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Existence of a Renormalized Solution of Nonlinear Parabolic Equations with Lower Order Term and General Measure Data

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ABSTRACT: We give an existence result of a renormalized solution for a class of nonlinear parabolic equations $\dfrac{\partial b(u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) + H(x,t,\nabla u) = \mu$, where the right side is a general measure, b is a strictly increasing C^1-function, $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u and $H(x,t,\nabla u)$ is a nonlinear lower order term which satisfy the growth condition with respect to ∇u.

Key Words: Nonlinear parabolic equations, Existence, Renormalized solutions, Lower order term, Measure data.

Contents

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 1)$, $T > 0$ and let $Q_T := \Omega \times (0,T)$. We prove the existence of a renormalized solution for a class of nonlinear parabolic equations of the type:

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) + H(x,t,\nabla u) = \mu \ \text{ in } Q_T, \tag{1.1}$$

$$b(u)(t=0) = b(u_0) \ \text{ in } \Omega, \tag{1.2}$$

$$u = 0 \ \text{ on } \partial\Omega \times (0,T). \tag{1.3}$$

In Problem (1.1)-(1.3) the framework is the following: the data μ is a general measure, b is a strictly increasing C^1-function, the operator $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions operator which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u and $H(x,t,\nabla u)$ is a nonlinear lower order. In the case where $b(u) = u$, $H = 0$, and

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the right hand side is a bounded measure, the existence of a distributional solution was proved in [3], but due the lack of regularity of solution, the distributional formulation is not strong enough to provide uniqueness (see [27] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by R.J. DiPerna and P.-L.Lions in [8] for the study of Boltzmann equation was adapted to parabolic equations and elliptic equations with L^1 data. When μ is measure data that does not charge the sets of zero p-capacity (the so called diffuse measure, see the definition in the section 2 below) a notion of renormalized solution for Problem (1.1)-(1.3) was introduced in [11] for $b(u) = u$ and $H = 0$. Similar result was proved in [5] when $H = 0$. In [22] the existence of renormalized solution was proved in the case where $b(u) = u$ and $H = 0$ and μ is a general measure (see also [6]). In [10] a similar notion of entropy solution is also defined and proved to be equivalent to the renormalized one. In this paper we use a new definition of renormalized solution which is adopted in [25] and [24] for the study of parabolic p-Laplacian equations with general measure data.
The paper is organized as follows. In section 2 we give some preliminaries on the concept of p-capacity. Section 3 will be devoted to set our main assumptions and definition of renormalized solution and the statement of the existence result, while in Section 4 we give the proof of our main result.

2. Preliminaries on parabolic capacity

We introduce the notion of p-capacity associated to our problem (for further details see [21], [11]). Let $Q_T = \Omega \times (0, T)$ for any fixed $T > 0$ and $1 < p < \infty$, and let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p}(\Omega)} + \|.\|_{L^2(\Omega)}$ and

$$W = \Big\{ u \in L^p(0, T; V), u_t \in L^{p'}(0, T; V') \Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;V)} + \|.\|_{L^{p'}(0,T;V')}$, remark that W is continuously embedded in $C([0, T], L^2(\Omega))$, and if $1 < p < \infty$, then $\mathcal{C}_c^\infty(Q_T)$ is dense in W. Let $U \subseteq Q_T$ be an open set, we define the parabolic p-capacity of U as

$$cap_p(U) = \inf \Big\{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q_T \Big\},$$

where as usual we set $inf\{\emptyset\} = +\infty$, then for any Borel set $B \subseteq Q_T$ we define

$$cap_p(B) = \inf \Big\{ cap_p(U) : U \text{ open set of } Q_T, B \subseteq U \Big\}.$$

We will denote by $\mathcal{M}(Q_T)$ the set of all Radon measures with bounded variation on Q_T, while, as we already mentioned, $\mathcal{M}_0(Q_T)$ the set of all measures with bounded total variation over Q_T that do not charge the sets of zero p-capacity, that is if $\mu \in \mathcal{M}_0(Q_T)$, then $\mu(E) = 0$, for all $E \subseteq Q_T$ such that $cap_p(E) = 0$.
In [11] the authors proved the following decomposition theorem:

Theorem 2.1. *Let μ be a bounded measure on Q_T. If $\mu \in \mathcal{M}_0(Q_T)$ then there exists (f, g_1, g_2) such that $f \in L^1(Q_T)$, $g_1 \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $g_2 \in L^p(0, T; V)$ and*

$$\int_{Q_T} \phi \, d\mu = \int_{Q_T} f\phi \, dx \, dt + \int_0^T \langle g_1, \phi \rangle \, dt - \int_0^T \langle \phi_t, g_2 \rangle \, dt \quad \phi \in \mathcal{C}_c^\infty(Q_T).$$

Such a triplet (f, g_1, g_2) will called decomposition of μ.

Definition 2.2. *A sequence of measures (μ_n) in Q_T is equidiffuse if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every Borel set $E \subseteq Q_T$,*

$$cap_p(E) < \eta \Rightarrow \ |\mu_n|(E) < \varepsilon \ \forall n \geq 1.$$

Let ρ_n be a sequence of mollifiers on Q_T, the following result is proved in [25]

Proposition 2.3. *If $\mu \in \mathcal{M}_0(Q_T)$, then the sequence $\rho_n * \mu$ is equidiffuse.*

If $\mu \in \mathcal{M}(Q_T)$, thanks to a well known decomposition result (see for instance [14]), we can split it into a sum (uniquely determined) of its absolutely continuous part μ_d with respect to p-capacity and its singular part μ_s, that is μ_s is concentrated on a set E of zero p-capacity. Hence, if $\mu \in \mathcal{M}(Q_T)$, we have

$$\mu = \mu_d + \mu_s = \mu_d + \mu_s^+ - \mu_s^-.$$

3. Assumptions and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q_T = \Omega \times (0, T)$

$$b : \mathbb{R} \to \mathbb{R} \text{ and} \tag{3.1}$$

is a strictly increasing C^1-function with $b(0) = 0$, and there exists $\gamma > 0$ and $\Lambda > 0$ such that

$$\gamma \leq b'(s) \leq \Lambda, \ \forall s \in \mathbb{R}. \tag{3.2}$$

$$a : Q_T \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function} \tag{3.3}$$

$$a(x, t, \xi).\xi \geq \alpha|\xi|^p, \tag{3.4}$$

for almost every $(x, t) \in Q_T$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number.

$$|a(x, t, \xi)| \leq \beta(L(x, t) + |\xi|^{p-1}), \tag{3.5}$$

for almost every $(x, t) \in Q_T$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ is a given real number, L is a non negative function in $L^{p'}(Q_T)$.

$$[a(x, t, \xi) - a(x, t, \xi')][\xi - \xi'] > 0. \tag{3.6}$$

Let $H : Q_T \times \mathbb{R}^N \to \mathbb{R}$ be Carathéodory function such that for a.e. $(x,t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,\xi)| \le g(x,t)|\xi|^{\delta}, \tag{3.7}$$

is satisfied, with $\delta = \frac{p(N+1)-N}{N+2}$ and g belongs to $L^{N+2,1}(Q_T)$.

