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On the Uniform Ergodic for $\alpha-$times Integrated Semigroups

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ABSTRACT: Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$. We study the uniform ergodicity of $(S(t))_{t\geq 0}$ and we show that the range of A is closed if and only if $\lambda R(\lambda, A)$ is uniformly ergodic. Moreover, we obtain that $(S(t))_{t\geq 0}$ is uniformly ergodic if and only if $\alpha = 0$. Finally, we get that $\frac{1}{t^{\alpha+1}}\int_0^t S(s)ds$ converge uniformly for all $\alpha \geq 0$.

Key Words: Uniformly ergodic, $\alpha-$times integrated semigroup, Generator.

Contents

1. Introduction

Throughout, X denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, let A be a closed linear operator on X with domain $D(A)$, we denote by $ker(A)$, $\mathcal{R}(A)$, $\rho(A)$ and $R(., A)$, respectively the kernel, the range, the resolvent set and the resolvent function of A.
The family $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called C_0-semigroup [7] if it has the following properties:

1. The map $t \to S(t)x$ from $[0, +\infty[$ into X is continuous for all $x \in X$,
2. S(t)S(s)=S(t+s),
3. S(0)=I.

In this case, its generator A is defined by

$$D(A) = \{x \in X \,/\, \lim_{t\to 0^+} \frac{S(t)x - x}{t}\},$$

with

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}.$$

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Now, we recall the notion of α-times integrated semigroup which is a generalization of C_0-semigroup. Let $\beta \geq -1$ and f be a continuous function. The convolution $j_\beta * f$ is defined for all $t \geq 0$ by

$$j_\beta * f(t) = \begin{cases} \int_0^t \frac{(t-s)^\beta}{\Gamma(\beta+1)} f(s)ds & \text{if } \beta > -1, \\ \int_0^t f(t-s)d\delta_0(s) & \text{if } \beta = -1, \end{cases}$$

where Γ is the Euler integral giving by $\Gamma(\beta+1) = \int_0^{+\infty} x^\beta e^{-x}dx$, $j_{-1} = \delta_0$ the Dirac measure and for all $\beta > -1$

$$\begin{array}{rccc} j_\beta & : &]0,+\infty[& \to & \mathbb{R} \\ & & t & \mapsto & \frac{t^\beta}{\Gamma(\beta+1)}. \end{array}$$

A strongly continuous $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called an α-times integrated semigroup where $\alpha > 0$ [3], if $S(0) = 0$ and for all $t, s \geq 0$

$$S_n(t)S_n(s) = \int_t^{t+s} \frac{(s+t-r)^{n-1}}{\Gamma(n)} S_n(r)dr - \int_0^s \frac{(s+t-r)^{n-1}}{\Gamma(n)} S_n(r)dr, \qquad (*)$$

where $n-1 < \alpha \leq n$ and $S_n(t)(x) = (j_{n-\alpha-1} * S)(x)$ for all $x \in X$.
By $(*)$ the following equality hold for all $t, s \geq 0$

$$S(t)S(s) = S(s)S(t).$$

Conversely, let $\alpha \geq 0$ and let A be a linear operator on a Banach space X.
We recall that A is the generator of an α-times integrated semigroup [1] if for some $\omega \in \mathbb{R}$, we have $]\omega,+\infty[\subseteq \rho(A)$ and there exists a strongly continuous mapping $S : [0,+\infty[\to \mathcal{B}(X)$ satisfying

$$\begin{aligned} \|S(t)\| &\leq Me^{\omega t} \text{ for all } t \geq 0 \text{ and some } M > 0 \\ R(\lambda, A) &= \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} S(t)dt \text{ for all } \lambda > \max\{\omega, 0\}, \end{aligned}$$

in this case, $(S(t))_{t\geq 0}$ is called the α-times integrated semigroup and the domain of its generator A is defined by

$$D(A) = \{x \in X \ : \ \int_0^t S(s)Axds = S(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)}\}.$$

From the uniqueness Theorem of Laplace Transforms, $(S(t))_{t\geq 0}$ is uniquely determined. In particular, an integrated semigroup is also an 1-times integrated semigroup.

