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An Efficient Numerical Method Based on Variational Iteration Method for Solving the Kuramoto-Sivashinsky Equations

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ABSTRACT: In this paper we consider variational iteration method to investigate solution of Kuramoto-Sivashinsky equations. Comparison of the results of this method obtained just in 2-iterations with RBF based mesh -free method and local continuous Galerkin methods, shows the efficiency of this method. Moreover, numerical solution of this equation by spectral collocation method is investigated. Numerical experiments are included to show the efficiency of these methods.

Key Words: Variational iteration method, Kuramoto-Sivashinsky equation, Lagrange multiplier, Spectral collocation method.

Contents

1. Introduction

The Kuramoto-Sivashinsky equation is one of the most important nonlinear evolution equations that has many applications in physical process such as unstable drift waves in plasma, reaction diffusion systems, thin hydrodynamics films [18], long waves on the interface between two viscous fluids [10]. It is a partial

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differential equation which exhibits chaotic behavior and has a solution like traveling waves which moves without change of shape over a finite spatial domain (see [11] and [15]). Several authors have investigated numerical solutions of Kuramoto-Sivashinsky (see [13] and [14]). Traveling wave solutions of this equation have been studied by Hooper, Grimshaw and Yang [9]. Moreover, Akrivis and Smyrlis studied implicit-explicit BDF methods for Kuramoto-Sivashinsky equation [1]; Uddin, Haq and Siraj-ul-Islam used RBF based mesh-free method for numerical solution of Kuramoto-Sivashinsky equation [19]; Lai and Ma used Lattice Boltzmann method [12]; Xu, Shu present Local discontinuous Galerkin method for solving this equation [20]. In this work we discuss variational iteration method for solving Kuramoto-Sivashinsky equation. The variational iteration method was first proposed by He ([5]-[8]). The main advantage of this method is that it can be applied directly for all types of equations such as autonomous ordinary differential equations [8], non-linear wave equations [3], circuit theory [4], non-linear polycrystalline solids [2]. Moreover Rafei applied this method to approximate the solutions of the epidemic and the predator and prey models [17]. Momani, Abuasad and Odibat used variational iteration method for solving nonlinear boundary value problems [16]. The paper has been organized as follows. In section 2, Kuramoto-Sivashinsky equations are introduced. In section 3, we explain general variational iteration method. In section 4, Kuramoto-Sivashinsky equations are analyzed by using variational iteration method. Section 5 is related to the numerical solution of Kuramoto-Sivashinsky equations by spectral collocation method. Finally in the last section numerical examples are presented and comparison with some other methods are explained.

2. Kuramoto-Sivashinsky equations

There are two standard forms for the kuramoto-Sivashinsky equation. The first form is

$$v_t + \frac{1}{2}(v_x)^2 = -v_{xx} - v_{xxxx}. \tag{2.1}$$

Initial condition for this equation is L-periodic initial condition with $L > 0$. Another kind of Kuramoto-Sivashinsky equation is as follows:

$$u_t + uu_x - u_{xx} + u_{xxxx} = 0 \tag{2.2}$$

The initial condition for this equation is:

$$\int_0^L u(x,0)dx = 0.$$

In some sense, the Kuramoto-Sivashinsky equation is similar to Burgers equation, however, because of presence of second and fourth order derivatives, this equation has more complicated behavior. The sign of second derivative term is such that it operate as an energy source. The nonlinear part of the equation uu_x transfers energy between low to high level wave numbers.

3. Variational iteration method

Basic concept of variational iteration method is as follows. We consider the following general differential equation

$$Lu + Nu = g(x), \tag{3.1}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is a known analytic function. According to variational iteration method ([5]- [8]), we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_o^x \lambda[Lu_n(\tau) + N\widehat{u_n}(\tau) - g(\tau)]d\tau, \tag{3.2}$$

where λ is a general Lagrangian multiplier ([5]- [8]), which can be identified optimally via the variational theory and integration by parts. The subscript n denotes the nth-order approximation, $\widehat{u_n}$ is considered as a restricted variation ([5]- [8]) so that its variation is zero, $\delta u_n = 0$, i.e., $\delta\widehat{u_n} = 0$. It is required first to determine the Lagrangian multiplier optimally. Employing the restricted variation in equation (2.2) makes it easy to compute this multiplier. By using an initial function u_0, the approximations $u_{n+1}, n \geq 0$ of the solution $u(x,t)$ can be obtained via the calculated Lagrange multiplier and the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n.$$

