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The Necessary and Sufficient Conditions for Wavelet Frames in Sobolev Space Over Local Fields

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ABSTRACT: In this paper we construct wavelet frame on Sobolev space. A necessary condition and sufficient conditions for wavelet frames in Sobolev space are given.

Key Words: Wavelet, Wavelet frame, Local fields, Sobolev space, Fourier transform.

Contents

1. Introduction

Let $\mathbb{F}$ be an algebraic field and topological space with the topological properties of non-discrete, complete, locally compact and totally disconnected.Let $\mathbb{F}^*$ and $\mathbb{F}^+$ are the multiplicative and additive groups of $\mathbb{F}$ respectively. Now we define Haar measure $d\xi$ for $\mathbb{F}^+$. Then for $\beta \neq 0(\beta \in \mathbb{F})$, $d(\beta\xi)$ is also a Haar measure. Let $d(\beta\xi) = |\beta|d\xi$ and we say $|\beta|$ is the absolute value or valuation of β.Let $|0| = 0$. The valuation or absolute value has following properties :
(a) $|\xi| \geq 0$ and $|\xi| = 0$ if and only if $\xi = 0$;
(b) $|\xi\eta| = |\xi||\eta|$;
(c) $|\xi + \eta| \leq max(|\xi|, |\eta|)$.
The last property is called ultrametric inequality. The set $\mathfrak{D} = \{\xi \in \mathbb{F} : |\xi| \leq 1\}$ is the ring of integers in $\mathbb{F}$ and is the unique maximal compact subring of $\mathbb{F}$. Define $\mathfrak{P} = \{\xi \in \mathbb{F} : |\xi| < 1\}$ The set $\mathfrak{P}$ is called the prime ideal in $\mathbb{F}$. The prime ideal in $\mathbb{F}$ is the unique maximal ideal in $\mathfrak{D}$. Then set $\mathfrak{P}$ is principal and prime.
Let A be a measurable subset of $\mathbb{F}$ and $|A| = \int_{\mathbb{F}} \zeta_A(\xi)d\xi$, where ζ_A is the characteristic function of A and $d\xi$ is the Haar measure of $\mathbb{F}$ normalized so that $|\mathfrak{D}| = 1$. Then we observe that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$. Therefore for $\xi \neq 0(\xi \in \mathbb{F})$, $|\xi| = q^k$ for some $k \in \mathbb{Z}$.
Define $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{\xi \in \mathbb{F} : |\xi| \leq q^{-k}, k \in \mathbb{Z}\}$. These are called fractional ideals. Each $\mathfrak{P}^k$ is a subgroup of $\mathbb{F}^+$.It is to see that $\mathfrak{P}^k$ is open as well as compact. If $\mathbb{F}$ is a local field, then there is a nontrivial, unitary, continuous character χ on $\mathbb{F}^+$.

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It can be proved that $\mathbb{F}^+$ is self dual.
Let χ be a fixed character on $\mathbb{F}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We will define fixed character χ for a local field of positive characteristic by $\chi_\eta(\xi) = \chi(\eta\xi), \quad \text{for } \xi, \eta \in \mathbb{F}$.

Definition 1.1. *If $g \in L^1(\mathbb{F})$, then the Fourier transform of g is the function $\hat{g}$ defined by*

$$\hat{g}(\eta) = \int_{\mathbb{F}} g(\xi)\overline{\chi_\eta(\xi)}d\xi = \int_{\mathbb{F}} g(\xi)\chi(-\eta\xi)d\xi.$$

The Fourier transform in $L^p(\mathbb{F})$, $1 < p \leq 2$, can be defined similarly as in $L^p(\mathbb{R})$. The inner product is defined by

$$\langle g, f\rangle = \int_{\mathbb{F}} g(\xi)\overline{f(\xi)}d\xi \ \text{ for } f, g \in L^2(\mathbb{F}).$$

The "natural" order on the sequence $\{v(n) \in \mathbb{F}\}_{n=0}^{\infty}$ is described as follows. Recall that $\mathfrak{P}$ is the prime ideal in $\mathfrak{D}$, $\mathfrak{D}/\mathfrak{P} \cong GF(q) = \tau$, $q = p^c$, p is a prime, c a positive integer and $\Omega : \mathfrak{D} \to \tau$ the canonical homomorphism of $\mathfrak{D}$ on to τ. Note that $\tau = GF(q)$ is a c- dimensional vector space over $GF(p) \subset \tau$. We choose a set $\{1 = \epsilon_0, \epsilon_1, ..., \epsilon_{c-1}\} \subset \mathfrak{D}^* = \mathfrak{D}\backslash\mathfrak{P}$ such that $\{\Omega(\epsilon_k)\}_{k=0}^{c-1}$ is a basis of $GF(q)$ over $GF(p)$.