An important example of generators of an $\alpha-$times integrated semigroup is the adjoint A^* on X^* for all $\alpha > 0$, where A is the generator of a C_0-semigroup on a Banach space X. In particular [3, Examples 3.8], we consider $X = L^1(\mathbb{R})$ and for all $f \in D(A) := \{f \in X : f$ is continuous and $f' \in X\}$, we define the linear operator by

$$Af = -f'.$$

Since $X^* = L^\infty(\mathbb{R})$ and for all $f \in D(A^*) = \{f \in X^* : f$ continuous and $f' \in X^*\}$, the adjoint A^* of A is defined by

$$A^* f = f'.$$

Then for all $\alpha > 0$, A^* is a generator of an α-times integrated semigroup.

An operator $T \in \mathcal{B}(X)$ is called uniformly ergodic if the averages $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ converge in the uniform operator topology (see [4, Chapitre II]).
In 1974, M. Lin showed in [5, Theorem] (called uniform ergodic theorem), that when $\lim_{n\to\infty} \left\|\frac{T^n}{n}\right\| = 0$, the operator T is uniformly ergodic if and only if $(I - T)X$ is closed. In [2] the authors proved that, if 1 is a pole of the resolvent function, then $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ converge in norm if and only if $\lim_{n\to\infty} \frac{T^n}{n} = 0$.

A semigroup $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly ergodic if $\frac{1}{t}\int_0^t S(s)ds$ converge uniformly when $t \to \infty$. Also in [6, Theorem], M. Lin shows for a C_0-semigroup $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ satisfying $\lim_{t\to\infty} \left\|\frac{S(t)}{t}\right\| = 0$, then the following conditions are equivalents:

1. $(S(t))_{t\geq 0}$ is uniformly ergodic,
2. the infinitesimal generator A has a closed range,
3. $\frac{1}{n}\sum_{k=0}^{n-1} R(1, A)^k$ converge uniformly,
4. $\lim_{\lambda\to 0^+} \lambda R(\lambda, A)$ exists in the uniform operator topology.

In this paper, we are motivated by application to the ergodic theory for an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$. We prove that when we assume the same conditions of M. Lin's theorem [6] for an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$, the integral $\frac{1}{t}\int_0^t S(s)ds$ converge uniformly if and only if $\alpha = 0$.
Moreover, we obtain that if $\mathcal{R}(A)$ is closed, then for all $\alpha \geq 0$, $\frac{1}{t^{\alpha+1}}\int_0^t S(s)ds$ converge uniformly when $t \to \infty$.

2. Main results

The next lemma was investigated by W. Arendt [1, Proposition 3.3] in the case of $n-$times integrated semigroup, $n \in \mathbb{N}$. This results has been generalized by M. Heiber [3, Proposition 2.4] to the $\alpha-$times integrated semigroup with $\alpha \geq 0$ (for interested reader we refer to [8, Lemma 2.1]).

Lemma 2.1. *[3, Proposition 2.4] Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ where $\alpha \geq 0$. Then for all $x \in D(A)$ and all $t \geq 0$ we have*

1. *$S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*
2. *$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t S(s)Axds$.*
3. *For all $x \in X$, $\int_0^t S(s)xds \in D(A)$ and*

$$A\int_0^t S(s)xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Lemma 2.2. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ satisfying $\lim\limits_{t\to+\infty} \left\|\frac{S(t)}{t}\right\| = 0$. Then for every $\lambda > \max\{w, 0\}$,*

$$\lim_{n\to+\infty} \frac{\|[\lambda R(\lambda, A)]^n\|}{n} = 0.$$

Proof: We put $T(t) := \lambda^\alpha S(t)$, since $\lim_{t\to+\infty} \left\|\frac{S(t)}{t}\right\| = 0$, then we have

$$\lim_{t\to+\infty} \left\|\frac{T(t)}{t}\right\| = 0.$$

Since

$$R(\lambda, A)x = \lambda^\alpha \int_0^\infty e^{-\lambda t}S(t)xdt \text{ for all } \lambda > max\{w, 0\}.$$

Then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt.$$

From [2, Lemma VIII.1.12], we have

$$R(\lambda, A)^n x = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)xdt \text{ for all } \lambda > max\{w, 0\}.$$

