3,2} + 20\lambda_{4,0} - 14\lambda_{4,2} + 8\lambda_{4,4} - 36\lambda_{5,0} \\
 & - 28\lambda_{5,2} - 20\lambda_{5,4} - 36\lambda_{5,6} \\
 & + 56\lambda_{6,0} - 46\lambda_{6,2} + 36\lambda_{6,4} - 26\lambda_{6,6}) a \\
 & + (-4\lambda_{1,0} + 6\lambda_{2,0} - 4\lambda_{2,2} - 8\lambda_{3,0} + 6\lambda_{3,2} - 4\lambda_{3,4} \\
 & + 10\lambda_{4,0} - 8\lambda_{4,2} + 6\lambda_{4,4} \\
 & - 12\lambda_{5,0} - 10\lambda_{5,2} - 8\lambda_{5,4} - 12\lambda_{5,6} + 14\lambda_{6,0} - 12\lambda_{6,2} + 10\lambda_{6,4} - 8\lambda_{6,6}),
 \end{aligned}$$

$$\begin{aligned}
 b_0 = & (-4\lambda_{5,0} - 4\lambda_{5,2} - 4\lambda_{5,4} - 4\lambda_{5,6} + 14\lambda_{6,0} - 14\lambda_{6,2} + 14\lambda_{6,4} - 14\lambda_{6,6}) a^3 \\
 & + (-4\lambda_{3,0} + 4\lambda_{3,2} - 4\lambda_{3,4} + 10\lambda_{4,0} - 10\lambda_{4,2} + 10\lambda_{4,4} - 18\lambda_{5,0} - 18\lambda_{5,2} \\
 & - 18\lambda_{5,4} - 18\lambda_{5,6} + 28\lambda_{6,0} - 28\lambda_{6,2} + 28\lambda_{6,4} - 28\lambda_{6,6}) a^2 \\
 & + (-4\lambda_{1,0} + 4\lambda_{1,2} + 6\lambda_{2,0} - 6\lambda_{2,2} - 8\lambda_{3,0} + 8\lambda_{3,2} - 8\lambda_{3,4} \\
 & + 10\lambda_{4,0} - 10\lambda_{4,2} + 10\lambda_{4,4} \\
 & - 12\lambda_{5,0} - 12\lambda_{5,2} - 12\lambda_{5,4} - 12\lambda_{5,6} + 14\lambda_{6,0} \\
 & - 14\lambda_{6,2} + 14\lambda_{6,4} - 14\lambda_{6,6}) a \\
 & + (-2\lambda_{1,0} + 2\lambda_{1,2} + 2\lambda_{2,0} - 2\lambda_{2,2} - 2\lambda_{3,0} + 2\lambda_{3,2} \\
 & - 2\lambda_{3,4} + 2\lambda_{4,0} - 2\lambda_{4,2} \\
 & + 2\lambda_{4,4} - 2\lambda_{5,0} - 2\lambda_{5,2} - 2\lambda_{5,4} - 2\lambda_{5,6} + 2\lambda_{6,0} \\
 & - 2\lambda_{6,2} + 2\lambda_{6,4} - 2\lambda_{6,6}),
 \end{aligned}$$

$$c_6 = 2\lambda_{6,0},$$

$$c_4 = (6\lambda_{6,0} - 2\lambda_{6,2}) a + (2\lambda_{4,0} - 6\lambda_{5,0} - 2\lambda_{5,2} - 2\lambda_{5,6} + 12\lambda_{6,0} - 6\lambda_{6,2} + 2\lambda_{6,4})$$

and

$$\begin{aligned}
 c_2 = & (6\lambda_{6,0} - 4\lambda_{6,2} + 2\lambda_{6,4}) a^2 \\
 & + (4\lambda_{4,0} - 2\lambda_{4,2} - 12\lambda_{5,0} - 8\lambda_{5,2} - 4\lambda_{5,4} - 12\lambda_{5,6} + 24\lambda_{6,0}) a \\
 & (-18\lambda_{6,2} + 12\lambda_{6,4} - 6\lambda_{6,6}) a \\
 & + (2\lambda_{2,0} - 4\lambda_{3,0} + 2\lambda_{3,2} + 6\lambda_{4,0} - 4\lambda_{4,2} + 2\lambda_{4,4} - 8\lambda_{5,0} \\
 & - 6\lambda_{5,2} - 4\lambda_{5,4} - 8\lambda_{5,6} \\
 & + 10\lambda_{6,0} - 8\lambda_{6,2} + 6\lambda_{6,4} - 4\lambda_{6,6}).
 \end{aligned}$$

In fact, there are only ten independent parameters between the a_i , b_i and c_i with respect to p_{ij} , q_{ij} and a . In order to bound the zeros number of numerator of