$$\mu \in \mathcal{M}(Q_T), \tag{3.8}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{3.9}$$

We use in the present paper the two Lorentz spaces $L^{q,1}(Q_T)$ and $L^{q,\infty}(Q_T)$, see for example ([18], [19]) for references about Lorentz spaces $L^{q,s}$. if f^* denotes the decreasing rearrangement of a measurable function f,

$$f^*(r) = \inf\Big\{s \ge 0 : meas\{(x,t) \in Q_T : |f(x,t)| > s\} < r\Big\}, \quad r \in [0, meas(Q_T)],$$

$L^{q,1}(Q_T)$ is the space of Lebesgue measurable functions such that

$$\|f\|^q_{L^{q,1}(Q_T)} = \int_0^{meas(Q_T)} f^* r^{\frac{1}{q}} \frac{dr}{r} < \infty,$$

while $L^{q,\infty}(Q)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,\infty}(Q_T)} = \sup_{r>0} r[meas\{(x,t) \in Q_T : |f(x,t)| > r\}]^{\frac{1}{q}} < \infty.$$

If $1 < q < \infty$ we have the generalized Hölder inequality

$$\forall f \in L^{q,\infty}(Q_T), \forall g \in L^{q',1}(Q_T) \text{ such that } \frac{1}{q} + \frac{1}{q'} = 1,$$

$$\int_Q |fg| \le \|f\|_{L^{q,\infty}(Q_T)} \|g\|_{L^{q',1}(Q_T)}. \tag{3.10}$$

Now we give the definition of renormalized solution of Problem (1.1) – (1.3).

Definition 3.1. *A measurable function u is a renormalized solution of Problem (1.1)-(1.3) if*

$$T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega)), \text{ for every } k > 0, \quad H(x,t,\nabla u) \in L^1(Q_T), \tag{3.11}$$

and if there exists a sequence of measures $\Gamma^k \in \mathcal{M}(Q_T)$ such that:

$$\Gamma^k \to \mu_s \text{ tightly as } k \to \infty, \tag{3.12}$$

$$(3.13)$$

$$B_k(u)_t - \operatorname{div}\Big(a(x,t,\nabla T_k(u))\Big) + H(x,t,\nabla T_k(u)) = \mu_d + \Gamma^k \text{ in } \mathcal{D}'(Q_T),$$

where $B_k(s) = \int_0^s T_k'(r) b'(r)\, dr, \quad \forall s \in \mathbb{R}.$

Remark 3.2. *Observe that (3.2) and (3.11) imply that each term in (3.13) is well defined and that (3.13) implies that* $B_k(u)_t - \operatorname{div}\Big(a(x,t,\nabla T_k(u))\Big) + H(x,t,\nabla T_k(u))$ *is a bounded measure, then we have*

$$B_k(u)_t - \operatorname{div}\Big(a(x,t,\nabla T_k(u))\Big) + H(x,t,\nabla T_k(u)) = \mu_d + \Gamma^k \ \text{in } \mathcal{M}(Q_T).$$

A remark on the assumption (3.2) is also necessary. As one could check later, since the data is a measure μ, *we are forced to assume* $\gamma \le b'(s) \le \Lambda$. *We conjecture that this assumption is only technical and could be removed in order to deal with more general elliptic-parabolic problems (see for instance [1], [7]).*

In order to prove the existence result we give the following Lemma

Lemma 3.3. *Let* u *be a measurable function satisfying* $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ *for every* $k > 0$ *such that:*

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u)|^2\,dx + \int_{Q_T} |\nabla T_k(u)|^p\,dxdt \le Mk \quad \forall k > 0,$$

where M *is a positive constant. Then*

$$\||u|^{p-1}\|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)} \le CM^{(\frac{p}{N}+1)\frac{N}{N+p'}} |Q_T|^{\frac{1}{p'}\frac{N}{N+p'}},$$

$$\||\nabla u|^{p-1}\|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)} \le CM^{\frac{(N+2)(p-1)}{p(N+1)-N}},$$

where C *is a constant which depends only on* N *and* p.

Proof. See [13] and [12] . □

4. Existence result

Let us introduce the following regularization of the data: for $n \ge 1$ fixed

$$u_0^n \in C_c^\infty(Q_T), \ \text{such that } u_0^n \to u_0 \text{ in } L^1(\Omega), \tag{4.1}$$

$$\mu^n \in C^\infty(Q_T), \ \mu^n = \mu_d^n + \mu_s^n, \tag{4.2}$$

where $\mu_d^n = \rho_n * \mu_d$ and $\mu_s^n = \rho_n * \mu_s^+ - \rho_n * \mu_s^- = \lambda_+^n - \lambda_-^n$.
Moreover we have

$$\|\mu^n\|_{L^1(Q_T)} \le |\mu|_{\mathcal{M}(Q_T)},$$

and

$$\mu^n \text{ converges to } \mu \text{ in the narrow topology of measures.}$$

Let us now consider the following regularized problem

$$u^n \in L^p(0,T;W_0^{1,p}(\Omega)), \tag{4.3}$$

$$\int_0^T \langle \frac{\partial v^n}{\partial t}, \varphi \rangle \, dt + \int_{Q_T} a(x,t,\nabla u^n)\nabla\varphi \, dxdt + \int_{Q_T} H(x,t,\nabla u^n)\varphi \, dxdt = \int_{Q_T} \mu^n \varphi \, dxdt$$
$$\forall \varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T),$$

$$b(u^n)(t=0) \;=\; b(u_0^n) \text{ in } \Omega, \tag{4.4}$$

where $v^n = b(u^n)$.
As a consequence, proving existence of a weak solution $u^n \in L^p(0,T;W_0^{1,p}(\Omega))$ of (4.3)-(4.4) is classical (see for instance [15]).
Now we give the following proposition which gives some compactness results.

Proposition 4.1. *Let u^n and v^n be defined as before. Then*

$$\| |\nabla u^n|^\delta \|_{L^{\frac{N+2}{N+1},\infty}(Q_T)} \le C, \tag{4.5}$$

$$\|u^n\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{4.6}$$

$$\int_Q |\nabla T_k(u^n)|^p \, dxdt \le Ck, \tag{4.7}$$

$$\int_Q |\nabla T_k(v^n)|^p \, dxdt \le Ck, \tag{4.8}$$

$$u^n \text{ is bounded in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \forall\ 1 < q < p - \frac{N}{N+1}, \tag{4.9}$$

Moreover, there exists a measurable function u and $v = b(u)$ such that $T_k(u)$ and $T_k(v)$ belong to $L^p(0,T;W_0^{1,p}(\Omega))$, and u belongs to $L^\infty(0,T;L^1(\Omega))$, up to a subsequence, for any $k > 0$ and for any $1 < q < p - \frac{N}{N+1}$ we have

$$\tag{4.10}$$

$$u^n \to u \text{ a.e. on } Q_T \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q_T),$$

$$T_k(u^n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q_T, \tag{4.11}$$

$$T_k(v^n) \rightharpoonup T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. in } Q_T. \tag{4.12}$$