Definition 1.2. *For k, $0 \leq k < q$, $k = a_0 + a_1p + ... + a_{c-1}p^{c-1}$, $0 \leq a_i < p$, $i = 0, 1, ..., c-1$, we define*

$$v(k) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1} \ \ (0 \leq k < q).$$

For $k = b_0 + b_1q + ... + b_sq^s$, $0 \leq b_i < q$, $k \geq 0$, we set

$$v(k) = v(b_0) + \mathfrak{p}^{-1}v(b_1) + ... + \mathfrak{p}^{-s}v(b_s).$$

Note that for $k, l \geq 0$, $v(k + l) \neq v(k) + v(l)$. However, it is true that for all $r, s \geq 0$, $v(rq^s) = \mathfrak{p}^{-s}v(r)$, and for $r, s \geq 0$, $0 \leq t < q^s$, $v(rq^s + t) = v(rq^s) + v(t) = \mathfrak{p}^{-s}v(r) + v(t)$.

We will denote $\chi_{v(n)}$ by $\chi_n (n \geq 0)$ and use the notation $\mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\mathbb{N} = \{1, 2, 3, ...\}$ throughout this paper.

1.1. Distributions over local fields

We denote $\mathscr{S}(\mathbb{F})$ the spaces of all finite linear combinations of characteristics functions of ball of $\mathbb{F}$. The Fourier transform is homeomorphism of $\mathscr{S}(\mathbb{F})$ onto $\mathscr{S}(\mathbb{F})$. The distribution space of $\mathscr{S}(\mathbb{F})$ is denoted by $\mathscr{S}'(\mathbb{F})$.
The Fourier transform of $g \in \mathscr{S}(\mathbb{F})$ is denoted by $\hat{g}(\omega)$ and defined by

$$\hat{g}(\omega) = \int_{\mathbb{F}} g(\xi)\overline{\chi_\omega(\xi)}d\xi = \int_{\mathbb{F}} g(\xi)\chi(-\omega\xi)d\xi, \ \omega \in \mathbb{F}, \tag{1.1}$$

and the inverse Fourier transform defined by,

$$g(\xi) = \int_{\mathbb{F}} \hat{g}(\omega)\chi_\xi(\omega)d\omega,\ \xi \in \mathbb{F}. \tag{1.2}$$

The Fourier transform and inverse Fourier transforms of a distributions $g \in \mathscr{S}'(\mathbb{F})$ is defined by

$$\langle \hat{g}, \varphi \rangle = \langle g, \hat{\varphi} \rangle,\ \langle g^\vee, \varphi \rangle = \langle g, \varphi^\vee \rangle,\ \text{for all } \varphi \in \mathscr{S}(\mathbb{F}). \tag{1.3}$$

Definition 1.3. ***Sobolev space over local fields***.
Let $s \in \mathbb{R}$. Sobolev space over local fields denote by $H^s(\mathbb{F})$, defined by the space of all $g \in \mathscr{S}'(\mathbb{F})$ such that

$$\hat{\nu}^{\frac{s}{2}}(\omega)\hat{g}(\omega) \in L^2(\mathbb{F}),\ \textit{where } \hat{\nu}^s(\omega) = (Max(1, |\omega|))^s.$$

We equip $H^s(\mathbb{F})$ with the inner product

$$\langle g, h \rangle_s = \langle g, h \rangle_{H^s(\mathbb{F})} = \int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega)\overline{\hat{h}(\omega)}d\omega,$$

which induces the norm

$$||g||^2_{H^s(\mathbb{F})} = \int_{\mathbb{F}} \hat{\nu}^s(\omega)|\hat{g}(\omega)|^2 d\omega.$$

