Bol. Soc. Paran. Mat. (3s.) **v. 39** 4 (2021): 153–164.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.41077

Initial Coefficient Bounds for a Subclass of m-fold Symmetric Bi-univalent Functions

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ABSTRACT: In this paper, we introduce and investigate a subclass $\mathcal{B}^{h,p}_{\Sigma_m}(\tau,\lambda)$ of analytic and bi-univalent functions which both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric in the open unit disk $\mathbb{U}$. Furthermore, we find upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this subclass. The results presented in this paper would generalize and improve some recent works.

Key Words: Analytic functions, Bi-univalent functions, Coefficient estimates, m-fold symmetric functions, m-fold symmetric bi-univalent functions.

Contents

1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

The Koebe one-quarter Theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

2010 *Mathematics Subject Classification:* Primary: 30C45; Secondary: 30C50.
Submitted December 23, 2017. Published June 11, 2018

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [6] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Recently there interest to study the bi-univalent functions class Σ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ ([2,3,10,11,12,16,17]). The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \Sigma$ is still an open problem.

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)} \ (z \in \mathbb{U}, m \in \mathbb{N}),$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [7,9]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \ (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.3}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (1.3). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric.

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (1.3) and the series expansion for f^{-1} which was recently proven by Srivastava et al. [13], is given as follows:

$$\begin{aligned} g(w) = {} & w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \ldots, \end{aligned} \tag{1.4}$$

where $g = f^{-1}$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}} \ \textit{and} \ [-\log(1 - z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}$$

respectively.

Recently, Srivastava et al. [14] investigated the following two subclasses $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$ and $\mathcal{B}^*_{\Sigma_m}(\tau,\lambda,\beta)$ of Σ_m consisting of m-fold symmetric bi-univalent functions in the open unit disk $\mathbb{U}$ and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

Definition 1.1. *(see [14]) Let $0<\alpha\le 1$, $\tau\in\mathbb{C}\setminus\{0\}$ and $\lambda\ge 1$. A function $f(z)$ given by (1.3) is said to be in the class $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$ if the following conditions are satisfied:*

$$f\in\Sigma_m \text{ and } \left|arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]\right)\right| < \frac{\alpha\pi}{2}\ (z\in\mathbb{U}),$$

and

$$\left|arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right]\right)\right| < \frac{\alpha\pi}{2}\ (w\in\mathbb{U}),$$

where the function g is given by (1.4).

Theorem 1.2. *(see [14]) Let the function $f(z)$ given by (1.3) be in the class $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$. Then*

$$|a_{m+1}|\le\frac{2\alpha|\tau|}{\sqrt{|\tau\alpha(m+1)(2\lambda m+1)+(1-\alpha)(\lambda m+1)^2|}},$$

and

$$|a_{2m+1}|\le\frac{2\alpha^2|\tau|^2(m+1)}{(\lambda m+1)^2}+\frac{2\alpha|\tau|}{2\lambda m+1}.$$

Definition 1.3. *(see [14]) Let $0\le\beta<1$, $\tau\in\mathbb{C}\setminus\{0\}$ and $\lambda\ge 1$. A function $f(z)$ given by (1.3) is said to be in the class $\mathcal{B}^*_{\Sigma_m}(\tau,\lambda,\beta)$ if the following conditions are satisfied:*

$$f\in\Sigma_m \text{ and } \mathfrak{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]\right) > \beta\ (z\in\mathbb{U}),$$

and

$$\mathfrak{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right]\right) > \beta\ (w\in\mathbb{U}),$$

where the function g is given by (1.4).