Fix $\epsilon > 0$, then there exists $t_0 \geq 0$ such that for all $t \geq t_0$,

$$\|T(t)\| \leq \epsilon t.$$

Since $\|T(t)x\|$ is continuous on $[0, t_0]$, then there exists $K > 0$ such that for all $t \in [0, t_0]$,

$$\|T(t)\| \leq K.$$

Therefore

$$\frac{\|[\lambda R(\lambda, A)]^n\|}{n} \leq \frac{\lambda^n}{n!}K\int_0^{t_0} e^{-\lambda t}t^{n-1}dt + \epsilon\frac{\lambda^n}{n!}\int_{t_0}^\infty e^{-\lambda t}t^n dt.$$

Finally as for all $\lambda > 0$,

$$\frac{\lambda^n}{n!}\int_0^{\infty} e^{-\lambda t} t^n dt = \frac{1}{\lambda},$$

consequently

$$\frac{\|[\lambda R(\lambda, A)]^n\|}{n} \leq \frac{K}{n} + \frac{\epsilon}{\lambda}.$$

Hence we obtain when $n \to \infty$ and $\epsilon \to 0$,

$$\lim_{n\to+\infty} \frac{\|[\lambda R(\lambda, A)]^n\|}{n} = 0.$$

□

Lemma 2.3. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$. Then we have the following assertions:*

1. $\mathcal{R}(A) = \Big(\lambda R(\lambda, A) - I\Big)X$.
2.

$$\begin{aligned} Ker(A) &= \{x \in X \,:\, S(t)x = \frac{t^{\alpha}}{\Gamma(\alpha+1)}x \text{ for all } t \geq 0\} \\ &= \{x \in X : \lambda R(\lambda, A)x = x\}. \end{aligned}$$

Proof: It is known that for all $\lambda > \max\{w, 0\}$ we have

$$(\lambda I - A)R(\lambda, A) = I$$

and for every $x \in D(A)$

$$R(\lambda, A)(\lambda I - A)x = x.$$

1. Let $y \in \Big(\lambda R(\lambda, A) - I\Big)X$, then there exists $x \in X$ such that

$$y = \lambda R(\lambda, A)x - x.$$

Since $x = (\lambda - A)R(\lambda, A)x$, then

$$\lambda R(\lambda, A)x - x = AR(\lambda, A)x.$$

Therefore $y = AR(\lambda, A)x \in \mathcal{R}(A)$, hence $\Big(\lambda R(\lambda, A) - I\Big)X \subset \mathcal{R}(A)$. Conversely, let $y \in \mathcal{R}(A)$, then there exists $x \in D(A)$ such that $y = Ax$, since

$$\begin{aligned} x &= R(\lambda, A)(\lambda I - A)x \\ &= \lambda R(\lambda, A)x - R(\lambda, A)Ax \\ &= \lambda R(\lambda, A)x - R(\lambda, A)y. \end{aligned}$$

Thus

$$\begin{aligned} R(\lambda, A)y &= \lambda R(\lambda, A)x - x \\ &= \Big(\lambda R(\lambda, A) - I\Big)x. \end{aligned}$$

Since $(\lambda I - A)$ and $\Big(\lambda R(\lambda, A) - I\Big)$ commute on $D(A)$, we get

$$\begin{aligned} y &= (\lambda I - A)R(\lambda, A)y \\ &= (\lambda I - A)\Big(\lambda R(\lambda, A) - I\Big)x \\ &= \Big(\lambda R(\lambda, A) - I\Big)(\lambda I - A)x \\ &= \Big(\lambda R(\lambda, A) - I\Big)z, \end{aligned}$$

where $z = (\lambda I - A)x$, hence $\mathcal{R}(A) \subset \Big(\lambda R(\lambda, A) - I\Big)X$.
Then we conclude that $\mathcal{R}(A) = \Big(\lambda R(\lambda, A) - I\Big)X$.