Proof. The proof of this Proposition relies on standard techniques for problems of type (4.3)-(4.4). Let $k > 0$, we take $T_k(u^n)\chi_{(0,t)}$ as test function in (4.3) for every $t \in (0,T)$ and we have

$$\tag{4.13}$$

$$\int_\Omega \overline{B}_k(u^n)(t) \, dx + \int_{Q_t} a(x,t,\nabla u^n)\nabla T_k(u^n) \, dxdt$$

$$\le \int_{Q_t} |H(x,t,\nabla u^n)||T_k(u^n)| \, dxdt + \int_{Q_t} \mu^n T_k(u^n) \, dxdt + \int_\Omega \overline{B}_k(u_0^n) \, dx,$$

where $\overline{B}_k(s) = \displaystyle\int_0^s T_k(r)b'(r)\,dr$.
Using (3.4) and (3.7) we obtain

$$\int_\Omega \overline{B}_k(u^n)(t)\,dx + \alpha\int_{Q_t} |\nabla T_k(u^n)|^p\,dxdt$$

$$\leq k\Big(\int_{Q_t} |g(x,t)||\nabla u^n|^\delta\,dxdt + \|\mu^n\|_{L^1(Q_t)} + \|b(u_0^n)\|_{L^1(\Omega)}\Big),$$

if we take the supremum for $t \in (0, t_1)$, where $t_1 \in (0,T)$ will be choosen later, by (3.2) we have

$$\frac{\gamma}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u^n)|^2\,dx + \alpha\int_{Q_{t_1}} |\nabla T_k(u^n)|^p\,dxdt$$

$$\leq k\Big(\int_{Q_{t_1}} |g(x,t)||\nabla u^n|^\delta\,dxdt + \|\mu^n\|_{L^1(Q_T)} + \|b(u_0^n)\|_{L^1(\Omega)}\Big),$$

and thanks to the generalized Hölder inequality we obtain

$$\frac{\gamma}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u^n)|^2\,dx + \alpha\int_{Q_{t_1}} |\nabla T_k(u^n)|^p\,dxdt \tag{4.14}$$

$$\leq k\Big(\||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|g\|_{L^{N+2,1}(Q_{t_1})} + \|\mu^n\|_{L^1(Q_T)} + \|b(u_0^n)\|_{L^1(\Omega)}\Big) \leq Mk,$$

where $M = \||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|g\|_{L^{N+2,1}(Q_{t_1})} + \|\mu^n\|_{L^1(Q_T)} + \|b(u_0^n)\|_{L^1(\Omega)}$,
by Lemma 3.3 we obtain

$$\||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})} = \||\nabla u^n|^{p-1}\|^{\frac{\delta}{p-1}}_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_{t_1})} \tag{4.15}$$

$$\leq C(\||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})}\|g\|_{L^{N+2,1}(Q_{t_1})} + \|\mu^n\|_{L^1(Q_T)} + \|b(u_0^n)\|_{L^1(\Omega)}).$$

If we choose t_1 such that

$$1 - C\|g\|_{L^{N+2,1}(Q_{t_1})} > 0, \tag{4.16}$$

holds, then we have

$$\||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_{t_1})} \leq C, \tag{4.17}$$

which yields (4.5).
Since $\overline{B}_k(s) \geq \gamma\displaystyle\int_0^s T_1(r)\,dr \geq \gamma(|s|-1)\ \forall s \in \mathbb{R}$, we obtain

$$\|u^n\|_{L^\infty(0,t_1;L^1(\Omega))} \leq \frac{1}{\gamma}M + \ meas(\Omega).$$

From (4.17) it follows that

$$\|u^n\|_{L^\infty(0,t_1;L^1(\Omega))} \leq C. \tag{4.18}$$

Now we use the same technique as in ([23]). We consider a partition of the interval $[0,T]$ into a finite number of intervals $[0,t_1]$, $[t_1,t_2]$,...,$[t_{n-1},T]$ such that for each $[t_{i-1},t_i]$ the condition (4.16) holds.
In this way in each cylindre $\Omega\times[t_{i-1},t_i]$ we obtain a priori estimates of type (4.5) and (4.6). From (4.14) and (4.17) with T in place of t_1 we obtain (4.7).
By using (4.6) and (4.7), and thanks to L. Boccardo and T. Gallouët (see [3]) we obtain (4.9). By (3.2), (4.9), and since μ^n is bounded in $L^1(Q_T)$, one obtain that $\frac{\partial v^n}{\partial t}$ is bounded in $L^1(0,T;W^{-1,q'}(\Omega))$ for every $q' < 1 + \frac{1}{(p-1)(N+1)}$, using a standard compactness arguments (see [26]) yield (4.10), (4.11) and (4.12). □

Let us introduce for $k \geq 0$ fixed, the time regularization of the function $T_k(v)$. This kind of regularization has been first introduced by R. Landes. More recently, it has been exploited to solve a few nonlinear evolution problems with L^1 or measure data. This specific time regularization of $T_k(v)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\nu)_\nu$ in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\nu\|_{L^\infty(\Omega)} \leq k$, for all $\nu > 0$, and $v_0^\nu \to T_k(b(u_0))$ a.e. in Ω with $\frac{1}{\nu}\|v_0^\nu\|_{L^p(\Omega)} \to 0$ as $\nu \to +\infty$.
For fixed $k \geq 0$ and $\nu > 0$, let us consider the unique solution $T_k(v)_\nu \in L^\infty(Q_T)\cap L^p(0,T,W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_k(v)_\nu}{\partial t} + \nu(T_k(v)_\nu - T_k(v)) = 0 \text{ in } \mathcal{D}'(Q_T),$$

$$T_k(v)_\nu(t=0) = v_0^\nu \text{ in } \Omega.$$

The behavior of $T_k(v)_\nu$ as $\nu \to +\infty$ is investigated in [17] and we just recall here that:

$$T_k(v)_\nu \to T_k(v) \text{ strongly in } L^p(0,T,W_0^{1,p}(\Omega)) \text{ a.e. in } Q_T \; as \; \nu \to +\infty$$

with $\|T_k(v)_\nu\|_{L^\infty(\Omega)} \leq k$ for any $\nu > 0$, and $\frac{\partial T_k(v)_\nu}{\partial t} \in L^p(0,T,W_0^{1,p}(\Omega))$.
We will denote $\omega(n,\nu,k,\varepsilon)$ any quantity that vanishes as the parameters go to their limit point with in the same order in which they appear, that is, for example

$$\overline{\lim_{\varepsilon\to 0}}\;\overline{\lim_{k\to\infty}}\;\overline{\lim_{\nu\to\infty}}\;\overline{\lim_{n\to\infty}} \quad |\omega(n,\nu,k,\varepsilon)| = 0.$$

We give the following result which has been proved in [2].

Lemma 4.2. *Let v^n be a sequence in $L^p(0,T;W_0^{1,p}(\Omega))\cap C^0([0,T];L^2(\Omega))$, and $(v^n)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, suppose that v^n converges almost everywhere in Q_T to a function v such that $T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. then we have*

$$\int_0^T \langle \frac{\partial v^n}{\partial t}, T_\varepsilon(v^n - T_k(v)_\nu))\rangle \, dt \geq \omega(n,\nu,k,\varepsilon).$$

Proposition 4.3. *The sequence* (∇u^n) *converges to* ∇u *a.e. in* Q_T.

