Bol. Soc. Paran. Mat. (3s.) **v. 39** 3 (2021): 115–122.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.41488

On Generalized Drazin-Riesz Operators

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abstract: In this paper, we investigate the clsses of operators as class of generalized Drazin Riesz operators. We give some results for these classes throught localized single valued extension property (SVEP). Some applications are given.

Key Words: Generalized Drazin invertible operator, Riesz operator, Single-valued extension property, Generalized kato Riesz decomposition, Generalized Drazin-Riesz decomposition.

Contents

1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $r(T)$, $\rho(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_p(T)$ and $\sigma(T)$, respectively the adjoint, the null space, the range, the hyper-range, the spectral radius of T, the resolvent set, the approximate point spectrum, the surjectivity spectrum, the point spectrum and the spectrum of T.
A bounded linear operator is called Kato operator if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. A bounded linear operator is nilpotent when $T^n = 0$ for some $n \in \mathbb{N}$, while T is quasi-nilpotent if $\|T^n\|^{1/n} \longrightarrow 0$, that is $T - \lambda$ is invertible for all complex $\lambda \neq 0$.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $dimN(T) < \infty$ and $R(T)$ is closed (resp, $codimR(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm. The index of a semi Fredholm operator T is defined by $ind(T) = dimN(T) - codimR(T)$. T is a Fredholm operator if is lower and upper semi-Fredholm, and T is called a Weyl operator if it is a Fredholm of index zero.
The upper, lower and semi-Fredholm spectra of T are closed and defined by :

$$\sigma_{uF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not an upper semi-Fredholm operator}\},$$

2010 *Mathematics Subject Classification:* 47A10, 47A11.
Submitted January 27, 2018. Published April 16, 2018

$$\sigma_{lF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a lower semi-Fredholm operator}\},$$

$$\sigma_{sF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a semi-Fredholm operator}\}.$$

The essential and Weyl spectra of T are closed and defined by :

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a Fredholm operator}\},$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not a Weyl operator}\}.$$

An operator $T \in \mathcal{B}(X)$ is Riesz, we write $T \in \mathcal{R}(X)$, if $T - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C} \setminus \{0\}$.
If M is a subspace of X such that $T(M) \subset M$, $T \in \mathcal{B}(X)$, it is said that M is T- invariant. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in X$. If M and N are two closed T- invariant subspaces of X such that $X = M \oplus N$, we say that T is completely reduced by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case, we write $T = T_M \oplus T_N$.

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exists we let $a(T) = \infty$. Analogously the descent of T is $d(T) = min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exists we let $d(T) = \infty$ [9]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.
The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_{des}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ has not finite descent}\},$$

$$\sigma_{acc}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ has not finite ascent}\}.$$

Let $T \in \mathcal{B}(X)$, T is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that:

$$ST = TS, \ \ T^{k+1}S = T^k \ \ and \ S^2 T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. It is well known that T is Drazin invertible if it has a finite ascent and descent.
The concept of Drazin invertible operators has been generalized by Koliha [7]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$ ($acc(\sigma(T))$ is the set of all points of accumulation of $\sigma(T)$), which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasi-nilpotent. The generalized Drazin invertible spectrum defined by :

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not generalized Drazin invertible}\}.$$

An operator $T \in \mathcal{B}(X)$ is said to admit a generalized Kato decomposition, abbreviated as GKD, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasi-nilpotent. A relevant case is obtained if we assume that T_N is nilpotent. In this case, T is said to be of Kato type. An operator is said to be essentially Kato if it admits a $GKD(M, N)$ such that N is finite dimensional. If T