2. Firstly, let $x \in Ker(A)$, then by Lemma 2.1, we obtain

$$\begin{aligned} S(t)x &= \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t S(s)Axds \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)}x. \end{aligned}$$

Hence $x \in \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x$ for all $t \geq 0\}$.
Conversely, let $x \in X$ such that for all $t \geq 0$

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Then by Lemma 2.1, we obtain

$$A\int_0^t S(s)xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = 0.$$

Hence for every $t \geq 0$,

$$A\int_0^t S(s)xds = 0.$$

Thus we conclude that $\int_0^t S(s)xds \in Ker(A)$, hence $x \in Ker(A)$. Therefore

$$Ker(A) = \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x \text{ for all } t \geq 0\}.$$

Let $x \in Ker(A)$. Since $R(\lambda, A)(\lambda - A)x = x$, then

$$\lambda R(\lambda, A)x = x.$$

Conversely, let $x \in X$ such that $\lambda R(\lambda, A)x = x$, then $x \in D(A)$.
Since $(\lambda I - A)R(\lambda, A)x = x$, we deduce that

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x = 0.$$

Hence

$$\begin{aligned} Ax &= A\big(\lambda R(\lambda, A)x\big) \\ &= \lambda AR(\lambda, A)x \\ &= 0. \end{aligned}$$

Therefore we conclude that $x \in Ker(A)$ and finally

$$Ker(A) = \{x \in X : \lambda R(\lambda, A)x = x\}.$$

□

Now, we give a new characterization of $Ker(A)$.

Corollary 2.4. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ such that $\lim\limits_{t\to\infty} \left\| \frac{S(t)}{t} \right\| = 0$.*

If $\alpha \geq 1$, then A is one to one.

Proof: Let $(S(t))_{t\geq 0}$ be an $\alpha-$times integrated semigroup in $\mathcal{B}(X)$ where $\alpha \geq 0$ such that $\lim\limits_{t\to\infty} \left\| \frac{S(t)}{t} \right\| = 0$. Let $x \in ker(A)$, then

$$S(t)x = \frac{t^{\alpha}}{\Gamma(\alpha+1)}x \ ; \text{ for all } t \geq 0.$$

Therefore we obtain

$$\lim_{t\to\infty} \left\| \frac{S(t)}{t} \right\| = \lim_{t\to\infty} \left\| \frac{t^{\alpha-1}}{\Gamma(\alpha+1)}x \right\| = 0.$$

Which means that if $\alpha \geq 1$, then $x = 0$. □

Theorem 2.5. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ such that $\lim\limits_{t\to\infty} \left\| \frac{S(t)}{t} \right\| = 0$. Then the following conditions are equivalents:*

1. *$\mathcal{R}(A)$ is closed,*
2. *$\lambda R(\lambda, A)$ is uniformly ergodic, $\lambda \in \rho(A)$.*

Proof: (1) $\Rightarrow$ (2) Assume that $\mathcal{R}(A)$ is closed, then by Lemma 2.3, we obtain

$$Y = \mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)X.$$

Hence, by Lemma 2.1, we obtain

$$\lim_{n\to+\infty} \frac{\|[\lambda R(\lambda, A)]^n\|}{n} = 0.$$

Therefore, by [5, Theorem], we conclude that $\lambda R(\lambda, A)$ is uniformly ergodic.
(2) $\Rightarrow$ (1) By the uniform ergodic theorem for the operator $\lambda R(\lambda, A)$, we obtain

$$X = \ (I - \lambda R(\lambda, A))X \oplus Ker(I - \lambda R(\lambda, A)).$$

Since $\big(I-\lambda R(\lambda, A)\big)X$ is closed, then by Lemma 2.3, we deduce that $\mathcal{R}(A)$ is closed. □

We show in the next proposition that an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ is uniformly ergodic if and only if $\alpha = 0$.

Proposition 2.6. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ where $\alpha > 0$ such that $\lim_{t\to\infty} \left\|\frac{S(t)}{t}\right\| = 0$. If $\mathcal{R}(A)$ is closed, then $(S(t))_{t\geq 0}$ is not uniformly ergodic.*

Proof: Assume that $\mathcal{R}(A)$ is closed. Then, by Theorem 2.5 $\lambda R(\lambda, A)$ is uniformly ergodic. So

$$X = \big(I - \lambda R(\lambda, A)\big)X \oplus \{x \in X:\ \lambda R(\lambda, A)x = x\}.$$

Hence by Lemma 2.3, we obtain

$$X = \mathcal{R}(A) \oplus Ker(A).$$

Now, assume that $(S(t))_{t\geq 0}$ is uniformly ergodic and $0 < \alpha < 1$, let $x \in Ker(A)$, then by Lemma 2.2

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Therefore we obtain

$$\begin{aligned}\left\|\frac{1}{t}\int_0^t S(s)xds\right\| &= \left\|\frac{1}{t}\int_0^t \frac{s^\alpha}{\Gamma(\alpha+1)}xds\right\| \\ &= \left\|\frac{t^\alpha}{\Gamma(\alpha+2)}x\right\|.\end{aligned}$$

Hence $(S(t))_{t\geq 0}$ is not uniformly ergodic.