4. Application of variational iteration method for Kuramoto-Sivashinsky equation

Problem 1. Consider the following Kuramoto-Sivashinsky equation

$$u_t + uu_x - u_{xx} + u_{xxxx} = 0, \tag{4.1}$$

with initial condition

$$u(x,0) = c + \frac{15}{19\sqrt{19}}[\tanh^3(k(x-x_0)) - 3\tanh(k(x-x_0))]. \tag{4.2}$$

The exact solution of the above problem is given by

$$u(x,t) = c + \frac{15}{19\sqrt{19}}[\tanh^3(k(x-ct-x_0)) - 3\tanh(k(x-ct-x_0))]. \tag{4.3}$$

To solve this equation by variational iteration method, we construct a correction functional in x-direction as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda[\frac{\partial u_n}{\partial \tau} + \widehat{u_n\frac{\partial u_n}{\partial x}} - \widehat{\frac{\partial^2 u_n}{\partial x^2}} + \widehat{\frac{\partial^4 u_n}{\partial x^4}}]d\tau, \tag{4.4}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda[\frac{\partial u_n}{\partial \tau} + \widehat{u_n\frac{\partial u_n}{\partial x}} - \widehat{\frac{\partial^2 u_n}{\partial x^2}} + \widehat{\frac{\partial^4 u_n}{\partial x^4}}]d\tau. \tag{4.5}$$

Note that $\delta\widehat{u_n} = 0$. Calculus of variations and integration by parts give

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\tau)\delta u_n(x,\tau)|_{\tau=t} - \int_0^t \delta u_n(x,\tau)\lambda'(\tau)d\tau = 0. \quad (4.6)$$

This yields the stationary conditions:

$$\delta u_n : \lambda'(\tau)|_{\tau=t} = 0, \quad (4.7)$$

$$\delta u_n : 1 + \lambda(\tau)|_{\tau=t} = 0, \quad (4.8)$$

which give $\lambda(\tau) = -1$.
Thus, by inserting the Lagrange multiplier into (3.4) we obtain the following iteration formula in x-direction:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t [\frac{\partial u_n}{\partial \tau} + u_n \frac{\partial u_n}{\partial x} - \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^4 u_n}{\partial x^4}]d\tau, \quad (4.9)$$

with

$$u_0(x,t) = c + \frac{15}{19\sqrt{19}}[\tanh^3(k(x-x_0)) - 3\tanh(k(x-x_0))]. \quad (4.10)$$

From above equalities we can obtain the rest of the components of the iteration formula.
Problem 2. Consider the following Kuramoto-Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \quad (4.11)$$

with initial condition

$$u(x,0) = c + \frac{15}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x-x_0)) - 9\tanh(k(x-x_0))]. \quad (4.12)$$

The exact solution of the above problem is given by

$$u(x,t) = c + \frac{15}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x-ct-x_0)) - 9\tanh(k(x-ct-x_0))]. \quad (4.13)$$

To solve this equation by variational iteration method, in the same manner as above we have the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t [\frac{\partial u_n}{\partial \tau} + u_n \frac{\partial u_n}{\partial x} + \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^4 u_n}{\partial x^4}]d\tau, \quad (4.14)$$

with

$$u_0(x,t) = c + \frac{15}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x-x_0)) - 9\tanh(k(x-x_0))]. \quad (4.15)$$

5. The spectral collocation approximation for Kuramoto-Sivashinsky equation

One of the most important methods for numerical solution of partial differential equations, are spectral methods. They are a class of spatial discretization for differential equations. There are three kinds of these methods, Galerkin, collocation and tau. In this section we are interested in the application of spectral collocation method for numerical solution of Kuramoto-Sivashinsky equation. For this aim let $e_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx), k = 1, 2, ...,$ for $x \in [0, \pi]$, then by using $N - 1$ basis functions $e_k(x), k = 1, \cdots, N - 1$ and the following interpolation points in the interval $[0, \pi]$:

$$x_j = \frac{\pi j}{N} \quad (j = 1, ..., N - 1).$$

we approximate a function $f(x)$ by

$$F_N(x) = \sum_{j=1}^{N-1} f(x_j)C_j(x). \tag{5.1}$$

where

$$C_j(x) = \frac{2}{N}\sum_{m=1}^{N-1} \sin(mx_j)\sin(mx),$$

which satisfy $C_j(x_i) = \delta_{ij}, i, j = 1, ..., N - 1$. For approximating the derivatives of $f(x)$ at the collocation points $x_j,\ j = 1, ..., N-1$, by differentiating (5.1) we obtain