Proof. Adopting the method used in [2], we prove that for some $\theta > 0$, one has up to subsequences still denoted by u^n (for simplicity of notation, we will omit the dependence of a on x and t),

$$\Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta \to 0 \text{ a.e. in } Q. \tag{4.19}$$

Note that (4.19) will be true if we show that

$$\int_{Q_T} \Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta\, dxdt = \omega(n) \tag{4.20}$$

The same argument in [16] and under asumptions on $a(x,t,\xi)$ implies that ∇u^n converges to ∇u a.e. in Q_T.
Thanks to Proposition 4.1, the following estimate holds

$$meas(\{|v| \geq k\}) = \omega(k),$$

We can write

$$\begin{aligned} &\int_{Q_T} \Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta\, dxdt \\ &= \int_{\{|v|\geq k\}} \Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta\, dxdt \\ &+ \int_{\{|v|< k\}} \Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta\, dxdt \\ &= I_{n,k} + J_{n,k}. \end{aligned}$$

Since u^n is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for $q < p - \frac{N}{N+1}$, we can choose $\theta < \frac{q}{p} < 1$, so that using Hölder inequality, we obtain

$$|I_{n,k}| \leq c\ meas(\{|v| \geq k\})^{1-\theta p/q},$$

and then $I_{n,k} = \omega(k)$. Now we set

$$\Psi_{n,k} = \Big(a(\nabla u^n) - a(\nabla u \chi_{\{|v|<k\}})\Big)\Big(\nabla u^n - \nabla u \chi_{\{|v|<k\}}\Big),$$

and we have

$$\int_{Q_T} \Big[(a(\nabla u^n) - a(\nabla u)).(\nabla u^n - \nabla u)\Big]^\theta\, dxdt \tag{4.21}$$

$$\leq \int_{Q_T} \Psi^\theta_{n,k} \chi_{\{|v^n - T_k(v)_\nu| \leq \varepsilon\}} + \int_{Q_T} \Psi^\theta_{n,k} \chi_{\{|v^n - T_k(v)_\nu| > \varepsilon\}} + \omega(k),$$

since $\Psi^\theta_{n,k}$ is bounded in $L^{q/\theta p}(Q_T)$ independently of n and k, $\chi_{\{|v^n - T_k(v)_\nu| > \varepsilon\}}$ converges to $\chi_{\{|v - T_k(v)| > \varepsilon\}}$ almost everywhere in Q_T as n tends to $+\infty$ (see [2], Lemma 3.2) and $\chi_{\{|v-T_k(v)_\nu|>\varepsilon\}}$ converges to zero almost everywhere in Q_T as ν and k tends to $+\infty$ we obtain

$$\int_{Q_T} \Psi^\theta_{n,k}\chi_{\{|v^n-T_k(v)_\nu|>\varepsilon\}} = \omega(n,\nu,k),$$

using Hölder inequality, (4.21) becomes

$$\int_{Q_T} \Big[(a(\nabla u^n)-a(\nabla u)).(\nabla u^n-\nabla u)\Big]^\theta dxdt$$

$$\leq meas(Q_T)^{1-\theta}\Big(\int_{Q_T} \Psi_{n,k}\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}\Big)^\theta + \omega(n,\nu,k).$$

Then it remains to prove that

$$\int_{Q_T} \Psi_{n,k}\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}} = \omega(n,\nu,k,\varepsilon). \tag{4.22}$$

By assumption (3.2) we can write

$$\int_{Q_T} \Psi_{n,k}\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}} \tag{4.23}$$

$$\leq \frac{1}{\gamma}\Big(\int_{Q_T} b'(u^n)a(\nabla u^n)\Big(\nabla u^n-\nabla u\chi_{\{|v|\leq k\}}\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}\Big)$$

$$-\frac{1}{\gamma}\Big(\int_{Q_T} b'(u^n)a(\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla u^n-\nabla u\chi_{\{|v|\leq k\}}\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}\Big)$$

By Proposition 4.1 and since $|T_k(v)_\nu|\leq k$ we obtain

$$\int_{Q_T} b'(u^n)a(\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla u^n-\nabla u\chi_{\{|v|\leq k\}}\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}} \tag{4.24}$$

$$= \int_{Q_T} a(\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla v^n-b'(u^n)b'(u)^{-1}\nabla v\chi_{\{|v|\leq k\}}\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}$$

$$= \int_{Q_T} a(\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla v^n-\nabla T_k(v)_\nu\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}$$

$$+\int_{Q_T} a(\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla T_k(v)_\nu-b'(u^n)b'(u)^{-1}\nabla T_k(v)\Big)\chi_{\{|v^n-T_k(v)_\nu|\leq\varepsilon\}}$$

$$= A_1 + A_2.$$

For $\varepsilon<1$ and thanks to Propsition 4.1 we obtain

$$A_1 = \int_{Q_T} a(\nabla u\chi_{\{|v|\leq k\}})\nabla T_\varepsilon(v^n-T_k(v)_\nu)$$

$$= \int_{Q_T} a(\nabla u \chi_{\{|v|\le k\}}) \nabla T_\varepsilon (T_{k+1}(v^n) - T_k(v)_\nu)$$

$$= \int_{Q_T} a(\nabla u \chi_{\{|v|\le k\}}) \nabla T_\varepsilon (T_{k+1}(v) - T_k(v)_\nu) + \omega(n),$$

and the strong convergence of $\nabla T_k(v)_\nu$ to $\nabla T_k(v)$ in $(L^p(Q_T))^N$ leads to

$$A_1 = \int_{Q_T} a(\nabla u \chi_{\{|v|\le k\}}) \nabla T_\varepsilon (T_{k+1}(v) - T_k(v)) + \omega(n,\nu)$$

$$= \omega(n,\nu).$$

By Proposition 4.1 we have $b'(u^n)$ converges to $b'(u)$ almost everywhere in Q_T, since $a(\nabla u \chi_{\{|v|\le k\}})$ belongs to $(L^{p'}(Q_T))^N$, $\nabla T_k(v)_\nu$ and $\nabla T_k(v)$ belong to $(L^p(Q_T))^N$, the Lebesgue's convergence theorem leads to

$$|A_2| \le \int_{Q_T} |a(\nabla u \chi_{\{|v|\le k\}})||\nabla T_k(v)_\nu - b'(u^n) b'(u)^{-1} \nabla T_k(v)|,$$

$$\le \int_{Q_T} |a(\nabla u \chi_{\{|v|\le k\}})||\nabla T_k(v)_\nu - \nabla T_k(v)| + \omega(n),$$

and by the strong convergence of $\nabla T_k(v)_\nu$ to $\nabla T_k(v)$ in $(L^p(Q_T))^N$ we obtain

$$A_2 = \omega(n,\nu).$$

On the other hand we have

$$\int_{Q_T} b'(u^n) a(\nabla u^n) \Big(\nabla u^n - \nabla u \chi_{\{|v|\le k\}} \Big) \chi_{\{|v^n - T_k(v)_\nu| \le \varepsilon\}} \tag{4.25}$$

$$= \int_{Q_T} a(\nabla u^n) \nabla (v^n - T_k(v)_\nu) \chi_{\{|v^n - T_k(v)_\nu| \le \varepsilon\}}$$

$$+ \int_{Q_T} a(\nabla u^n) \Big(\nabla T_k(v)_\nu - b'(u^n) b'(u)^{-1} \nabla T_k(v) \Big) \chi_{\{|v^n - T_k(v)_\nu| \le \varepsilon\}}.$$