is essentially Kato, then T_N is nilpotent since every quasi-nilpotent operator on a finite dimensional space is nilpotent.
For $T \in \mathcal{B}(X)$, the Kato spectrum, the essentially Kato spectrum, the Kato type spectrum and the generalized Kato spectrum are defined by :
$\sigma_K(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I$ is not Kato $\}$,
$\sigma_{eK}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I$ is not essentially Kato $\}$,
$\sigma_{Kt}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I$ is not of Kato type $\}$,
$\sigma_{gK}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I$ does not admits a GKD $\}$, respectively.
Now for a bounded linear operator T and for each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm (resp. semi-Fredholm) operator, then T is called a B-Fredholm operator (resp. a semi-B-Fredholm) operator. In this case, T_m is a Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geq n$. This enables us to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and T_n is a Fredholm operator. A bounded linear operator $T \in \mathcal{B}(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0, see [4].
Next, let $T \in \mathcal{B}(X)$, T has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. In particular, T and T^* have the SVEP at every isolated point of the spectrum. We denote by $S(T)$ the open set of $\lambda \in \mathbb{C}$ where T fails to have SVEP at λ, and we say that T has SVEP if $S(T) = \emptyset$. It is easy to see that $S(T) \subset \sigma_p(T)$, see [1,8].

In accordance with some decomposition as Kato decompsition ($T = T_1 \oplus T_2$ such that T_1 is Kato and T_2 is nilpotent), generalized Kato decomposition ($T = T_1 \oplus T_2$ such that T_1 is Kato and T_2 is quasinilpotent),...it is natural to study various types of the direct sums. Namely, let $R_1(X)$, $R_2(X)$, $R_3(X)$, $R_4(X)$, $R_5(X)$, $R_6(X)$, $R_7(X)$, $R_8(X)$, $R_9(X)$, $R_{10}(X)$,$R_{11}(X)$ and $R_{12}(X)$ denoted respectively the class of upper semi Fredholm, lower semi Fredholm, Fredholm, upper semi Weyl, lower semi Weyl, Weyl, upper semi Browder, lower semi Browder, Browder, bounded below, surjective and invertible operators.
The purpose of this work is to investigate the new classes of operators introduced in [5], [13] as class of generalized Drazin Riesz operators. We give some results for these classes throught localized valued extension property (SVEP). At last some applications are given.

2. Main results

Definition 2.1. *For $1 \leq i \leq 12$, define the classes of bounded linear operators:*

$$DR_i(X) = \left\{ \begin{array}{c} T \in \mathcal{B}(X): \text{ there exists } (M,N) \in Red(T) \\ \text{such that } T_M \in R_i(X) \text{ and } T_N \text{ is nilpotent} \end{array} \right\},$$

$$gDR_i(X) = \left\{ \begin{array}{c} T \in \mathcal{B}(X): \text{ there exists } (M,N) \in Red(T) \\ \text{such that } T_M \in R_i(X) \text{ and } T_N \text{ is quasinilpotent} \end{array} \right\},$$

and

$$gDRR_i(X) = \left\{ \begin{array}{c} T \in \mathcal{B}(X): \text{ there exists } (M,N) \in Red(T) \\ \text{such that } T_M \in R_i(X) \text{ and } T_N \text{ is Riesz} \end{array} \right\}.$$

Remark 2.2. *1. $DR_i(X) \subset gDR_i(X)$ for $1 \leq i \leq 12$,*

2. *The classes $gDR_i(X)$ generalize the class of generalized Drazin invertible,*
3. *The classes $gDRR_{12}(X)$ is called the class of generalized Drazin-Riesz invertible,*
4. *The classes $gDRR_6(X)$ is called the class of generalized Drazin-Riesz Weyl, see [13].*

Set

$$\sigma_{gDRW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin gDRR_6(X)\}$$
$$\sigma_{gDR}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin gDRR_{12}(X)\}$$

we have the following

Theorem 2.3. *Let $T \in \mathcal{B}(X)$, then*

$$\sigma_{gDR}(T) = \sigma_{gDRW}(T) \cup (S(T) \cap S(T^*)).$$

Proof: We prove that $\sigma_{gDRW}(T) \cup (S(T) \cap S(T^*) \subset \sigma_{gDR}(T)$. Indeed, we have (S(T)$\cap$ $S(T^*) \subset \sigma_{gDR}(T)$ by [13, Theorem 2.3]. Since $\sigma_{gDRW}(T) \subset \sigma_{gDR}(T)$, therefore $\sigma_{gDRW}(T) \cup (S(T) \cap S(T^*) \subset \sigma_{gDR}(T)$.

