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A Pair of Generalized Derivations in Prime, Semiprime Rings and in Banach Algebras *

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Abstract: Let R be a prime ring with extended centroid C, I a non-zero ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G such that $(H(xy) + G(yx))^n = (xy \pm yx)$ for all $x, y \in I$, then one of the following holds:

1. R is commutative;
2. $n = 1$ and $H(x) = x$ and $G(x) = \pm x$ for all $x \in R$.

Moreover, we examine the case where R is a semiprime ring. Finally, we apply the above result to non-commutative Banach algebras.

Key Words: Prime ring, Semiprime ring, Generalized derivation, Utumi quotient ring, Banach algebra.

Contents

1. Introduction

Let R be an associative prime ring with center $Z(R)$, Q its Martindale quotient ring and U its left Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [3] for these objects).

We denote by $[x, y] = xy - yx$ the simple commutator of the elements $x, y \in R$ and by $x \circ y = xy + yx$ the simple anti-commutator of x, y. A linear mapping $d : R \to R$ is called a derivation, if it satisfies the Leibnitz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. More results about derivation can be found in [1,2,9,18,24,25].

In [4], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation if there exists a derivation

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$d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers the concepts of derivation. In [20], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus all generalized derivations of R will be implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

In the mean while many authors obtained more information about derivations and generalized derivations satisfying certain suitable conditions in rings.

A well-known result proved by Ashraf and Rehman [1], states that R must be commutative if I is a non-zero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$. More recently in [2], Argac and Inceboz generalized the result of [1] as following:

Let R be a prime ring, I a non-zero ideal of R and n a fixed positive integer.

(i) If R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

(ii) If $char(R) \neq 2$ and $(d(x \circ y))^n - x \circ y$ is central for all $x, y \in I$, then R is commutative.

On the other hand, in [23], Quadri, Khan and Rehman proved that if R is a prime ring, I a non-zero ideal of R and F a generalized derivation associated with a non-zero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative. Further, this result of Quadri et al. is studied in semiprime ring by Dhara in [8]. Recently in [7], De Filippis and Huang studied the situation $(F([x, y]))^n = [x, y]$ for all $x, y \in I$, where I is a non-zero ideal in a prime ring R, F a generalized derivation of R and $n \geq 1$ a fixed integer. In this case they conclude that either R is commutative or $n = 1$ and $F(x) = x$ for all $x \in R$. Recently, Huang [26] proved the following:

Let R be a prime ring, I a non-zero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a non-zero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

In the present paper, we generalize above results by considering the situation, when the prime ring R satisfies $(H(xy) + G(yx))^n = (xy \pm yx)$ for all $x, y \in I$, where I is a non-zero ideal of R, H, G are two generalized derivations of R and $n \geq 1$ a fixed integer.

2. Results in Prime Rings

To prove our theorem, we need the following Lemmas:

Lemma 2.1. *Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over the field F with $k \geq 2$ and $a, b, p, q \in R$. Suppose that*

$$(axy + byx + [p, xy] + [q, yx])^n - (xy \pm yx) = 0$$

for all $x, y \in R$*, where* $n \geq 1$ *is a fixed integer. Then* $a, b, p, q \in F \cdot I_k$*.*

Proof: Let $a = (a_{ij})_{k\times k}$, $b = (b_{ij})_{k\times k}$, $p = (p_{ij})_{k\times k}$ and $q = (q_{ij})_{k\times k}$ where $a_{ij}, b_{ij}, p_{ij}, q_{ij} \in F$. Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By assumption, we have

$$(axy + byx + [p, xy] + [q, yx])^n - (xy \pm yx) = 0 \tag{2.1}$$

for all $x, y \in R$.
By choosing $x = e_{ii}$, $y = e_{ij}$ for any $i \neq j$, we have

$$(ae_{ij} + [p, e_{ij}])^n - e_{ij} = 0 \tag{2.2}$$

Multiplying this equality from right by e_{ij}, we arrive at

$$0 = (ae_{ij} + [p, e_{ij}])^n(e_{ij}) = (-1)^n(p_{ji})^n e_{ij}.$$

This implies $p_{ji} = 0$. Thus for any $i \neq j$, we have $p_{ji} = 0$, which implies that p is diagonal matrix. Let $p = \sum_{i=1}^{k} p_{ii}e_{ii}$. For any F-automorphism θ of R, we have