We deal with the second term on the right side of (4.25), by assumption (3.1) it is clear that $\{|v^n| \le k+\varepsilon\} \subset \{|u^n| \le k_\varepsilon = \max\{b^{-1}(k+\varepsilon), |b^{-1}(-k-\varepsilon)|\}\}$ and by Hölder inequality we have

$$\Big| \int_{Q_T} a(\nabla u^n) \Big(\nabla T_k(v)_\nu - b'(u^n) b'(u)^{-1} \nabla T_k(v) \Big) \chi_{\{|v^n - T_k(v)_\nu| \le \varepsilon\}} \Big|$$

$$\le \|a(\nabla T_{k_\varepsilon}(u^n)\|_{L^{p'}(Q_T)} \|\nabla T_k(v)_\nu - b'(u^n) b'(u)^{-1} \nabla T_k(v)\|_{L^p(Q_T)},$$

the almost everywhere convergence of $b'(u^n)$ to $b'(u)$ and Lebesgue's convergence theorem imply that $b'(u^n) b'(u)^{-1} \nabla T_k(v)$ converges to $\nabla T_k(v)$ strongly in $(L^p(Q_T))^N$, since $|a(\nabla T_{k_\varepsilon}(u^n)|$ is bounded in $L^{p'}(Q_T)$ we obtain

$$\Big| \int_{Q_T} a(\nabla u^n) \Big(\nabla T_k(v)_\nu - b'(u^n) b'(u)^{-1} \nabla T_k(v) \Big) \chi_{\{|v^n - T_k(v)_\nu| \le \varepsilon\}} \Big|$$

$$\leq C\|\nabla T_k(v)_\nu - \nabla T_k(v)\|_{L^p(Q_T)} + \omega(n),$$

and the strong convergence of $\nabla T_k(v)_\nu$ to $\nabla T_k(v)$ in $(L^p(Q_T))^N$ leads to

$$\int_{Q_T} b'(u^n)a(\nabla u^n)\Big(b'(u^n)^{-1}\nabla T_k(v)_\nu - b'(u)^{-1}\nabla T_k(v)\Big)\chi_{\{|v^n - T_k(v)_\nu| \leq \varepsilon\}} = \omega(n,\nu).$$

Hence (4.23), (4.24) and (4.25) imply that

$$\int_{Q_T} \Psi_{n,k}\chi_{\{|v^n - T_k(v)_\nu| \leq \varepsilon\}} \leq \int_{Q_T} a(\nabla u^n)\nabla(v^n - T_k(v)_\nu)\chi_{\{|v^n - T_k(v)_\nu| \leq \varepsilon\}} + \omega(n,\nu).$$

Now we use the equation solved by u^n. Taking $T_\varepsilon(v^n - T_k(v)_\nu)$ in (4.3) we obtain

$$\int_0^T \langle \frac{\partial v^n}{\partial t}, T_\varepsilon(v^n - T_k(v)_\nu)\rangle\, dt + \int_{Q_T} a(\nabla u^n)\nabla T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt$$
$$+ \int_{Q_T} H(x,t,\nabla u^n)T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt = \int_{Q_T} \mu^n T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt.$$

By property of μ^n we have

$$\Big|\int_{Q_T} \mu^n T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt\Big| \leq \varepsilon\|\mu^n\|_{L^1(Q_T)} \leq \varepsilon|\mu|_{\mathcal{M}(Q_T)}.$$

By generalized Hölder inequality we have

$$\Big|\int_{Q_T} H(x,t,\nabla u^n)T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt\Big| \leq \varepsilon\|g\|_{L^{N+2,1}(Q_T)}\||\nabla u^n|^\delta\|_{L^{\frac{N+2}{N+1},\infty}(Q_T)}.$$

By Lemma 4.2 we obtain

$$\int_{Q_T} a(\nabla u^n)\nabla T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt \leq \varepsilon\Big(C\|g\|_{L^{N+2,1}(Q_T)} + |\mu|_{\mathcal{M}(Q_T)}\Big).$$

Hence

$$\int_{Q_T} a(\nabla u^n)\nabla T_\varepsilon(v^n - T_k(v)_\nu)\, dxdt \leq \omega(n,\nu,\varepsilon). \tag{4.26}$$

Then by (4.26) we obtain (4.22) and therefore (4.20) and (4.19). □

Remark 4.4. *Let us observe that from Proposition 4.3 we have $H(x,t,\nabla u^n)$ converges to $H(x,t,\nabla u)$ a.e. in Q_T and by Proposition 4.1 $H(x,t,\nabla u^n)$ is equi-integrable in $L^1(Q_T)$. Indeed if E is a measurable set of Q_T, due the growth assumption (3.7) on H, estimate (4.5) yields that*

$$\int_E |H(x,t,\nabla u^n)|\, dxdt \leq \int_E g(x,t)|\nabla u^n|^\delta\, dxdt$$

$$\leq C\|g\|_{L^{N+2,1}(E)}.$$

We conclude that $H(x,t,\nabla u^n)$ is equi-integrable in $L^1(Q_T)$. Then by Vitali's theorem we deduce that $H(x,t,\nabla u^n)$ converges to $H(x,t,\nabla u)$ strongly in $L^1(Q_T)$. Let also remark that from Proposition 4.3, assumption (3.5) on a and Vitali's theorem, we deduce that $a(x,t,\nabla u^n)$ is strongly compact in $L^1(Q_T)$.

Now we define the space $\mathcal{S}$ by

$$\mathcal{S} = \Big\{ z \in L^p(0,T;W_0^{1,p}(\Omega)), z_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T) \Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|.\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q_T)}$, and its sub-space $\mathcal{W}_1$ as

$$\mathcal{W}_1 = \Big\{ z \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T), z_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T) \Big\},$$

endowed with its natural norm

$$\|.\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|.\|_{L^\infty(Q_T)} + \|.\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q_T)},$$

for any $p > 1$.
Let us recall that a function z is called cap_p-quasi continuous if for every $\varepsilon > 0$ there exists an open set F_ε with $cap(F_\varepsilon) \leq \varepsilon$ such that the restriction of z to $Q_T \backslash F_\varepsilon$ is continuous. The following result shows that every functions in $\mathcal{W}_1$ satisfy a capacitary estimate for the parabolic capacity.

Theorem 4.5. *Let $z \in \mathcal{W}_1$, then z admits a unique cap_p-quasi continuous representative. Moreover, we have*

$$cap_p(\{|z| > k\}) \leq \frac{C}{k} max\{[z]_*^{\frac{1}{p}}, [z]_*^{\frac{1}{p'}}\},$$

where

$$\begin{aligned}[z]_* \;=\; & \|z\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} + \|z_t^1\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \\ & + \|z\|_{L^\infty(Q_T)}\|z_t^2\|_{L^1(Q_T)} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

such that $z_t^1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $z_t^2 \in L^1(Q_T)$ is any decomposition of z_t, that is $z_t = z_t^1 + z_t^2$.

