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Darboux Curves in Lorentzian Three Dimensional Heisenberg Group

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ABSTRACT: In this paper, a new characterization for darboux curves in $Heis_3$ is completely given. Then, a new classification for translation surface, which is generated by darboux curve in $Heis_3$ is obtained.

Key Words: Heisenberg group, Lorentz metric.

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1. Introduction

Darboux curves firstly were studied by Saban [17] in the Euclidean space, thereafter generalised by Ergin [2]. For a curve α on a surface in the $\mathbb{E}^3$ the function

$$D =< \alpha''', \mathbf{u} >= \kappa'_{\mathbf{u}} - \kappa_g \tau_g$$

is called Darboux function of α. Here $\mathbf{u}$ is normal vector field of surface, $\kappa_{\mathbf{u}}$, κ_g and τ_g are normal curvature, geodesic curvature and geodesic torsion. If Darboux function is equal to zero, then these curves called Darboux curves. In Minkowski 3-space timelike Darboux curves on a timelike surface were studied by Ergin [3].

Translation surfaces in $\mathbb{E}^3$, firstly studied by H. F. Scherk. He showed that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right| = \frac{1}{a} \log |\cos(ax)| - \frac{1}{a} \log |\cos(ay)|,$$

where a is a non-zero constant, [6]. Then, when the second fundamental form was considered as a metric on a non-developable surface, translation surfaces in the Euclidean space were obtained by, [15]. The translation surfaces which are generated by two space curves in $\mathbb{E}^3$ have been investigated by Çetin. Also they showed that Scherk surface is not only minimal translation surface, [1] D. W. Yoon has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = \Delta A$, where Δ denotes the Laplacian of the

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surface, [19]. Translation surfaces in terms of a pair of two planar curves lying in orthogonal planes defined by [6] in the Nil^3 with left invariant Riemannian metric. They classified minimal translation surfaces in Nil^3. Translation surfaces in Sol^3 constructed by [13] and they investigated properties of minimal one. Also some curves and surfaces studied in [7-12].

The purpose of this paper is to study and classify modifieded translation surfaces in $Heis_3$ and investigate conditions of being minimal surface. Also, obtain characterizations of points on this surface.

2. The Heisenberg Group

The Heisenberg group $Heis_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (x_1, y_1, z_1) = \left(x + x_1, y + y_1, z + z_1 + \frac{1}{2} (xy_1 - x_1 y) \right). \tag{2.1}$$

The left-invariant Lorentzian metric on $Heis_3$ is

$$g = ds^2 = -dx^2 + dy^2 + (xdy + dz)^2. \tag{2.2}$$

The orthonormal basis for the corresponding Lie algebra:

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x}. \tag{2.3}$$

Then, we have

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, \ \ [\mathbf{e}_1, \mathbf{e}_2] = [\mathbf{e}_1, \mathbf{e}_3] = 0 \tag{2.4}$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.5}$$

Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left -invariant metric g is

$$\nabla_{\mathbf{e}_i} \mathbf{e}_j = \begin{bmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{bmatrix}, \tag{2.6}$$

where the $(i, j)-$element in the table above equals for $\nabla_{\mathbf{e}_i} \mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Let $\gamma : I \longrightarrow Heis_3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \nabla_{\mathbf{T}} \mathbf{B} &= \tau \mathbf{N}, \end{aligned} \tag{2.7}$$

where κ, τ are curvature function and torsion function. With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{N} &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ \mathbf{B} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned} \tag{2.9}$$

3. Darboux Curves in 3-D Lorentzian Heisenberg Group

Let unit tangent vector field of the curve be

$$\mathbf{T} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$$

and the unit normal vector field of the surface be

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3.$$

Theorem 3.1. *Let $\alpha : I \longrightarrow M$ be a unit speed curve in $(Heis_3, g)$. If α is a darboux curve on surface M, then*

$$\delta' u_1 t_1 + \lambda' u_2 t_1 + \gamma' u_3 t_1 + \frac{1}{2}(u_1(t_2\gamma + t_3\lambda) - u_2(t_1\gamma + t_3\delta) + u_3(t_1\lambda - t_2\delta)) = 0.$$