Theorem 1.4. *(see [14]) Let the function $f(z)$ given by (1.3) be in the class $\mathcal{B}^*_{\Sigma_m}(\tau,\lambda,\beta)$. Then*

$$|a_{m+1}|\le\sqrt{\frac{4|\tau|(1-\beta)}{(m+1)(2\lambda m+1)}},$$

and

$$|a_{2m+1}| \leq \frac{2|\tau|^2(1-\beta)^2(m+1)}{(\lambda m+1)^2} + \frac{2|\tau|(1-\beta)}{2\lambda m+1}.$$

The objective of the present paper is to introduce a formula for computing the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses which improve the coefficient bounds obtained in Theorem 1.2 and Theorem 1.4. Our results generalize and improve some recent works as Srivastava [11,13,14], Eker [15] and Frasin and Aouf [5].

2. The subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$

In this section, we introduce and investigate the general subclass $\mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$.

Definition 2.1. *Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be analytic functions and*

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots,$$
$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots,$$

such that

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \ (z \in \mathbb{U}).$$

Let $\tau \in \mathbb{C} \setminus \{0\}$ and $\lambda \geq 1$. A function f given by (1.3) is said to be in the class $\mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_m \text{ and } 1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right] \in h(\mathbb{U}) \ (z \in \mathbb{U}), \tag{2.1}$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right] \in p(\mathbb{U}) \ (w \in \mathbb{U}), \tag{2.2}$$

where the function g is defined by (1.4).

Remark 2.2. *There are many choices of h and p which would provide interesting subclasses of class $\mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$. For example,*

1. *For $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots$, where $0 < \alpha \leq 1$, it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in \mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$, then*

$$\left|arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right]\right)\right| < \frac{\alpha\pi}{2} \ (z \in \mathbb{U}),$$

and

$$\left|arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right]\right)\right| < \frac{\alpha\pi}{2} \ (w \in \mathbb{U}).$$

Therefore in this case, the class $\mathcal{B}_{\Sigma_m}^{h,p}(\tau,\lambda)$ reduce to class $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$ in Definition 1.1.

2. *For* $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots$, *where* $0 \leq \beta < 1$, *the functions* $h(z)$ *and* $p(z)$ *satisfy the hypotheses of Definition* 2.1. *Now if* $f \in \mathcal{B}^{h,p}_{\Sigma_m}(\tau, \lambda)$, *then*

$$\mathfrak{Re}\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right]\right) > \beta \ (z \in \mathbb{U}),$$

and

$$\mathfrak{Re}\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right]\right) > \beta \ (w \in \mathbb{U}).$$

Therefore in this case, the class $\mathcal{B}^{h,p}_{\Sigma_m}(\tau, \lambda)$ *reduce to class* $\mathcal{B}^{*}_{\Sigma_m}(\tau, \lambda, \beta)$ *in Definition 1.3.*

3. Coefficient Estimates

Now, we obtain the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for subclass $\mathcal{B}^{h,p}_{\Sigma_m}(\tau, \lambda)$.

Theorem 3.1. *Let the function* $f(z)$ *given by* (1.3) *be in the class* $\mathcal{B}^{h,p}_{\Sigma_m}(\tau, \lambda)$. *Then*

$$|a_{m+1}| \leq \min\left\{\sqrt{\frac{|\tau|^2(|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{2(m!)^2(\lambda m + 1)^2}}, \sqrt{\frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{(2m)!(2\lambda m + 1)(m+1)}}\right\}, \tag{3.1}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{2(2m)!(2\lambda m + 1)} + \frac{|\tau|^2(m+1)(|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{4(m!)^2(\lambda m + 1)^2}, \frac{|\tau||h^{(2m)}(0)|}{(2m)!(2\lambda m + 1)}\right\}. \tag{3.2}$$

Proof: First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1\right] = h(z) \ (\lambda \geq 1, z \in \mathbb{U}), \tag{3.3}$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) - 1\right] = p(w) \ (\lambda \geq 1, w \in \mathbb{U}), \tag{3.4}$$

respectively, where functions h and p satisfy the conditions of Definition 2.1. Also, the functions h and p have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots , \tag{3.5}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots . \tag{3.6}$$