Proof. See [22], Theorem 3 and Lemma 2. □

Now we prove the following theorem

Theorem 4.6. *Let $u^n \in L^p(0,T;W_0^{1,p}(\Omega))$ be a solution of Problem (4.3)-(4.4) then*

$$cap_p(\{|v^n| > k\}) \leq \frac{C}{k} max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\}. \;\; \forall k \geq 1$$

Proof. Due to the presence of the lower order term H, the approach used in [25] in the proof of Theorem 1.2 does not apply here, to overcome this difficulty we are going to exploit the method used in [22] Theorem 4. Let us first introduce the following function

$$G_k(s) = \begin{cases} 1 & \text{if } |s| \le k, \\ k+1-|s| & \text{if } \ k < |s| \le k+1, \\ 0 & \text{if } \ |s| > k+1. \end{cases}$$

let us denote by $\overline{G}_k(s)$ the primitive function of $G_k(s)$. Since we have

$$\int_{Q_T} |\nabla T_k(v^n)|^p \, dxdt \le Ck,$$

we obtain

$$\int_{Q_T} |\nabla \overline{G}_k(v^n)|^p \, dxdt \le Ck. \tag{4.27}$$

Given $\varphi \in C_c^\infty(Q_T)$ and taking $G_k(v^n)\varphi$ as test function in (4.3) we have in the sense of distribution

$$\overline{G}_k(v^n)_t = \operatorname{div}\Big(G_k(v^n)a(x,t,\nabla u^n)\Big) \tag{4.28}$$

$$-b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{k \le v^n < k+1\}} + b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{-k-1 < v^n \le -k\}}$$
$$-H(x,t,\nabla u^n)G_k(v^n) + G_k(v^n)\mu_n,$$

therefore by assumption (3.2) and Proposition 4.1, we have

$$\overline{G}_k(v^n)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q),$$

and

$$\overline{G}_k(v^n) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T),$$

thanks to Theorem 4.5, $\overline{G}_k(v^n)$ has a cap_p-quasicontinuous representative. To conclude the proof is enough to prove the capacitary estimate of v_n.
Since $\{|v^n| > k\} = \{\overline{G}_k(v^n) > k\}$, by Theorem 4.5 we obtain

$$cap_p(\{|v^n| > k\}) \le \frac{C}{k} max\{[\overline{G}_k(v^n)]_*^{\frac{1}{p}}, [\overline{G}_k(v^n)]_*^{\frac{1}{p'}}\}.$$

Taking $\theta_k(v^n) = T_{k+1}(v^n) - T_k(v^n)$ as test function in (4.3) leads to

$$\int_\Omega \Theta_k(v^n)(T)\,dx + \int_{\{k<|v^n|\le k+1\}} b'(u^n)a(x,t,\nabla u^n).\nabla u^n \,dxdt$$
$$+ \int_{Q_T} H(x,t,\nabla u^n)\theta_k(v^n)\,dxdt = \int_{Q_T} \theta_k(v^n)\mu^n\,dxdt + \int_\Omega \Theta_k(b(u_0^n))\,dx,$$

where $\Theta_k(s) = \int_0^s \theta_k(r)\,dr \;\forall s \in \mathbb{R}$.
Since $\|\theta_k(v^n)\|_{L^\infty(Q_T)} \leq 1$ and $H(x,t,\nabla u^n)$ is strongly compact in $L^1(Q)$ one obtains

$$\int_{Q_T} b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{k\leq v^n<k+1\}}\,dxdt \leq C,$$

$$\int_{Q_T} b'(u^n)a(x,t,\nabla u^n).\nabla u^n \chi_{\{-k-1<v^n\leq -k\}}\,dxdt \leq C,$$

$$\int_{Q_T} |H(x,t,\nabla u^n)G_k(v^n)|\,dxdt \leq C.$$

Then, from (4.28) it follows that

$$\|\overline{G}_k(v^n)^1_t\|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq Ck,$$

$$\|\overline{G}_k(v^n)^2_t\|_{L^1(Q)} \leq C,$$

using the following estimate

$$\|\overline{G}_k(v^n)\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \|\overline{G}_k(v^n)\|_{L^\infty(Q_T)}\|\overline{G}_k(v^n)\|_{L^\infty(0,T;L^1(\Omega))},$$

since v^n is bounded in $L^\infty(0,T;L^1(\Omega))$, we conclude that

$$cap_p(\{|v^n|>k\}) \leq \frac{C}{k} max\{k^{\frac{1}{p}}, k^{\frac{1}{p'}}\}.$$

□

We have the following technical result

Lemma 4.7. *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}(Q_T)$ where μ_s^+ and μ_s^- are concentrated respectively, on two disjoint E^+ and E^- of zero p-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+\backslash K_\delta^+) \leq \delta, \quad \mu_s^-(E^+\backslash K_\delta^-) \leq \delta, \tag{4.29}$$

and there exist ψ_δ^+, $\psi_\delta^- \in C_c^\infty(Q_T)$, such that

$$\psi_\delta^+ \equiv 1 \text{ and } \psi_\delta^- \equiv 1 \text{ respectively on } K_\delta^+ \text{ and } K_\delta^-, \tag{4.30}$$

$$0 \leq \psi_\delta^+, \quad \psi_\delta^- \leq 1, \tag{4.31}$$

$$supp(\psi_\delta^+) \cap supp(\psi_\delta^-) \equiv \emptyset. \tag{4.32}$$

Moreover

$$\|\psi_\delta^+\|_{\mathcal{S}} \leq \delta, \quad \|\psi_\delta^-\|_{\mathcal{S}} \leq \delta, \tag{4.33}$$

and in particular, there exists a decomposition of $(\psi_\delta^+)_t$ *and a decomposition of* $(\psi_\delta^-)_t$ *such that*

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q_T)} \le \frac{\delta}{3}, \tag{4.34}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q_T)} \le \frac{\delta}{3}. \tag{4.35}$$

Both ψ_δ^+ *and* ψ_δ^- *converges to zero* $*-$*weakly in* $L^\infty(Q_T)$*, in* $L^1(Q_T)$*, and up to subsequences, almost everywhere as* δ *vanishes. Moreover, if* λ_+^n *and* λ_-^n *are as in (4.2) we have*

$$\int_{Q_T} \psi_\delta^- \, d\lambda_+^n = \omega(n,\delta), \quad \int_{Q_T} \psi_\delta^- \, d\mu_s^+ \le \delta, \tag{4.36}$$

$$\int_{Q_T} \psi_\delta^+ \, d\lambda_-^n = \omega(n,\delta), \quad \int_{Q_T} \psi_\delta^+ \, d\mu_s^- \le \delta, \tag{4.37}$$

$$\int_{Q_T} (1-\psi_\delta^+) \, d\lambda_+^n = \omega(n,\delta), \quad \int_{Q_T} (1-\psi_\delta^+) \, d\mu_s^+ \le \delta, \tag{4.38}$$

$$\int_{Q_T} (1-\psi_\delta^-) \, d\lambda_-^n = \omega(n,\delta), \quad \int_{Q_T} (1-\psi_\delta^-) \, d\mu_s^- \le \delta. \tag{4.39}$$