Proof. If we notice

$$\begin{aligned} \delta &= t_1', \\ \lambda &= t_2' + 2t_1t_3, \\ \gamma &= t_3' + 2t_1t_2, \end{aligned} \tag{3.1}$$

then we have

$$\begin{aligned} \nabla_{\mathbf{T}}(\nabla_{\mathbf{T}}\mathbf{T}) = \; & (t_1'' + t_2(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3))\mathbf{e}_1 \\ & + (t_2'' + 2(t_1t_3)' + t_1(t_3' + 2t_1t_2) - t_3t_1')\mathbf{e}_2 \\ & + (t_3'' + 2(t_1t_2)' - t_1(t_2' + 2t_1t_3) + t_2t_1')\mathbf{e}_3. \end{aligned} \tag{3.2}$$

So, the darboux function is

$$\begin{aligned} \mathbb{D} &= g(\nabla_{\mathbf{T}}(\nabla_{\mathbf{T}}\mathbf{T}), \mathbf{u}) \\ &= (t_1'' + t_2(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3)u_1 \\ & \quad + (t_2'' + 2(t_1t_3)' + t_1(t_3' + 2t_1t_2) - t_3t_1')u_2 \\ & \quad - (t_3'' + 2(t_1t_2)' - t_1(t_2' + 2t_1t_3) + t_2t_1')u_3. \end{aligned} \tag{3.3}$$

If α is a darboux curve on surface M,from the equation (3.3) we have

$$\begin{aligned}&(t_1'' + t_2\,(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3)u_1 \\ &+(t_2'' + 2(t_1t_3)' + t_1\,(t_3' + 2t_1t_2) - t_3t_1')u_2 \\ &-(t_3'' + 2(t_1t_2)' - t_1\,(t_2' + 2t_1t_3) + t_2t_1')u_3 = 0.\end{aligned} \tag{3.4}$$

Corollary 3.2. *Let α be an unit speed spacelike curve and M be a spacelike ruled surface in $(Heis_3, g)$ which is parametrized as*

$$M\,(x,y) = \alpha\,(x) + y\mathbf{T}\,(x)\,. \tag{3.5}$$

If α is a darboux curve, then

$$\begin{aligned}&(t_1'' + t_2\,(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3)(t_2\,(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3)) \\ &+(t_2'' + 2(t_1t_3)' + t_1\,(t_3' + 2t_1t_2) - t_3t_1')(t_3t_1' - t_1\,(t_3' + 2t_1t_2)) \\ &-(t_3'' + 2(t_1t_2)' - t_1\,(t_2' + 2t_1t_3) + t_2t_1')(t_2t_1' - t_1\,(t_2' + 2t_1t_3)) = 0.\end{aligned} \tag{3.6}$$

Proof. From equation (3.5), we have

$$M_x\,(x,y) = \mathbf{T}\,(x) + y\nabla_{\mathbf{T}}\mathbf{T} \tag{3.7}$$

$$M_y\,(x,y) = \mathbf{T}\,(x)\,. \tag{3.8}$$

From equations (3.6)-(3.7), then the unit normal vector field of the surface M is

$$\begin{aligned}\mathbf{u} &= \frac{1}{\omega}\{(t_2\,(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3))\mathbf{e}_1 \\ &\quad +(t_3t_1' - t_1\,(t_3' + 2t_1t_2))\mathbf{e}_2 \\ &\quad +(t_2t_1' - t_1\,(t_2' + 2t_1t_3))\mathbf{e}_3\}.\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\omega &= (t_2\,(t_3' + 2t_1t_2) - t_3(t_2' + 2t_1t_3)^2 + (t_3t_1' - t_1\,(t_3' + 2t_1t_2))^2 \\ &\quad -(t_2t_1' - t_1\,(t_2' + 2t_1t_3))^2.\end{aligned}$$

If α is a darboux curve, from equations (3.2), (3.3) and (3.9), we have (3.6).

Example 3.3. In $(Heis_3, g)$

$$\alpha\,(x) = (c_1, \frac{x^2}{2} + c_2, x^2 - \frac{c_1x^2}{2} + c_3) \tag{3.10}$$

is a unit speed curve where c_i $(i = 1, 2, 3)$ are constant.The unit spacelike tangent vector field of the α is

$$\mathbf{T}\,(x) = (0, x, 2x) \tag{3.11}$$

From equations (3.10), (3.11), the ruled surface which is parametrized (3.5) is

$$M\,(x,y) = (c_1, \frac{x^2}{2} + xy + c_2, x^2 - \frac{c_1 x^2}{2} + 2xy + c_3).$$

The unit normal vector field of the $M\,(x,y)$ is

$$\mathbf{u} = \frac{1}{\varpi}(0, -2x(c_1 x + 2x)^2, 2x^2(c_1 x + 2x)(1 - c_1 x - 2x)).$$

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