$$\frac{d^k F_N(x)}{dx^k} = \sum_{j=1}^{N-1} f(x_j)\frac{d^k}{dx^k}C_j(x),$$

so that

$$\frac{d^k F_N(x_i)}{dx^k} = \sum_{j=1}^{N-1} f(x_j)(D_k)_{i,j},\ i = 1, ..., N - 1,$$

$$(D_k)_{i,j} = \frac{d^k}{dx^k}C_j(x_i),\ i, j = 1, ..., N - 1,$$

where

$$(D_1)_{ij} = \begin{cases} -0.5\cot(x_j), & i = j\ , \\ \frac{(-1)^{i+j+1}\sin(x_j)}{\cos(x_i)-\cos(x_j)}, & \text{otherwise}\ . \end{cases}$$

and $D_2 = D'D_1$, where

$$(D')_{ij} = \begin{cases} 0.5\cot(x_j), & i = j \\ \frac{(-1)^{j+1}\sin(x_i)\cos(Nx_i)}{[\cos(x_i)-\cos(x_j)]}, & \text{otherwise.} \end{cases}$$

Now we describe the numerical solution of the Kuramoto-Sivashinsky equation by using the spectral collocation method for spatial discretization and the implicit

Euler scheme for time discretization.
Assume $T > 0, N \in \mathbb{N}$ and M is an integer. Let $\Delta t = \frac{T}{M}$ be a time step, the numerical method is expressed as

$$\mathbf{u}_{n+1}^N = \mathbf{u}_n^N - \Delta t(D_2\mathbf{u}_{n+1}^N + \mathbf{u}_n^N \times D_1\mathbf{u}_n^N + D_4\mathbf{u}_n^N), \cdots, n = 0, \cdots, N \tag{5.2}$$

where

$$\mathbf{u}_j^N = [u_j^N(\Delta x), ..., u_j^N((N-1)\Delta x)]^T,$$

and $u_j^N(i\Delta x)$ means the approximate value of the solution function u at $(i\Delta x, j\Delta t)$ for $j = 0, 1, ..., i = 1, ..., N-1$, where $\mathbf{u} \times \mathbf{v}$ is the component-wise product of two vectors $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{u}_0^N$ is the initial vector .

6. Numerical solution

We now obtain numerical solution of problems 1,2 by the variational iteration method. In order to verify the efficiency of the proposed method in comparison with exact solution, we report the error norm L_∞ for $N = 121$ points x in $[-30, 30]$ and different values of t, which is defined by

$$L_\infty = \|u_{exact} - u_{app}\|_\infty = Max|u_{exact} - u_{app}|. \tag{6.1}$$

Error norm L_∞ of two iterations of variational iteration method (VIM) and mesh-free collocation method based on RBFs such as multiquadric (MQ), thin plate spline (TPS), Gaussian (GA), and spline basis (r^7) [19], are shown in tables 1,3, and those of error of LDG method [20] are shown in tables 2,4 .

6.1. Numerical solution of problem 1

Table 1: The error norm L_∞ for mesh-free method [19] and variational iteration method for equation (4.1)

Time	0.1	0.3	0.5	0.7	1.0
MQ	$2.24e-005$	$3.60-e005$	$4.52e-005$	$5.16e-005$	$5.84e-005$
r^7	$6.23e-006$	$6.23e-006$	$8.45e-006$	$1.86e-005$	$1.24e-005$
TPS	$2.35e-003$	$4.94e-003$	$7.36e-003$	$9.79e-003$	$1.36e-002$
GA	$7.06e-003$	$1.12e-002$	$1.45e-002$	$1.68e-002$	$1.94e-002$
VIM	$7.20e-009$	$1.90e-007$	$9.10e-007$	$2.5e-006$	$7.3e-006$

$c = 0.2, k = \frac{1}{2\sqrt{19}}, x_0 = -10, in[-30, 30].$

Table 2: The error norm L_∞ for LDG method [20] for equation (4.1)

N	40	80	160	320
P^0	$7.56e-002$	$3.87-e002$	$1.95e-002$	$9.8e-003$
P^1	$1.04e-002$	$2.59e-003$	$6.54e-004$	$1.64e-004$
P^2	$7.93e-004$	$1.06e-004$	$1.34e-005$	$1.67e-006$