Now, upon substituting from (3.5) and (3.6) into (3.3) and (3.4), respectively, and equating the coefficients, we get

$$\left(\frac{\lambda m + 1}{\tau}\right) a_{m+1} = h_m, \tag{3.7}$$

$$\left(\frac{2\lambda m + 1}{\tau}\right) a_{2m+1} = h_{2m}, \tag{3.8}$$

$$-\left(\frac{\lambda m + 1}{\tau}\right) a_{m+1} = p_m, \tag{3.9}$$

and

$$\left(\frac{2\lambda m + 1}{\tau}\right) [(m+1)a_{m+1}^2 - a_{2m+1}] = p_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$h_m = -p_m, \tag{3.11}$$

and

$$2\left(\frac{\lambda m + 1}{\tau}\right)^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{3.12}$$

Adding (3.8) and (3.10), we get

$$\left(\frac{2\lambda m + 1}{\tau}\right) (m+1)a_{m+1}^2 = p_{2m} + h_{2m}. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we have

$$a_{m+1}^2 = \frac{\tau^2 (h_m^2 + p_m^2)}{2(\lambda m + 1)^2}, \tag{3.14}$$

and

$$a_{m+1}^2 = \frac{\tau (p_{2m} + h_{2m})}{(2\lambda m + 1)(m+1)}, \tag{3.15}$$

respectively. Therefore, we find from the equations (3.14) and (3.15), that

$$|a_{m+1}|^2 \leq \frac{|\tau|^2 (|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{2(m!)^2(\lambda m + 1)^2},$$

and

$$|a_{m+1}|^2 \leq \frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{(2m)!(2\lambda m+1)(m+1)},$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$ as asserted in (3.1).

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$2\left(\frac{2\lambda m+1}{\tau}\right) a_{2m+1} - \left(\frac{2\lambda m+1}{\tau}\right)(m+1)a_{m+1}^2 = h_{2m} - p_{2m}. \tag{3.16}$$

Upon substituting the value of a_{m+1}^2 from (3.14) into (3.16), it follows that

$$a_{2m+1} = \frac{\tau^2(m+1)(h_m^2 + p_m^2)}{4(\lambda m+1)^2} + \frac{\tau(h_{2m} - p_{2m})}{2(2\lambda m+1)},$$

Therefore, we get

$$\begin{aligned} |a_{2m+1}| \quad &\leq \quad \frac{|\tau|^2(m+1)(|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{4(m!)^2(\lambda m+1)^2} \\ &\quad + \frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{2(2m)!(2\lambda m+1)}. \end{aligned} \tag{3.17}$$

On the other hand, upon substituting the value of a_{m+1}^2 from (3.15) into (3.16), it follows that

$$a_{2m+1} = \frac{\tau(m+1)(p_{2m} + h_{2m})}{2(2\lambda m+1)(m+1)} + \frac{\tau(h_{2m} - p_{2m})}{2(2\lambda m+1)} = \frac{\tau h_{2m}}{2\lambda m+1},$$

Therefore, we get

$$|a_{2m+1}| \leq \frac{|\tau||h^{(2m)}(0)|}{(2m)!(2\lambda m+1)}. \tag{3.18}$$

So we obtain from (3.17) and (3.18) the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in (3.2). This completes the proof. □

4. Conclusions

If we take

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots,$$

in Theorem 3.1, we conclude the following result which is an improvement of Theorem 1.2.

Corollary 4.1. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$. Then*

$$|a_{m+1}| \leq \min\left\{\frac{2\alpha|\tau|}{\lambda m+1}, 2\alpha\sqrt{\frac{|\tau|}{(2\lambda m+1)(m+1)}}\right\},$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2|\tau|}{2\lambda m+1}.$$

Remark 4.2. *It is easy to see, for the coefficient $|a_{2m+1}|$, that*

$$\frac{2\alpha^2|\tau|}{2\lambda m+1} \leq \frac{2\alpha^2|\tau|^2(m+1)}{(\lambda m+1)^2} + \frac{2\alpha|\tau|}{2\lambda m+1}.$$

Thus, clearly, Corollary 4.1 is an improvement of Theorem 1.2.