Also we prove that $\sigma_{gDR}(T) \subset \sigma_{gDRW}(T) \cup (S(T) \cap S(T^*)$. It suffices to show that $\sigma_{gDR}(T) \setminus \sigma_{gDRW}(T) \subset (S(T) \cap S(T^*)$. Let $\lambda \in \sigma_{gDR}(T) \setminus \sigma_{gDRW}(T)$ and without loss of generality we assume that $\lambda = 0$. Suppose that $0 \notin (S(T) \cap S(T^*))$.

Case 1: $0 \notin$ S(T).

Since $0 \notin \sigma_{gDRW}(T)$ then $T = T_1 \oplus T_2$ with T_1 is a Weyl operator and T_2 is a Riesz operator. Since T has SVEP at 0 then T_1 and T_2 have SVEP at 0 . On the other hand , T_1 is Weyl operator then it's B-Weyl , and T_1 has SVEP at 0 . So according to [2, Theorem 2.3] we have T_1 is a Drazin invertible operator . This implies that there exists $(E, F) \in Red(T_1)$ such that $T_1 = T_E \oplus T_F$ where T_E is an invertible operator and T_F is a nilpotent operator. Therefore $T = T_E \oplus T_F \oplus T_2$ where T_E is an invertible operator and according to [13, lemma 2.2], it follows that $T_F \oplus T_2$ is a Riesz operator . Therefore T is a generalized Drazin-Riesz invertible operator . This is a contradiction.

Case 2: $0 \notin S(T^*)$. The proof follows similarly . □

Corollary 2.4. *Let $T \in \mathcal{B}(X)$. If T or T^* has SVEP then $\sigma_{gDR}(T) = \sigma_{gDRW}(T)$.*

Theorem 2.5. *Assume that H is separable, infinite dimensional, complex Hilbert space. Then for every $T \in \mathcal{B}(H)$ The following conditions are equivalent :*

1. *T is in the norm closure of $gDR_i(H)$,*
2. *T is in the norm closure of $DR_i(H)$.*

Proof: (2) $\Longrightarrow$ (1) because $DR_i(H) \subset gDR_i(H)$. Now we show that (1) $\Longrightarrow$ (2): Let $T \in gDR_i(H)$, so there exists $(M, N) \in Red(T)$ such that $T = T_M \oplus T_N$ with $T_1 = T_M \in R_i(X)$ and $T_2 = T_N$ is a quasinilpotent operator. According to [3], there exists $(T_{2,n})_n$ sequence of nilpotent operators which converges in norm to T_2 . Hence $T_1 \oplus T_{2,n}$ is a sequence of operators in $DR_i(H)$ which converges in norm to T= $T_1 \oplus T_2$. So T is in the norm closure of $DR_i(H)$. Therefore, if T is in the norm closure of $gDR_i(H)$, then T is in the norm closure of $DR_i(H)$. □

Corollary 2.6. *Assume that H is a separable , infinite dimensional, complex Hilbert space . Then $\overline{gDR_i(H)}^{\|\|\|}$=$\overline{DR_i(H)}^{\|\|\|}$.*

Theorem 2.7. *Let $T \in \mathcal{B}(X)$. If $T \in gDRR_i(X)$ then there exists $S \in \mathcal{B}(X)$ such that $T + S \in R_i(X)$, TS is Riesz and $[T, S] = 0$.*

Proof: Let $T \in \mathcal{B}(X)$. If $T \in gDRR_i(X)$, then there exists $(M, N) \in Red(T)$ such that T_1= $T_M \in R_i(X)$ and T_2 =T_N is a Riesz operator. Let $S \in \mathcal{B}(X)$ such that $S = 0 \oplus (I_N - T_2)$. Since $T_2 \in R_i(X)$, then $T + S = T_1 \oplus I_N = \begin{pmatrix} T_1 & 0 \\ 0 & I_N \end{pmatrix}$ $\in R_i(X)$.