$c = 0.2, k = \frac{1}{2\sqrt{19}}, x_0 = -10, in[-30, 30], t = 1.$

6.2. Numerical solution of problem 2

Table 3: The error norm L_∞ for mesh-free method [19] and variational iteration method for equation (4.11)

Time	0.1	0.3	0.5	0.7	1.0
MQ	$1.02e-004$	$1.85-e004$	$2.89e-004$	$3.85e-004$	$5.22e-004$
r^7	$3.17e-004$	$3.99e-004$	$4.21e-004$	$9.90e-004$	$1.31e-002$
TPS	$1.32e-002$	$1.98e-002$	$2.21e-002$	$2.26e-002$	$2.39e-002$
GA	$4.00e-002$	$8.14e-002$	$1.09e-001$	$1.37e-001$	$1.75e-001$
VIM	$7.53e-006$	$2.03e-004$	$1.02e-004$	$2.0e-004$	$7.5e-003$

$c = 0.1, k = \frac{1}{2}\sqrt{(\frac{11}{19})}, x_0 = -10, in[-30, 30].$

Table 4: The error norm L_∞ for LDG method [20] for equation (4.11)

N	40	80	160	320
P^0	1.37	$8.81e-001$	$5.21e-001$	$2.91e-001$
P^1	$6.64e-001$	$1.82e-001$	$4.64e-002$	$1.19e-002$
P^2	$1.49e-001$	$1.73e-002$	$2.43e-003$	$3.05e-004$

$c = 0.1, k = \frac{1}{2}\sqrt{(\frac{11}{19})}, x_0 = -10, in[-30, 30], t = 1.$

Moreover, numerical solution of the Problem 1 and problem 2, for $x \in [-30, 30], t \in [0, 1]$ by spectral collocation method are shown in Figure 1 and Figure 2 respectively.

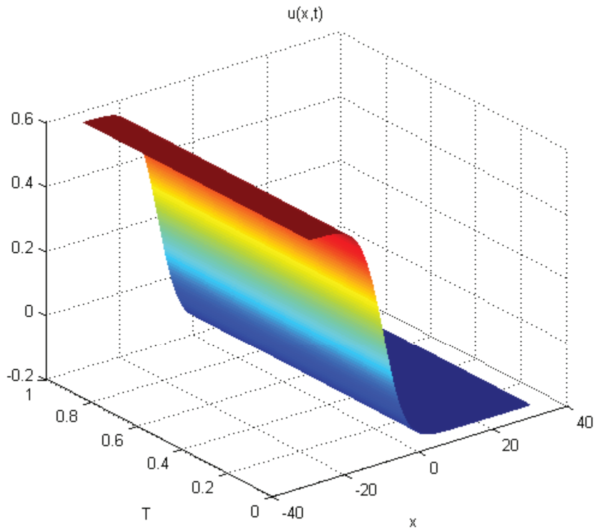

Figure 1: Numerical solution of the Problem 1, for $x \in [-30, 30], t \in [0, 1]$ by spectral collocation method.

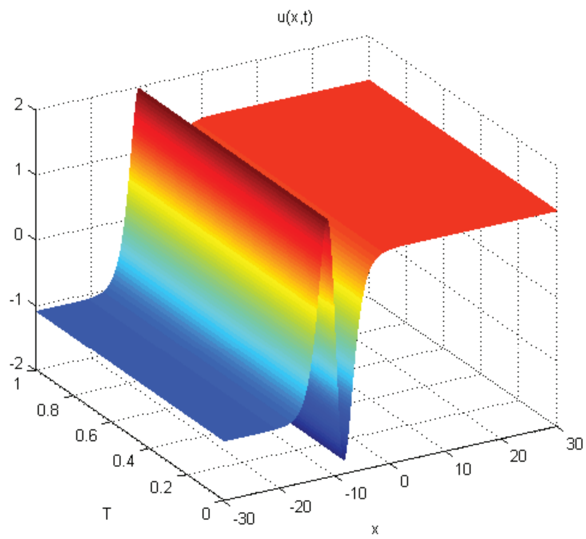

Figure 2: Numerical solution of the Problem 2, for $x \in [-30, 30], t \in [0, 1]$ by spectral collocation method.

7. Conclusion

In this paper, we applied the Variational iteration method for solving Kuramoto-Sivashinsky equations, The results indicate the accuracy of the proposed method, and show that this method is more efficient than those methods proposed in [19], [20].

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