Theorem 1.4. *The space $\mathscr{S}(\mathbb{F})$ is dense in $H^s(\mathbb{F})$.*

Proof. See [15]. □

2. A necessary condition of wavelet frame for $H^s(\mathbb{F})$

Let $\psi \in H^s(\mathbb{F})$, $\psi_{j,k}(\xi) = q^{\frac{j}{2}}\psi(\mathfrak{p}^{-j}\xi - v(k))$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$. The function system $\{\psi_{j,k}(\xi)\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}_0}$ a wavelet frame for $H^s(\mathbb{F})$, if there are two constants $C, D \geq 0$ such that

$$C||g||^2_{H^s(\mathbb{F})} \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle g, \psi_{j,k} \rangle|^2 \leq D||g||^2_{H^s(\mathbb{F})} \tag{2.1}$$

satisfies for all $g \in H^s(\mathbb{F})$.

Theorem 2.1. *If $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is a wavelet frame for $H^s(\mathbb{F})$ with bounds C and D then*

$$C \leq \hat{\nu}^s(\omega) \sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathfrak{p}^j\omega)|^2 \leq D\ \ a.e. \omega \in \mathbb{F}.$$

Proof. For $g \in \mathscr{S}(\mathbb{F})$ and $\psi \in H^s(\mathbb{F})$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2 &= \sum_{k=0}^{\infty} \Big| \int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega) q^{\frac{j}{2}} \overline{\psi(\mathfrak{p}^{-j}\omega - v(k))^\wedge} d\omega\Big|^2 \\
&= \sum_{k=0}^{\infty} \Big| \int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega) q^{-\frac{j}{2}} \overline{\hat{\psi}(\mathfrak{p}^j\omega)} \chi_k(\mathfrak{p}^j\omega) d\omega\Big|^2 \\
&= \sum_{k=0}^{\infty} q^{-j} \int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega)}\chi_k(\mathfrak{p}^j\omega) d\omega \\
&\quad \times \left\{ \int_{\mathbb{F}} \hat{\nu}^s(\omega)\overline{\hat{g}(\omega)}\hat{\psi}(\mathfrak{p}^j\omega)\overline{\chi_k(\mathfrak{p}^j\omega)} d\omega \right\} \\
&= \sum_{k=0}^{\infty} q^{j} \int_{\mathbb{F}} \hat{\nu}^s(\mathfrak{p}^{-j}\omega)\hat{g}(\mathfrak{p}^{-j}\omega)\overline{\hat{\psi}(\omega)}\chi_k(\omega) d\omega \\
&\quad \times \left\{ \int_{\mathbb{F}} \hat{\nu}^s(\mathfrak{p}^{-j}\omega)\overline{\hat{g}(\mathfrak{p}^{-j}\omega)}\hat{\psi}(\omega)\overline{\chi_k(\omega)} d\omega \right\} \\
&= \sum_{k=0}^{\infty} q^{j} \int_{\mathbb{F}} \{ \sum_{l=0}^{\infty} \int_{\mathfrak{D}} \hat{\nu}^s(\mathfrak{p}^{-j}(\omega + v(l)))\hat{g}(\mathfrak{p}^{-j}(\omega + v(l))) \\
&\quad \times \chi_k(\omega + v(l))\overline{\hat{\psi}(\omega + v(l))} d\omega\} \times \{\hat{\nu}^s(\mathfrak{p}^{-j}\omega)\overline{\hat{g}(\mathfrak{p}^{-j}\omega)}\hat{\psi}(\omega) \\
&\quad \times \overline{\chi_k(\omega)}\} d\omega.
\end{aligned}$$

Since $g \in \mathscr{S}(\mathbb{F})$ so the $\sum_{l=0}^{\infty}$ contains only finite non-zero terms and $\chi_k(v(l)) = 1$ for all $k, l \in \mathbb{N}_0$, then we get

$$\begin{aligned}
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2 &= \sum_{k=0}^{\infty} q^{j} \int_{\mathbb{F}} (\int_{\mathfrak{D}} \{ \sum_{l=0}^{\infty} \hat{\nu}^s(\mathfrak{p}^{-j}(\omega + v(l)))\hat{g}(\mathfrak{p}^{-j}(\omega + v(l)))\chi_k(\omega) \\
&\quad \times \overline{\hat{\psi}(\omega + v(l))}\} d\omega) \times \{\hat{\nu}^s(\mathfrak{p}^{-j}\omega)\overline{\hat{g}(\mathfrak{p}^{-j}\omega)}\hat{\psi}(\omega)\overline{\chi_k(\omega)}\} d\omega.
\end{aligned}$$