If we set $\tau = 1$ in Corollary 4.1, then the class $\mathcal{B}_{\Sigma_m}(\tau,\lambda,\alpha)$ reduces to the class $\mathcal{A}^{\alpha,\lambda}_{\Sigma_m}$ which introduced and studied by Sumer Eker [15].

Corollary 4.3. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{A}^{\alpha,\lambda}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leq \begin{cases} \frac{2\alpha}{\lambda m+1}, & \lambda \geq 1+\sqrt{\frac{m+1}{m}} \\ \dfrac{2\alpha}{\sqrt{(2\lambda m+1)(m+1)}}, & 1 \leq \lambda < 1+\sqrt{\frac{m+1}{m}} \end{cases} \tag{4.1}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{2\lambda m+1}.$$

Remark 4.4. *It is easy to see that*

$$\frac{2\alpha}{\lambda m+1} \leq \frac{2\alpha}{\sqrt{(\lambda m+1)^2+\alpha m(1+2\lambda m-m\lambda^2)}},$$

if

$$\lambda \geq 1+\sqrt{\frac{m+1}{m}}$$

and

$$\frac{2\alpha}{\sqrt{(2\lambda m+1)(m+1)}} \leq \frac{2\alpha}{\sqrt{(\lambda m+1)^2+\alpha m(1+2\lambda m-m\lambda^2)}}$$

if

$$1 \leq \lambda < 1+\sqrt{\frac{m+1}{m}}.$$

On the other hand, for the coefficient $|a_{2m+1}|$,

$$\frac{2\alpha^2}{2\lambda m+1} \leq \frac{2\alpha^2(m+1)}{(1+\lambda m)^2} + \frac{2\alpha}{2\lambda m+1}.$$

Thus, clearly Corollary 4.3 provides an improvement of a result which obtained by Sumer Eker [15, Theorem 1].

If we set $\tau = \lambda = 1$ in Corollary 4.1, then the class $\mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha)$ reduces to the class $\mathcal{H}^{\alpha}_{\Sigma_m}$ which introduced and studied by Srivastava et al. [13].

Corollary 4.5. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{H}^{\alpha}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(2m+1)(m+1)}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{2m+1}.$$

Remark 4.6. *Corollary 4.5 provides a refinement of a result which obtained by Srivastava et al. [13, Theorem 2].*

Remark 4.7. *If we set $m = 1$ in Corollary 4.5, then the class $\mathcal{H}^{\alpha}_{\Sigma_m}$ reduces to the class $\mathcal{H}_{\alpha}$ which introduced and studied by Srivastava et al. [11].*

Corollary 4.8. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}^{\alpha}_{\Sigma}$. Then*

$$|a_2| \leq \sqrt{\frac{2}{3}}\alpha, \tag{4.2}$$

and

$$|a_3| \leq \frac{2\alpha^2}{3}. \tag{4.3}$$

Remark 4.9. *Corollary 4.8 provides an improvement of a result which obtained by Srivastava [11, Theorem 1].*

For one-fold symmetric bi-univalent functions and for $\tau = 1$, the class $\mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha)$ reduces to the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$ and we obtain the following result which is an improvement of a result which were proven by Frasin and Aouf [5, Theorem 2.2].

Corollary 4.10. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2}{2\lambda+1}}\alpha, & 1 \leq \lambda < 1+\sqrt{2} \\ \frac{2\alpha}{\lambda+1}, & \lambda \geq 1+\sqrt{2} \end{cases} \tag{4.4}$$

and

$$|a_3| \leq \frac{2\alpha^2}{2\lambda+1}.$$

Remark 4.11. *Corollary 4.10 provides a refinement of a result which were obtained by Frasin and Aouf [5, Theorem 2.2].*

By setting

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots,$$

in Theorem 3.1, we deduce the following result.