We have:

$$\begin{aligned} TS &= [T_1 \oplus T_2][0 \oplus (I_N - T_2)] = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & I_N - T_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & T_2(I_N - T_2) \end{pmatrix}. \end{aligned}$$

Also

$$\begin{aligned} ST &= [\, 0 \ \oplus \ (I_N - T_2)][T_1 \oplus T_2] = \begin{pmatrix} 0 & 0 \\ 0 & I_N - T_2 \end{pmatrix} \cdot \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (I_N - T_2)T_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T_2(I_N - T_2) \end{pmatrix}. \end{aligned}$$

Thus $TS = ST$. Since $I_N - T_2$ commutes with T_2, then $I_N - T_2$ is a Riesz operator. Therefore TS is a Riesz operator and $[T, S] = 0$. □

Similarly, we have:

Theorem 2.8. *Let $T \in \mathcal{B}(X)$. If $T \in gDR_i(X)$ then there exists $S \in \mathcal{B}(X)$ such that $T + S \in R_i(X)$, TS is quasinilpotent and $[T, S] = 0$.*

Theorem 2.9. *Let $\mathcal{F}(X)$ denote the ideal of finite rank operators in $\mathcal{B}(X)$, and $T \in \mathcal{B}(X)$. Then $\bigcap\limits_{F\in\mathcal{F}(X)} \sigma_{gDR}(T + F) \subset \sigma_{gDRW}(T)$.*

Proof: Let $\lambda \notin \sigma_{gDRW}(T)$, without loss of generality we can assume that $\lambda = 0$.

$0 \notin \sigma_{gDRW}(T)$, then there exists $(M, N) \in Red(T)$ such that $T = T_1 \oplus T_2$ with $T_1 = T_M$ is a Weyl operator and $T_2 = T_N$ is a Riesz operator . According to [6, Theorem 6.5.2] there exists $F_1 \in \mathcal{F}(X)$ such that $T_1 + F_1$ is a invertible operator. Let $F = F_1 \oplus 0$. Then

$$T + F = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} + \begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 + F_1 & 0 \\ 0 & T_2 \end{pmatrix} = (T_1 + F_1) \oplus T_2.$$

Therefore $T + F$ is in $gDR(X)$. This implies that $0 \in \rho_{gDR}(T + F)$, so $0 \notin \bigcap\limits_{F\in\mathcal{F}(X)} \sigma(T + F)$. □

3. Applications

A bounded linear operator T is called supercyclic provided there is some $x \in X$ such that the set $\{\lambda T^n x,\ \lambda \in \mathbb{C}\ , n = 0, 1, 2, ..\}$ is dense in X. It is well Known that if T is supercyclic then $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\alpha\}$ for some nonzero $\alpha \in \mathbb{C}$. Since an operator with countable point spectrum has SVEP, then we have the following:

Proposition 3.1. *Let $T \in \mathcal{B}(X)$, a supercyclic operator. Then :*

$$\sigma_{gDR}(T) = \sigma_{gDRW}(T).$$

Since, every hyponormal operator T on a Hilbert space has the single valued extension property, we have:

Proposition 3.2. *Let T a hyponormal operator on a Hilbert space then:*

$$\sigma_{gDR}(T) = \sigma_{gDRW}(T).$$

In particular, If T is auto-adjoint.

Let A be a semi-simple commutative Banach algebra.
The mapping $T : A \longrightarrow A$ is said to be a multiplier of A if $T(x)y = xT(y)$ for all $x, y \in A$.
It is well known each multiplier on A is a continuous linear operator and that the set of all multiplier on A is a unital closed commutative subalgebra of $B(A)$ [8, Proposition 4.1.1]. Also the semi-simplicity of A implies that every multiplier has the SVEP (see [8, Proposition 2.2.1]).

Now if assume that A is regular and Tauberian (see [8, Definition 4.9.7]), then T and T^* have the SVEP. Hence, we have the following Proposition.

Proposition 3.3. *Let T be a multiplier on semi-simple regular and Tauberian commutative Banach algebra A, then the following assertions are equivalent*

1. *T is generalized Drazin-Riesz Weyl ,*
2. *T is generalized Drazin-Riesz invertible.*

Let G a locally compact abelian group, with group operation $+$ and Haar measure μ, let $L^1(G)$ consist of all $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have the following:

Corollary 3.4. *Let G be a locally compact abelian group, $\mu \in M(G)$. Then every convolution operator $T_\mu : L^1(G) \longrightarrow L^1(G)$, $T_\mu(k) = \mu \star k \in gDRW$ if and only if is generalized Drazin- Riesz invertible.*

Acknowledgement: The authors thank the referees for their suggestions, remarks and comments thorough reading of the manuscript.

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