By the convergence theorem of Fourier Series on $\mathfrak{D}$, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2 &= \int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega)\overline{\hat{\psi}}(\mathfrak{p}^j\omega) \{ \sum_{k=0}^{\infty} \nu^s(\omega + \mathfrak{p}^{-j}v(k))\overline{\hat{g}}(\omega + \mathfrak{p}^{-j}v(k)) \\
&\quad \times \hat{\psi}(\mathfrak{p}^j\omega + v(k))\} d\omega. \qquad (2.2)
\end{aligned}$$

Let A_j is the set of regular point of $\hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2$, so for all $\omega \in A_j$

$$q^l \int_{\omega - \omega_0 \in \mathfrak{P}^l} \hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2 d\omega \to \hat{\nu}^s(\omega_0)|\hat{\psi}(\mathfrak{p}^j\omega_0)|^2, \text{ as } l \to +\infty.$$

If $A = \cup_{j\in\mathbb{Z}} A_j^c$, then $|A| = 0$.
Suppose that $\omega_0 \in \mathbb{F} - A$. So for each fixed positive integer M, set

$$\hat{g}(\omega) = \frac{q^{\frac{l}{2}}\varphi_l(\omega-\omega_0)}{\hat{\nu}^{\frac{s}{2}(\omega)}} \quad \text{for all} \quad l \geq M,$$

where φ_l is the characteristic function of $\omega_0 + \mathfrak{P}^l$.
Then for $l \in \mathbb{N}$ and $j \geq -M$, $\hat{g}(\omega)\bar{\hat{g}}(\omega + \mathfrak{p}^{-j}v(l)) = 0$. Since ω and $(\omega + \mathfrak{p}^{-j}v(l))$ can not be in $\omega_0 + \mathfrak{P}^m$ simultaneously. Now, we have

$$\sum_{j\geq -M}\sum_{k=0}^{\infty}|\langle g, \psi_{j,k}\rangle|^2 = \sum_{j\geq -M}\int_{\omega+\mathfrak{P}^l} q^l\hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2 d\omega \leq D. \tag{2.3}$$

Let $l \to +\infty$ and $M \to +\infty$, we have

$$\sum_{j\in\mathbb{Z}}\hat{\nu}^s(\omega_0)|\hat{\psi}(\mathfrak{p}^j\omega_0)|^2 \leq D. \tag{2.4}$$

To prove the left hand inequality,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle g, \psi_{j,k}\rangle|^2 = T_1 + T_2, \tag{2.5}$$

where

$$T_1 = \sum_{j\geq -M}\sum_{k\in\mathbb{N}_0}|\langle g, \psi_{j,k}\rangle|^2 \ \textit{and}\ T_2 = \sum_{j\leq -M}\sum_{k\in\mathbb{N}_0}|\langle g, \psi_{j,k}\rangle|^2.$$

By condition of frame, $T_1 \geq C - T_2$. Since we have already show that $T_1 = \sum_{j>-M}\hat{\nu}^s(\omega_0)|\hat{\psi}(\mathfrak{p}^{-j}\omega_0)|^2$. So we only need to show that $T_2 \to 0$ as $M \to \infty$. Now, using the fact $\mathscr{S}'(\mathbb{F})$ is dence in $H^s(\mathbb{F})$ in (2.2) and Schwarz's inequality, we have

$$\begin{aligned} T_2 \quad \leq \quad & \sum_{j\geq -M}\sum_{k=0}^{\infty}\left\{\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{f}(\omega)|^2|\hat{\psi}(\mathfrak{p}^j\omega)|^2 d\omega\right\}^{\frac{1}{2}} \\ & \times\left\{\int_{\mathbb{F}}\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(k))|\hat{f}(\omega+\mathfrak{p}^{-j}v(k)))|^2|\hat{\psi}(\mathfrak{p}^j\omega+v(k))|^2 d\omega\right\}^{\frac{1}{2}}. \end{aligned}$$