$$(a^\theta xy + b^\theta yx + [p^\theta, xy] + [q^\theta, yx])^n - (xy \pm yx) = 0$$

for every $x, y \in R$. Hence p^θ must also be diagonal. We have

$$(1 + e_{ij})p(1 - e_{ij}) = \sum_{i=1}^{k} p_{ii}e_{ii} + (p_{jj} - p_{ii})e_{ij}$$

diagonal. Therefore, $p_{jj} = p_{ii}$ and so $p \in F \cdot I_k$. Multiplying the equality (2.2) from left by e_{ij}, we have

$$0 = (e_{ij})(ae_{ij} + [p, e_{ij}])^n = ((a + p)_{ji})^n e_{ij}.$$

Since $p_{ji} = 0$, we have from above that $a_{ji} = 0$ for any $i \neq j$, that is, a is diagonal and hence central by same argument as above.

By the same manner by choosing $x = e_{ii}, y = e_{ji}$ we have $b, q \in F \cdot I_k$.

□

Lemma 2.2. *Let* R *be a prime ring with extended centroid* C*,* I *a non-zero ideal of* R *and* $a, b, p, q \in R$*. Suppose that* $(axy + byx + [p, xy] + [q, yx])^n - (xy \pm yx) = 0$ *for all* $x, y \in I$*, where* $n \geq 1$ *is a fixed integer. Then* $a, b, p, q \in C$*.*

Proof: By assumption, I satisfies the generalized polynomial identity

$$f(x, y) = (axy + byx + [p, xy] + [q, yx])^n - (xy \pm yx). \tag{2.3}$$

By [6], this generalized polynomial identity (GPI) is also satisfied by U, that is $f(x,y)=0$ for all $x,y\in U$.

We assume first that U does not satisfy any non-trivial (GPI). Then from (2.3), we have that

$$((a+p)xy+(b+q)yx-xyp-yxq)^n-(xy\pm yx)=0 \tag{2.4}$$

is a trivial (GPI) for U, that is, zero element in $T=U*_C C\{x,y\}$, the free product of U and $C\{x,y\}$, the free C-algebra in non-commuting indeterminates x and y. Let $p\notin C$. Then $\{1,p\}$ is linearly independent over C. If $q\notin \mathrm{Span}_c\{1,p\}$, then $\{1,p,q\}$ is linearly independent over C. In this case expansion of (2.4) yields that $(-xyp)^n$ appears non-trivially, a contradiction. If $q\in \mathrm{Span}_c\{1,p\}$, then $q=\alpha+\beta p$ for some $\alpha,\beta\in C$. Then expansion of (2.4) yields that $\{(-xy-yx\beta)p\}^n$ appears non-trivially, a contradiction. Thus we conclude that $p\in C$. Similarly, we can prove that $q, a+p, b+q\in C$ and hence $a,b,p,q\in C$.

Next we assume that (2.3) is a non-trivial (GPI) for U. In this case, if C is infinite, we have $f(x,y)=0$ for all $x,y\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Moreover, both U and $U\otimes_C\overline{C}$ are prime and centrally closed algebras [10]. Hence, replacing R by U or $U\otimes_C\overline{C}$ according to C finite or infinite, without loss of generality we may assume that $C=Z(R)$ and R is C-algebra centrally closed. By Martindale's theorem [21], R is then a primitive ring having non-zero socle $\mathrm{soc}(R)$ with C as the associated division ring. Hence, by Jacobson's theorem [14, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Let $\dim_C V=k$. Then $R\cong M_k(C)$ for some $k\geq 1$. If $k=1$, then R is commutative and so $a,b,p,q\in C$. If $k\geq 2$, then by Lemma 2.1, $a,b,p,q\in C$.