Proof. See [22], Lemma 5. □

Now we prove the following theorem

Theorem 4.8. *Under assumptions (3.1)-(3.8), there exists at least a renormalized solution* u *of Problem (1.1)-(1.3).*

Let us fix $\sigma > 0$ and define

$$S_{k,\sigma}(s) = \begin{cases} 1 & \text{if } |s| \le k, \\ 0 & \text{if } |s| > k+\sigma, \\ \text{affine} & \text{if otherwise.} \end{cases}$$

Proof. **Step 1**. Estimates in $L^1(Q_T)$ on the energy term. Using $h_{k,\sigma}(u^n) = \frac{1}{\sigma}(T_{k+\sigma}(u^n) - T_k(u^n))$ as test function in (4.3) we obtain

$$\int_\Omega B^\star_{h_{k,\sigma}}(u^n)(T)\,dx + \frac{1}{\sigma}\int_{\{k<|u^n|\le k+\sigma\}} a(x,t,\nabla u^n).\nabla u^n \,dxdt$$
$$+ \int_{Q_T} H(x,t,\nabla u^n) h_{k,\sigma}(u^n)\,dxdt = \int_{Q_T} h_{k,\sigma}(u^n)\mu^n\,dxdt + \int_\Omega B^\star_{h_{k,\sigma}}(u_0^n)\,dx,$$

where $B^\star_{h_{k,\sigma}}(s) = \displaystyle\int_0^s b'(r) h_{k,\sigma}(r)\,dr \ \forall s \in \mathbb{R}$.
So that dropping positive terms

$$\frac{1}{\sigma}\int_{\{k<|u^n|\le k+\sigma\}} a(x,t,\nabla u^n).\nabla u^n\,dxdt \tag{4.40}$$

$$\leq \int_{\{|u^n|>k\}} |\mu^n| \, dxdt + \int_{\{|u^n|>k\}} |H(x,t,\nabla u^n)| \, dxdt + \int_{\{|u_0^n|>k\}} b(u_0^n) \, dx.$$

which implies, in particular,

$$\frac{1}{\sigma} \int_{\{k<|u^n|\leq k+\sigma\}} a(x,t,\nabla u^n).\nabla u^n \, dxdt \leq C. \tag{4.41}$$

Step 2. Equation for the truncations. Given $\varphi \in C_c^\infty(Q_T)$, taking $S_{k,\sigma}(u^n)\varphi$ as test function in (4.3), we obtain

$$\begin{aligned} & B^\star_{S_{k,\sigma}}(u^n)_t - \operatorname{div}\Big(S_{k,\sigma}(u^n)a(x,t,\nabla u^n)\Big) + H(x,t,\nabla u^n)S_{k,\sigma}(u^n) \\ = \; & \mu_d^n + \mu_s^n S_{k,\sigma}(u^n) + \frac{1}{\sigma} sign(u^n) a(x,t,\nabla u^n).\nabla u^n \chi_{\{k<|u^n|\leq k+\sigma\}} \\ & + \mu_d^n (S_{k,\sigma}(u^n) - 1) \text{ in } \mathcal{D}'(Q_T), \end{aligned} \tag{4.42}$$

where $B^\star_{S_{k,\sigma}}(s) = \displaystyle\int_0^s b'(r) S_{k,\sigma}(r) \, dr$.

From (4.41), there exists a bounded Radon measure ς_k^n such that, as σ goes to zero

$$\frac{1}{\sigma} sign(u^n) a(x,t,\nabla u^n).\nabla u^n \chi_{\{k<|u^n|\leq k+\sigma\}} \rightharpoonup \varsigma_k^n \;\; \star \text{ weakly in } \mathcal{M}(Q_T).$$

Taking the limit as σ vanishes in (4.42) it follows that

$$B_k(u^n)_t - \operatorname{div}\Big(a(x,t,\nabla T_k(u^n))\Big) + H(x,t,\nabla T_k(u^n)) = \mu_d^n + \mu_s^n \chi_{\{|u^n|\leq k\}}$$
$$+ \varsigma_k^n - \mu_d^n \chi_{\{|u^n|\geq k\}} \text{ in } \mathcal{D}'(Q_T),$$

where $B_k(s) = \displaystyle\int_0^s T_k'(r) b'(r) \, dr, \;\; \forall s \; \in \mathbb{R}$.

We define the measure Γ_n^k as

$$\Gamma_n^k = \mu_s^n \chi_{\{|u^n|\leq k\}} + \varsigma_k^n - \mu_d^n \chi_{\{|u^n|\geq k\}}.$$

Notice that

$$\|\Gamma_n^k\|_{L^1(Q_T)} \leq C,$$

so that there exist $\Gamma^k \in \mathcal{M}(Q)$ such that

$$\Gamma_n^k \rightharpoonup \Gamma^k \;\; \star \text{ weakly in } \mathcal{M}(Q_T).$$

Therefore, using Proposition 4.1 and Proposition 4.3 , in the sense of distribution we have

$$B_k(u)_t - \operatorname{div}\Big(a(x,t,\nabla T_k(u))\Big) + H(x,t,\nabla T_k(u^n)) = \mu_d + \Gamma^k \text{ in } \mathcal{D}'(Q_T). \tag{4.43}$$

Step 3. The limit of Γ^k. By substracting (4.43) from the distributional formulation of (4.3) we obtain for any $\varphi \in C_c^\infty(Q_T)$

$$\int_{Q_T} (v^n - B_k(u))\varphi_t \, dxdt + \int_{Q_T} (a(x,t,\nabla u^n) - a(x,t,\nabla T_k(u))\nabla\varphi \, dxdt \tag{4.44}$$

$$+ \int_{Q_T} (H(x,t,\nabla u^n) - H(x,t,\nabla T_k(u)))\varphi \, dxdt$$

$$= \int_{Q_T} (\mu_d^n - \mu_d)\varphi \, dxdt + \int_Q (\mu_s^n - \Gamma^k)\varphi \, dxdt.$$