Let $\alpha \geq 1$. Then by Corollary 2.4, A is one to one. Hence by the ergodic decomposition and Lemma 2.3, we obtain

$$\begin{aligned} X &= \mathcal{R}(A) \oplus ker(A) \\ &= \mathcal{R}(A). \end{aligned}$$

Hence A is bijective and A^{-1} is defined for all X. Then by the Closed Graph Theorem, we obtain A^{-1} is continuous.

Assume that $(S(t))_{t\geq 0}$ is uniformly ergodic, then there exists an operator P such that $\lim\limits_{t\to\infty} \|t^{-1}\int_0^t S(s)ds - P\| = 0$, $P^2 = P$ and $X = P(X) \oplus Ker(P)$. Thus we conclude that

$$P(X) = Ker(I - \lambda R(\lambda, A)) = Ker(A) = \{0\}.$$

Therefore $X = Ker(P) = \mathcal{R}(A)$ and $\lim\limits_{t\to\infty} \|t^{-1}\int_0^t S(s)ds\| = 0$.
For $x \neq 0$, we applied Lemma 2.1, we get

$$A\int_0^t S(s)xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

Then

$$\frac{1}{t}A\int_0^t S(s)xds = \frac{S(t)}{t}x - \frac{t^{\alpha-1}}{\Gamma(\alpha+1)}x.$$

Since A is invertible, we get

$$\begin{aligned} \lim_{t\to\infty}\big(A\frac{1}{t}\int_0^t S(s)xds\big) &= A\big(\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)xds\big) \\ &= \lim_{t\to\infty}\frac{S(t)}{t}x - \lim_{t\to\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}x \\ &= -\lim_{t\to\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}x. \end{aligned}$$

It follows that

$$\lim_{t\to\infty}\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}x = 0,$$

which is absurd because $\alpha \geq 1$ and $x \neq 0$. Finally, we deduce that $(S(t))_{t\geq 0}$ is not uniformly ergodic. □

Eventually, we give the following theorem.

Theorem 2.7. *Let A be the generator of an $\alpha-$times integrated semigroup $(S(t))_{t\geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ such that $\lim\limits_{t\to\infty}\left\|\frac{S(t)}{t}\right\| = 0$. If $\mathcal{R}(A)$ is closed, then*

$$\frac{1}{t^{\alpha+1}}\int_0^t S(s)ds$$

converge uniformly for all $\alpha \geq 0$.

Proof: Assume that $\mathcal{R}(A)$ is closed and denoted by Y.
From Lemma 2.1, we have for all $x \in D(A)$; $AS(t)x = S(t)Ax$, hence $S(t)Y \subset Y$. We denote by A_1 the generator of the restriction of $S(t)$ to Y, that is the restriction of A to $Y \cap D(A)$. Since $Y = (I - \lambda R(\lambda, A))X$, the uniform ergodic theorem shows that $(I - \lambda R(\lambda, A))$ is invertible on Y.
If $A_1 y = 0$ for $y \in Y \cap D(A)$, then by

$$R(\lambda, A)(\lambda I - A)x = x \text{ for all } x \in D(A)$$

we obtain

$$R(\lambda, A)(\lambda I - A)y = y.$$

Hence

$$\lambda R(\lambda, A)y = y.$$

Then

$$y \in Ker(I - \lambda R(\lambda, A)).$$

That implies $y = 0$, thus A_1 is one to one.
Since $(I - \lambda R(\lambda, A))Y \subset \mathcal{R}(A_1)$, we conclude that

$$Y \supset \mathcal{R}(A_1) \supset (I - \lambda R(\lambda, A))Y = (I - \lambda R(\lambda, A))X = Y = \mathcal{R}(A).$$

Hence $\mathcal{R}(A_1) = Y$, so A_1^{-1} is defined for all Y, and by the Closed Graph Theorem, we obtain A_1^{-1} is continuous.