If V is infinite dimensional over C, then for any $e^2=e\in \mathrm{soc}(R)$, we have $eRe\cong M_t(C)$ with $t=\dim_C Ve$. If $a,b,p,q\in C$, we have our conclusion. So assume that not all of a,b,p,q are in C. Then at least one of a,b,p,q does not centralize the non-zero ideal $\mathrm{soc}(R)$. Hence, there exist $h_1,h_2,h_3,h_4\in \mathrm{soc}(R)$ such that either $[a,h_1]\neq 0$ or $[b,h_2]\neq 0$ or $[p,h_3]\neq 0$ or $[q,h_4]\neq 0$. By Litoff's theorem (see [11]), there exists an idempotent $e\in\mathrm{soc}(R)$ such that $ah_1,h_1a,bh_2,h_2b,ph_3,h_3p,qh_4,h_4q,h_1,h_2,h_3,h_4\in eRe$. We have $eRe\cong M_k(C)$ with $k=\dim_C Ve$. Replacing $x=e$ and $y=ex(1-e)$, we have that R satisfies

$$((a+p)ex(1-e)-ex(1-e)p)^n-ex(1-e)=0. \tag{2.5}$$

Left multiplying by $(1-e)$, we have $(1-e)((a+p)ex(1-e))^n=0$ that is $((1-e)(a+p)ex)^{n+1}=0$ for all $x\in R$. By Levitzki's lemma [12, Lemma 1.1], we have $(1-e)(a+p)eR=0$ implying $(1-e)(a+p)e=0$. Analogously, we can prove that $(1-e)(b+q)e=0$. Therefore, $(a+p)e=e(a+p)e$ and $(b+q)e=e(b+q)e$. Moreover, since R satisfies

$$e\{((a+p)exeye+(b+q)eyexe-exeyep-eyexeq)^n-(exeye\pm eyexe)\}e=0 \tag{2.6}$$

eRe satisfies

$$(e(a+p)exy+e(b+q)eyx-xyepe-yxeqe)^n-(xy\pm yx)=0. \tag{2.7}$$

Then by the above finite dimensional case, eae, ebe, epe, eqe are central elements of eRe. Thus, $ah_1 = (eae)h_1 = h_1eae = h_1a$, $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$, $ph_3 = (epe)h_3 = h_3epe = h_3p$ and $qh_4 = (eqe)h_4 = h_4(eqe) = h_4q$, a contradiction. □

Theorem 2.3. *Let R be a prime ring with extended centroid C, I a non-zero ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G such that $(H(xy) + G(yx))^n - (xy \pm yx) = 0$ for all $x, y \in I$, then either R is commutative or $n = 1$, $H(x) = x$ and $G(x) = \pm x$ for all $x \in R$.*

Proof: By hypothesis, we have that R satisfies

$$(H(xy) + G(yx))^n - (xy \pm yx) = 0. \tag{2.8}$$

In view of [20, Theorem 3], we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $H(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$. Since I, R and U satisfy the same generalized polynomial identities (see [6]) as well as the same differential identities (see [19]), we may assume that

$$(axy + d(xy) + byx + \delta(yx))^n - (xy \pm yx) = 0 \tag{2.9}$$

for all $x, y, z \in U$, where d, δ are two derivations of U, that is

$$(axy + d(x)y + xd(y) + byx + \delta(y)x + y\delta(x))^n - (xy \pm yx) = 0 \tag{2.10}$$

for all $x, y \in U$. Here we divide the proof into two cases:

Case 1. Let d and δ both be inner derivations of U, that is $d(x) = [p, x]$ and $\delta(x) = [q, x]$ for all $x \in U$, for some $p, q \in U$. Then from (2.10) we get that U satisfies