Using Proposition 4.1 and Proposition 4.3 we obtain from (4.44) in the sense of distribution

$$\Gamma^k = \mu_s + \omega(n,k) \text{ in } \mathcal{D}'(Q_T).$$

To complete the proof we have to show that the previous limit is actually tight. Let us choose without loss of generality $\varphi \in C^1(\overline{Q_T})$ (then by density argument we show the result holds with $\varphi \in C(\overline{Q_T})$). We have

$$\int_{Q_T} \Gamma^k \varphi \, dxdt = \int_{Q_T} \Gamma^k \Psi_\delta \varphi \, dxdt + \int_{Q_T} \Gamma^k (1 - \Psi_\delta)\varphi \, dxdt,$$

where $\Psi_\delta = \psi_\delta^+ + \psi_\delta^-$ is chosen as in Lemma 4.7. Thanks to the previeous result we can write

$$\int_{Q_T} \Gamma^k \Psi_\delta \varphi \, dxdt = \int_{Q_T} \mu_s^+ \Psi_\delta \varphi \, dxdt - \int_{Q_T} \mu_s^- \Psi_\delta \varphi \, dxdt + \omega(n,k),$$

we have

$$\int_{Q_T} \mu_s^+ \Psi_\delta \varphi \, dxdt = \int_{K_\delta^+} \mu_s^+ \psi_\delta^+ \varphi \, dxdt + \int_{E^+\setminus K_\delta^+} \mu_s^+ \psi_\delta^+ \varphi \, dxdt + \int_{Q_T} \mu_s^+ \psi_\delta^- \varphi \, dxdt,$$

since $\psi_\delta^+ = 1$ on K_δ^+ by Lebesgue's theorem we have

$$\int_{Q_T} \mu_s^+ \Psi_\delta \varphi \, dxdt = \int_{Q_T} \mu_s^+ \varphi \, dxdt + \omega(\delta)$$

by Lemma 4.7 we obtain

$$\Big| \int_{E^+\setminus K_\delta^+} \mu_s^+ \psi_\delta^+ \varphi \, dxdt \Big| \leq \delta \|\varphi\|_{L^\infty(Q_T)},$$

and

$$\Big| \int_{Q_T} \mu_s^+ \psi_\delta^- \varphi \, dxdt \Big| \leq \|\varphi\|_{L^\infty(Q_T)} \int_{Q_T} \psi_\delta^- \varphi \, d\mu_s^+ = \omega(\delta).$$

Then we otain

$$\int_{Q_T} \mu_s^+ \Psi_\delta \varphi \, dxdt = \int_{Q_T} \mu_s^+ \varphi \, dxdt + \omega(\delta).$$

Similarly we obtain

$$\int_{Q_T} \mu_s^- \Psi_\delta \varphi \, dxdt = \int_{Q_T} \mu_s^- \varphi \, dxdt + \omega(\delta).$$

Hence

$$\int_{Q_T} \Gamma^k \Psi_\delta \varphi \, dxdt = \int_{Q_T} \varphi \, d\mu_s + \omega(k, \delta).$$

To conclude we have to prove that

$$\int_{Q} \Gamma^k (1 - \Psi_\delta) \varphi \, dxdt = \omega(k, \delta).$$

From the definition of Γ^k we have

$$\int_{Q_T} (1 - \Psi_\delta) \varphi \, d\Gamma^k = \lim_n \Big(\lim_\sigma \frac{1}{\sigma} \int_{\{k < |u^n| \le k+\sigma\}} sign(u^n) a(x, t, \nabla u^n).\nabla u^n (1 - \Psi_\delta) \varphi$$

$$+ \int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\mu_s^n - \int_{\{|u^n| > k\}} (1 - \Psi_\delta) \varphi \, d\mu_d^n \Big).$$

By Proposition 2.3 the sequence μ_d^n is equi-diffuse, thanks to assumption (3.2) and Theorem 4.6 we deduce that

$$\Big| \int_{\{|u^n| > k\}} (1 - \Psi_\delta) \varphi \, d\mu_d^n \Big| \le \|\varphi\|_{L^\infty(Q_T)} \int_{\{|v^n| > k\gamma\}} |\mu_d^n| \, dxdt = \omega(n, k).$$

We have

$$\int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\mu_s^n = \int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\lambda_+^n - \int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\lambda_-^n,$$

and

$$\int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\lambda_+^n = \int_{\{|u^n| \le k\}} (1 - \psi_\delta^+) \varphi \, d\lambda_+^n - \int_{\{|u^n| \le k\}} \psi_\delta^- \varphi \, d\lambda_+^n,$$

Thanks to Lemma 4.7 we obtain

$$\Big| \int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\lambda_+^n \Big| \le \|\varphi\|_{L^\infty(Q_T)} \Big(\int_{Q_T} (1 - \psi_\delta^+) d\lambda_+^n + \int_{Q_T} \psi_\delta^- \, d\lambda_+^n \Big) = \omega(n, \delta).$$

Similarly we obtain

$$\int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\lambda_-^n = \omega(n, \delta),$$

and then

$$\int_{\{|u^n| \le k\}} (1 - \Psi_\delta) \varphi \, d\mu_s^n = \omega(n, \delta).$$

It remains to prove that

$$\int_{\{k<|u^n|\le k+\sigma\}} \frac{1}{\sigma} sign(u^n) a(x,t,\nabla u^n).\nabla u^n (1-\Psi_\delta)\varphi \, dxdt = \omega(\sigma,n,k,\delta).$$

we use $h_{k,\sigma}(u^n)(1-\Psi_\delta)$ as test function in (4.3) we obtain

$$\int_{Q_T} B^\star_{h_{k,\sigma}}(u^n)(\Psi_\delta)_t + \int_\Omega B^\star_{h_{k,\sigma}}(u^n)(T) - \int_\Omega B^\star_{h_{k,\sigma}}(u_0^n)(1-\Psi_\delta(0)) \tag{4.45}$$

$$+\frac{1}{\sigma}\int_{\{k<|u^n|\le k+\sigma\}} a(x,t,\nabla u^n).\nabla u^n(1-\Psi_\delta) - \int_{Q_T} a(x,t,\nabla u^n).\nabla \Psi_\delta h_{k,\sigma}(u^n)$$

$$+\int_{Q_T} H(x,t,\nabla u^n) h_{k,\sigma}(u^n)(1-\Psi_\delta)$$

$$= \int_{Q_T} h_{k,\sigma}(u^n)(1-\Psi_\delta)\mu_d^n + \int_{Q_T} h_{k,\sigma}(u^n)(1-\Psi_\delta)\mu_s^n.$$

Using assumption (3.2), the convergence in $L^1(Q_T)$ of u^n, $a(x,t,\nabla u^n)$, $H(x,t,\nabla u^n)$ and the regularity of Ψ_δ we obtain

$$\int_\Omega B^\star_{h_{k,\sigma}}(u^n)(T) = \omega(\sigma,n,k),$$

$$\int_{Q_T} B^\star_{h_{k,\sigma}}(u^n)(\Psi_\delta)_t = \omega(\sigma,n,k),$$

$$\int_\Omega B^\star_{h_{k,\sigma}}(u_0^n)(1-\Psi_\delta(0)) = \omega(\sigma,n,k),$$

$$\int_{Q_T} a(x,t,\nabla u^n).\nabla \Psi_\delta h_{k,\sigma}(u^n) = \omega(\sigma,n,k),$$

$$\int_{Q_T} H(x,t,\nabla u^n) h_{k,\sigma}(u^n)(1-\Psi_\delta) = \omega(\sigma,n,k).$$

Thanks to Theorem 4.6 and equi-diffuse property of μ_d^n

$$\int_{Q_T} h_{k,\sigma}(u^n)(1-\Psi_\delta)\mu_d^n = \omega(\sigma,n,k),$$

finally by Lemma 4.7 we have

$$\int_{Q_T} h_{k,\sigma}(u^n)(1-\Psi_\delta)\mu_s^n = \omega(\sigma,n,\delta).$$

Hence we conclude that u is a renormalized solution of Problem (1.1)-(1.3). □

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