Let $z \in Y$ there is an $x \in Y \cap D(A)$ such that $A_1 x = z$ and $\|x\| \leq \|A_1^{-1}\|\|z\|$. By Lemma 2.1, we have

$$\int_0^t S(s)A_1 x ds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

If $0 \leq \alpha < 1$, we have the ergodic decomposition below

$$X = Y \oplus \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x \,; t \geq 0\}.$$

Therefore we obtain

$$\begin{aligned}
\left\|\frac{1}{t^{\alpha+1}}\int_0^t S(s)zds\right\| &= \left\|\frac{1}{t^{\alpha+1}}\int_0^t S(s)A_1xds\right\| \\
&= \left\|\frac{1}{t^{\alpha+1}}\Big(S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\Big)\right\| \\
&\leq \left\|\frac{S(t)}{\alpha+1}x\right\| + \left\|\frac{I}{t\Gamma(\alpha+1)}x\right\| \\
&\leq \Big(\left\|\frac{S(t)}{t^{\alpha+1}}\right\| + \left\|\frac{I}{t\Gamma(\alpha+1)}\right\|\Big)\|x\| \\
&\leq \Big(\left\|\frac{S(t)}{t^{\alpha+1}}\right\| + \left\|\frac{I}{t\Gamma(\alpha+1)}\right\|\Big)\|A_1^{-1}\|\|z\|.
\end{aligned}$$

For $t \to \infty$, we obtain the uniform convergence to 0 on Y.

Now, let $z \in \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x \text{ for all } t \geq 0\}$.
Therefore we obtain

$$\begin{aligned} \left\|\frac{1}{t^{\alpha+1}}\int_0^t S(s)zds\right\| &= \left\|\frac{1}{t^{\alpha+1}}\int_0^t \frac{s^\alpha}{\Gamma(\alpha+1)}zds\right\| \\ &= \left\|\frac{1}{t^{\alpha+1}}\Big[\frac{s^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}z\Big]_0^t\right\| \\ &= \left\|\frac{I}{\Gamma(\alpha+2)}z\right\|. \end{aligned}$$

Hence, we get the convergence to $\frac{z}{\Gamma(\alpha+2)}$ on $\{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x;\ t \geq 0\}$.
By the ergodic decomposition above, we conclude that $\frac{1}{t^{\alpha+1}}\int_0^t S(s)ds$ converge uniformly for $0 \leq \alpha < 1$.

If $\alpha \geq 1$, we find that $Ker(A) = \{0\}$.
By the ergodic decomposition and by Lemma 2.3,

$$X = \mathcal{R}(A) \oplus Ker(A) = \mathcal{R}(A).$$

Hence A^{-1} is defined for all X and by the Closed Graph Theorem, we obtain A^{-1} is continuous. Then for $z \in X$ there exists $x \in D(A)$ such that $Ax = z$ and $\|x\| \leq \|A^{-1}\|\ \|z\|$.
Hence

$$\begin{aligned} \left\|\frac{1}{t^{\alpha+1}}\int_0^t S(s)zds\right\| &= \left\|\frac{1}{t^{\alpha+1}}\int_0^t S(s)Axds\right\| \\ &= \left\|\frac{1}{t^{\alpha+1}}\Big(S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\Big)\right\| \\ &\leq \left\|\frac{S(t)}{\alpha+1}x\right\| + \left\|\frac{I}{t\ \Gamma(\alpha+1)}x\right\| \\ &\leq \Big(\left\|\frac{S(t)}{t^{\alpha+1}}\right\| + \left\|\frac{I}{t\ \Gamma(\alpha+1)}\right\|\Big)\|A^{-1}\|\|z\|. \end{aligned}$$

For $t \to \infty$, we obtain the uniform convergence to 0 on X. □

Remark 2.8. *As mentioned above, the uniform ergodicity implies the ergodic decomposition of* $X = Ker(A) \oplus \mathcal{R}(A)$. *But The convergence obtained in the last theorem does not imply this decomposition, which is means that the converse of implication above that is no satisfy in general.*

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