$$(axy + [p, xy] + byx + [q, yx])^n - (xy \pm yx) = 0. \tag{2.11}$$

Then by Lemma 2.2, $a, b, p, q \in C$. Then U satisfies

$$(axy + byx)^n - (xy \pm yx) = 0. \tag{2.12}$$

This is a polynomial identity for U. Then by [18, Lemma 2], there exists a field F such that $U \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, moreover U and $M_k(F)$ satisfy the same polynomial identities. If $k = 1$, then U and so R is commutative. If $k \geq 2$, then replacing $x = e_{ij}$ and $y = e_{jj}$ for $i \neq j$, we have $(ae_{ij})^n - e_{ij} = 0$. For $n \geq 2$, $e_{ij} = 0$, a contradiction. Hence $n = 1$ and so $(a - 1)xy + (b \mp 1)yx = 0$ for all $x, y \in M_k(F)$. Replacing $x = e_{ii}$ and $y = e_{ij}$ for $i \neq j$, we have $(a - 1)e_{ij} = 0$, implying $a = 1$. Again, replacing $x = e_{ij}$ and $y = e_{ii}$ for $i \neq j$, we get $(b \mp 1)e_{ij} = 0$, implying $b = \pm 1$. Therefore, we have $H(x) = ax + [p, x] = x$ and $G(x) = bx + [q, x] = \pm x$ for all $x \in U$ and so for $x \in R$.

Case 2. Assume that d and δ are not both inner derivations of U. Suppose that d and δ are linearly C-dependent modulo U_{int}, say $\alpha d + \beta\delta = ad_{q'}$, where $\alpha, \beta \in C$, $q' \in U$ and $ad_{q'}(x) = [q', x]$ for all $x \in U$.

Subcase-i: *Let* $\alpha \neq 0$.

Then $d(x) = \lambda\delta(x) + [c, x]$ for all $x \in U$, where $\lambda = -\beta\alpha^{-1}$ and $c = \alpha^{-1}q'$. Then δ can not be inner derivation of U. From (2.9), we obtain

$$(axy + \lambda\delta(x)y + \lambda x\delta(y) + [c, xy] + byx + \delta(y)x + y\delta(x))^n - (xy \pm yx) = 0 \quad (2.13)$$

for all $x, y, z \in U$. Then by Kharchenko's theorem [15], U satisfies

$$(axy + \lambda sy + \lambda xt + [c, xy] + byx + tx + ys)^n - (xy \pm yx) = 0. \quad (2.14)$$

If R is commutative, we have our conclusion (1). So let R be non-commutative. Then there exits $q \in U$ such that $q \notin C$. Thus by replacing s with $[q, x]$ and t with $[q, y]$, we have from (2.14) that U satisfies

$$(axy + \lambda[q, x]y + \lambda x[q, y] + [c, xy] + byx + [q, y]x + y[q, x])^n - (xy \pm yx) = 0 \quad (2.15)$$

that is

$$(axy + [\lambda q + c, xy] + byx + [q, yx])^n - (xy \pm yx) = 0. \quad (2.16)$$

Then by Lemma 2.2, we conclude that $q \in C$, a contradiction.

Subcase-ii: *Let* $\alpha = 0$.

Then $\beta \neq 0$ and so $\delta(x) = [c', x]$ for all $x \in U$, where $c' = q'\beta^{-1}$. From (2.9), we obtain

$$(axy + d(x)y + xd(y) + byx + [c', yx])^n - (xy \pm yx) = 0 \quad (2.17)$$

for all $x, y, z \in U$. By Kharchenko's theorem [15], U satisfies

$$(axy + sy + xt + byx + [c', yx])^n - (xy \pm yx) = 0. \quad (2.18)$$

In particular for $y = 0$, we have that U satisfies $(xt)^n = 0$. Since this is a polynomial identity, as above, there exists a field F such that $U \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and $M_k(F)$ satisfies the identity $(xt)^n = 0$. If $k \geq 2$, then for $x = t = e_{22}$, we have $0 = (xt)^n = e_{22}$, a contradiction. Hence $k = 1$ which implies R is commutative.

Case 3. Assume that d and δ are C-independent modulo U_{int}. Then by Kharchenko's theorem [15], we have from (2.9) that U satisfies

$$(axy + sy + xt + byx + yu + vx)^n - (xy \pm yx) = 0 \quad (2.19)$$

for all $x, y, z \in U$. Then in particular for $y = v = 0$, we have $(xt)^n = 0$ for all $x, t \in U$. Then by same argument as above, this implies the commutativity of R. □

In particular, when $G = H$, we have the following:

Corollary 2.4. *Let R be a prime ring with extended centroid C, I a non-zero ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivation H such that $(H(x \circ y))^n - (x \circ y) = 0$ for all $x, y \in I$, then either R is commutative or $n = 1$, $H(x) = x$ for all $x \in R$.*

In particular, when $G = -H$, we have the following:

Corollary 2.5. *Let R be a prime ring with extended centroid C, I a non-zero ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivation H such that $(H([x,y]))^n - [x,y] = 0$ for all $x, y \in I$, then either R is commutative or $n = 1$, $H(x) = x$ for all $x \in R$.*

In particular, when $n = 1$ and $G(x) = -2x$ and $H(x) = F(x) + x$, we have the following:

Corollary 2.6. *Let R be a prime ring with extended centroid C and I a non-zero ideal of R. If R admits the generalized derivation F such that $F(xy) - yx = 0$ for all $x, y \in I$, then R is commutative.*

3. Results in Semiprime Rings

Now, we prove our theorems in semiprime ring and non-commutative Banach algebras.

Theorem 3.1. *Let R be a semiprime ring and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G associated with derivations d, δ such that $(H(xy) + G(yx))^n = (xy \pm yx)$ for all $x, y \in R$, then (1) for $n \geq 2$, R is commutative and (2) for $n = 1$, $H(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$ for all $x \in R$, with $a, b \in C$ and $d(R) \subseteq Z(R)$ and $\delta(R) \subseteq Z(R)$.*

Proof: We know the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [19, Lemma 2]. Moreover R and U satisfy the same GPIs (see [6]) as well as same differential identities (see [19]).

Thus, by Lee [20], $H(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$ for some $a, b \in U$ and derivations d, δ on U and hence

$$(axy + d(xy) + byx + \delta(yx))^n - (xy \pm yx) = 0$$

for all $x, y, z \in U$. Let $M(C)$ be the set of all maximal ideals of C and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [19, p.31-32]), we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\cap\{PU | P \in M(C)\} = 0$. Set $\overline{U} = U/PU$. Then derivations d and δ canonically induce the derivations $\overline{d}$ and $\overline{\delta}$ on U defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$ respectively. Therefore,

$$((\overline{a}\,\overline{x}\,\overline{y} + \overline{d}(\overline{x}\,\overline{y}) + \overline{b}\overline{y}\,\overline{x} + \overline{\delta}(\overline{y}\,\overline{x}))^n - (\overline{x}\,\overline{y} \pm \overline{y}\,\overline{x}) = 0$$

for all $\overline{x}, \overline{y} \in U$. By Theorem 2.3 for prime ring case, we have one of the following: (1) when $n \geq 2$ for each $P \in M(C)$, $[U, U] \subseteq PU$. Since $\cap_P PU = 0$, we have $[U, U] = 0$, implying U and so R is commutative. (2) When $n = 1$, for each $P \in M(C)$, either $[U, U] \subseteq PU$ or $(a - 1) \in PU$, $(b \mp 1) \in PU$, $d(U) \subseteq PU$ and

$\delta(U) \subseteq PU$. This implies that $(a-1)[U,U] \subseteq PU$ for all $P \in M(C)$, $(b\mp 1)[U,U] \subseteq PU$ for all $P \in M(C)$, $d(U)[U,U] \subseteq PU$ for all $P \in M(C)$ and $\delta(U)[U,U] \subseteq PU$ for all $P \in M(C)$. Since $\cap_P PU = 0$, we obtain $(a-1)[U,U] = 0$, $(b\mp 1)[U,U] = 0$, $d(U)[U,U] = 0$ and $\delta(U)[U,U] = 0$. In particular, $(a-1)[R,R] = 0$, $(b\mp 1)[R,R] = 0$, $d(R)[R,R] = 0$ and $\delta(R)[R,R] = 0$. These cases imply that $a \in C$, $b \in C$, $d(R) \subseteq Z(R)$ and $\delta(R) \subseteq Z(R)$. □

4. Result in non-commutative Banach Algebras

In this section, we prove our last result in non-commutative Banach algebra. Here A will denote a complex non-commutative Banach algebras. By a Banach algebra we shall mean a complex normed algebra A whose underlying vector space is a Banach space. By rad(A) we denote the Jacobson radical of A, which is the intersection of all primitive ideals of A. A is said to be semisimple, if rad$(A) = 0$.