$f^0(r)$, it is sufficient to bound the zeros number of

$$\begin{aligned} K(r) &= A_r^2 h^2 g^2 - [B_r (g_1 \pm g_2) + C_r h (g_1 \mp g_2)]^2 \\ &= A_r^2 h^2 g^2 - 4B_r^2 (2a + 1 \pm 2g) - 4C_r^2 h^2 (2a + 1 \mp 2g) + 8B_r C_r h^2 \end{aligned}$$

since

$$(g_1 \pm g_2)^2 = 4(2a + 1 \pm 2g), g_1^2 - g_2^2 = -4h.$$

Finally, in order to bound the zeros number of the above expression, we should bound the zeros of the following polynomial

$$\begin{aligned} L(r) &= (A_r^2 h^2 g^2 + 8B_r C_r h^2 - 4(2a + 1)(B_r^2 + C_r^2 h^2))^2 - 64(B_r^2 - C_r^2 h^2)^2 g^2 \\ &= \left((g^2 A_r^2 + 8B_r C_r - 4C_r^2 (2a + 1))^2 - 64g^2 C_r^4 \right) h^4 \\ &\quad + (128g^2 B_r^2 C_r^2 - 8B_r^2 (2a + 1)(g^2 A_r^2 + 8B_r C_r - 4C_r^2 (2a + 1))) h^2 \\ &\quad + (16B_r^4 (2a + 1)^2 - 64g^2 B_r^4) \\ &= h^2 \left[(g^2 A_r^2 + 8B_r C_r - 4(2a + 1)C_r^2)^2 - 64g^2 C_r^4 \right] h^2 \\ &\quad + 128g^2 B_r^2 C_r^2 - 8(2a + 1)B_r^2 (g^2 A_r^2 + 8B_r C_r - 4(2a + 1)C_r^2) + 16B_r^4, \end{aligned}$$

we obtain

$$L(r) = (4r^2 + 4a + 1) (\delta_{28} r^{28} + \delta_{26} r^{26} \dots + \delta_2 r^2 + \delta_0)$$

where δ_i are polynomials in a, p_{ij} and q_{ij} . Hence, $f^0(r)$ has at most 14 simple zeros. Therefore, statement **(b)** in Theorem 1 holds.y.

Acknowledgments

The authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions which helped them to improve the paper.

References

1. Y.Bouattia, A.Makhlouf, Limit cycles of quartic and quintic polynomial differential systems via averaging theory, *Ann.of Diff. Eqs.*27, 1 (2011), 70-85.
2. A.Buica, J.Llibre, Averaging methods for finding periodic orbits vai Brouwer degree; *Bulletin des Sciences mathématiques* 128 (2004), 7-22.
3. B.Coll, A.Gasull, R.Prohens; Bifurcation of limit cycles from two families of centers, *Dyn. Contin. Discrete Impuls. Syst.* 12 (2005), 275-88.
4. L.Feng, W.Miao, Bifurcation of limit cycles in a quintic system with ten parameters; *Nonlinear Dynam.* 71 (2013), 213-222.
5. J.Giné, J.Llibre, Limit cycles of cubic polynomial vector fields via the averaging theory, *Nonlinear Analysis* 66 (2007), 1707-1721.
6. L.Hongwei, Limit cycles in a sextic Lyapunov system; *Nonlinear Dynam.* 72 (2013), 555-559.
7. J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector elds, *Int. J. Bifurcation and Chaos*, 13 (2003), 47-106.

8. J.Liber, A.Mereu; Limit cycles for generalized Kukles polynomial differential systems; *Nonlinear Analysis* 74 (2011), 1261-1271.
9. J.Llibre, R.M.Martins, M. A. Teixeira, Periodic orbits, invariant tori and cylinders of Hamiltonian systems near integrable ones having a return map equal to the identity; *J. Math. Phys.* 51 (2010), 082704.
10. S. Li, T. Huang, Limit cycles for piecewise smooth perturbations of a cubic polynomial differential center, *Electronic Journal of Differential Equations*, 108 (2015), 1-17.
11. L.S. Pontrjagin, Uber Autoschwingungssysteme, die den hamiltonshen nahe liegen, in: *Physikalische Zeitschrift der Sowjetunion*, Band 6, Heft (1-2), (1934), 25-28.
12. K.Xiaona, X.Yanqin, Limit cycle bifurcations in a class of quintic \mathbb{Z}_2 -equivariant polynomial systems; *Nonlinear Dynam.* 73 (2013), 1271-1281.
13. Z.Yulin, Z.Huaiping; Bifurcation of limit cycles from a non-Hamiltonian quadratic integrable system with homoclinic loop; *Infinite dimensional dynamical systems*, 445-479, *Fields Inst. Commun.*, 64, Springer, New York, 2013.

A. Menaceur,
Department of Mathematics,
Faculty of MISM Guelma University,
P O. Box 401, Guelma 24000, Algeria.
E-mail address: menaceur.amor@univ-guelma.dz

and

S. Boulaaras,
Department of Mathematics, College of Sciences and Arts,
Al-Ras, Qassim University, Kingdom Of Saudi Arabia.

Department of Mathematics,
Faculty of Sciences,
University of Science and Technology of Oran,
Algeria.
E-mail address: S.Boulaaras@qu.edu.sa