Corollary 4.12. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{B}^*_{\Sigma_m}(\tau,\lambda,\beta)$. Then*

$$|a_{m+1}| \le \min\left\{\frac{2(1-\beta)|\tau|}{\lambda m+1}, \sqrt{\frac{4(1-\beta)|\tau|}{(2\lambda m+1)(m+1)}}\right\},$$

and

$$|a_{2m+1}| \le \frac{2(1-\beta)|\tau|}{2\lambda m+1}.$$

Remark 4.13. *It is easy to see, for the coefficient $|a_{2m+1}|$, that*

$$\frac{2(1-\beta)|\tau|}{2\lambda m+1} \le \frac{2|\tau|^2(1-\beta)^2(m+1)}{(\lambda m+1)^2} + \frac{2|\tau|(1-\beta)}{2\lambda m+1}.$$

Thus, clearly, Corollary 4.12 is an improvement of Theorem 1.4.

If we set $\tau = 1$ in Corollary 4.12, then the class $\mathcal{B}^*_{\Sigma_m}(\tau,\lambda,\beta)$ reduces to the class $\mathcal{A}^\lambda(\beta)$ which introduced and studied by Sumer Eker [15].

Corollary 4.14. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{A}^\lambda(\beta)$. Then*

$$|a_{m+1}| \le \min\left\{\frac{2(1-\beta)}{\lambda m+1}, \sqrt{\frac{4(1-\beta)}{(2\lambda m+1)(m+1)}}\right\},$$

and

$$|a_{2m+1}| \le \frac{2(1-\beta)}{2\lambda m+1}.$$

Remark 4.15. *It is easy to see that*

$$\frac{2(1-\beta)}{2\lambda m+1} \le \frac{2(1-\beta)^2(m+1)}{(1+\lambda m)^2} + \frac{2(1-\beta)}{2\lambda m+1}.$$

Thus, Corollary 4.14 provides an improvement of a result which obtained by Sumer Eker [15, Theorem 2].

If we take $\lambda = 1$ in Corollary 4.14, then the class $\mathcal{A}^\lambda_{\Sigma_m}(\beta)$ reduces to the class $\mathcal{H}^\beta_{\Sigma_m}$ which introduced and studied by Srivastava et al. [13].

Corollary 4.16. *Let the function $f(z)$ given by (1.3) be in the class $\mathcal{H}^\beta$. Then*

$$|a_{m+1}| \le \min\left\{\frac{2(1-\beta)}{m+1}, \sqrt{\frac{4(1-\beta)}{(2m+1)(m+1)}}\right\}$$

and

$$|a_{2m+1}| \le \frac{2(1-\beta)}{2m+1}.$$

Remark 4.17. *Corollary 4.16 provides a refinement of a result which obtained by Srivastava [13, Theorem 3].*

If we take $m = 1$ in Corollary 4.16, then the class $\mathcal{H}_{\Sigma_m}^{\beta}$ reduces to the class $\mathcal{H}_{\Sigma}^{\beta}$ introduced and studied by Srivastava et al. [11].

Corollary 4.18. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\beta}$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & 0 \leq \beta \leq \frac{1}{3} \\ (1-\beta), & \frac{1}{3} \leq \beta < 1 \end{cases} \tag{4.5}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

Remark 4.19. *Corollary 4.18 provides a refinement of a result which obtained by Srivastava [11, Theorem 2].*

For one-fold symmetric bi-univalent functions and for $\tau = 1$, the class $\mathcal{B}_{\Sigma_m}^{*}(\tau, \lambda, \beta)$ reduces to the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$ and we obtain the following result which is an improvement of a result which were proven by Frasin and Aouf [5, Theorem 3.2].

Corollary 4.20. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{\lambda+1}, \sqrt{\frac{2(1-\beta)}{2\lambda+1}}\right\},$$

and

$$|a_3| \leq \frac{2(1-\beta)}{2\lambda+1}.$$

Remark 4.21. *Corollary 4.20 provides an improvement of a result which were obtained by Frasin and Aouf [5, Theorem 3.2].*

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