The classical result of Singer and Werner in [28] says that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. In this paper they conjectured that the continuity is not necessary. Thomas [29] verified this conjecture. Of course the same result of Singer and Werner dose not hold in non-commutative Banach algebras because of inner derivations. Hence in this context a very interesting question is how to obtain non-commutative version of Singer-Werner theorem. Some partial solutions of this open question have been obtained by a number of authors under certain conditions for non-commutative Banach algebras.

Let A be a non-commutative Banach algebra and D be a continuous derivation on A. Brešar and Vukman [5] proved that if $[D(x), x] \in$ rad(A) for all $x \in A$, then D maps A into rad(A). Vukman [30] proved that the same conclusion holds if $[D(x), x]_3 \in$ rad(A) for all $x \in A$. In [17], Kim proved that if D is a continuous linear Jordan derivation in a Banach algebra A, such that $[D(x), x]D(x)[D(x), x] \in$ rad(A), for all $x \in A$, then D maps A into $rad(A)$. In [22], Park proved that if $[[D(x), x], D(x)] \in$ rad(A), for all $x \in A$, then again D maps A into $rad(A)$. Recently, Kim [16] proved that if D is a continuous linear Jordan derivation in a Banach algebra A, such that $D(x)^3[D(x), x] \in$ rad(A), for all $x \in A$, then D maps A into rad(A).

In this line of investigation, we prove the following:

Theorem 4.1. *Let A be a non-commutative Banach algebra, n a fixed positive integer, $H = L_a + d$ and $G = L_b + \delta$ two continuous generalized derivations of A, where L_a and L_b denote the left multiplication by some element $a \in A$ and some $b \in A$, respectively. If $(H(xy) + G(yx))^n - (xy \pm yx) \in rad(A)$ for all $x, y \in A$, then $d(A) \subseteq rad(A)$ and $\delta(A) \subseteq rad(A)$.*

Proof: By the hypothesis H and G are continuous and moreover since it is well known that L_a and L_b also are continuous, we get d and δ are continuous, too. By [27], any continuous derivation of Banach algebra leaves the primitive ideals invariant. Hence for any primitive ideal P of A, we have $H(P) \subseteq ap + d(P) \subseteq P$ and $G(P) \subseteq bp + \delta(P) \subseteq P$. It means that continuous generalized derivations H and G leaves the primitive ideal invariant. Denote $\frac{A}{P} = \overline{A}$ for any primitive ideals P. Hence we may introduce the generalized derivations $H_P : \overline{A} \to \overline{A}$ and $G_P : \overline{A} \to \overline{A}$ defined by $F_P(\overline{x}) = F_p(x + P) = F(x) + P = ax + d(x) + P$ and $G_P(\overline{x}) = G_p(x + P) = G(x) + P = bx + \delta(x) + P$ for all $\bar{x} \in \bar{A}$, where $A/P = \bar{A}$ and $\overline{x} = x + P$, respectively. Since P is a primitive ideal, the factor algebra $\bar{A}$ is primitive and so it is prime. By $(F(xy) + G(yx))^n - (xy \pm yx) \in \text{rad}(A)$ for all $x, y \in A$ we have

$$(F_P(\bar{x}\bar{y}) + G_P(\bar{y}\bar{x})^n - (\bar{x}\bar{y} \pm \bar{y}\bar{x}) = \bar{0}.$$

for all $\overline{x}, \overline{y} \in \overline{A}$. Now, by Theorem 2.3, it is immediate that $\overline{A}$ is commutative or $d = \bar{0}$ and $\delta = \bar{0}$. Now, we assume that P is primitive ideal such that $\overline{A}$ is commutative. In [28], Singer and Werner proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into the radical. Furthermore by [13], any linear derivation on semisimple Banach algebra is continuous. We know that there are no non-zero linear continuous derivations on commutative semisimple Banach algebras. Therefore, $d = \bar{0}$ and $\delta = \bar{0}$ in $\bar{A}$. Hence we get $d(A) \subseteq P$ and $\delta(A) \subseteq P$ for all primitive ideal P of A. Since $\text{rad}(A)$ is the intersection of all primitive ideals, we get $d(A) \subseteq \text{rad}(A)$ and $\delta(A) \subseteq \text{rad}(A)$, and we get the required conclusion. □

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