where $\hat{g} = (\hat{\nu}^{-\frac{s}{2}}\hat{f})$ and $\hat{f} \in \mathscr{S}(\mathbb{F})$
Since $\hat{f} \in \mathscr{S}(\mathbb{F})$, so there exists a characteristic funtion $\varphi_r(\omega-\omega_0)$ of the set $\omega_0 + \mathfrak{P}^r$, where r is some integers. Now $\hat{f}$ can be written as $\hat{f}(\omega) = q^{\frac{r}{2}}\varphi_r(\omega-\omega_0)$. If $\omega + \mathfrak{p}^{-j}v(k) \in \omega_0 + \mathfrak{P}^r$, then $|\mathfrak{p}^{-j}v(k)| \leq q^{-r}$, hence $|v(k)| \leq q^{-r-j}$. Then summation index k is bounded by q^{-r-j}. So using this, we get

$$T_2 \quad \leq \quad q^{-r}\int_{\mathfrak{p}^{-j}\omega_0+\mathfrak{P}^{-j+r}}\hat{\nu}^s(\mathfrak{p}^{-j}\omega)|\hat{\psi}(\omega)|^2 d\omega.$$

Suppose that $\omega_0 \neq 0$. For any $\epsilon > 0$, choose $J < 0$ enough small satisfies the following two inequalities : $q^J < |\omega_0| = q^\rho$ such that $J + \rho < 0$ and $\int_{\mathfrak{P}^{-J-\rho}} \hat{\nu}^s(\mathfrak{p}^{-J}\omega)|\hat{\psi}(\omega)|^2 d\omega < \epsilon$.
We have

$$\mathfrak{p}^{-j}\omega_0 + \mathfrak{P}^{-j+r} \subset \mathfrak{P}^{-J-\rho} \text{ for all } j \leq J. \tag{2.6}$$

Since $|\mathfrak{p}^{-j}\omega_0| = q^j q^\rho \leq q^J q^\rho$ and $\mathfrak{P}^{-j+r} \subset \mathfrak{P}^{-J-\rho}$.
Hence, $T_2 \to 0$ as $j \to -\infty$. Therefore there exists j such that

$$T_2 < \epsilon.$$

Hence we obtain required result. □

3. Sufficient conditions of wavelet frame for $H^s(\mathbb{F})$

To find the sufficient conditions of wavelet frame for $H^s(\mathbb{F})$, we need the following Lemma

Lemma 3.1. *Let g be in $\mathscr{S}(\mathbb{F})$ and $\psi \in H^s(\mathbb{F})$.If $sup\{\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\omega)|^2 : \omega \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\} < +\infty$, then*

$$\sum_{j\in\mathbb{Z}}\sum_{k=0}^{\infty}|\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2 = \int_{\mathbb{F}} \hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\omega)|^2 d\omega + T_2, \tag{3.1}$$

where

$$\begin{aligned} T_2 \;=\; & \sum_{j\in\mathbb{Z}}\int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\big[\sum_{l=1}^{\infty}\hat{\nu}^s(\omega + \mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega + \mathfrak{p}^{-j}v(l)) \\ & \times\hat{\psi}(\mathfrak{p}^j\omega + v(l))\big]d\omega. \end{aligned} \tag{3.2}$$

Then iterated series in (3.2) is absolutely convergent.

Proof. Since $g \in \mathscr{S}(\mathbb{F})$ so the $\sum_{l=0}^{\infty}$ in (3.2) contains only finite non-zero terms. Hence,

$$\begin{aligned} & \sum_{j\in\mathbb{Z}}\int_{\mathbb{F}} \hat{\nu}^s(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\hat{g}(\omega)\big[\sum_{l=0}^{\infty}\hat{\nu}^s(\omega + \mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega + \mathfrak{p}^{-j}v(l))\hat{\psi}(\mathfrak{p}^j\omega + v(l))\big]d\omega \\ = & \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}} \hat{\nu}^s(\omega)\hat{g}(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\hat{\nu}^s(\omega + \mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega + \mathfrak{p}^{-j}v(l))\hat{\psi}(\mathfrak{p}^j\omega + v(l))d\omega. \end{aligned} \tag{3.3}$$

We claim that,

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2=\sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^s(\omega)|\bar{\hat{\psi}}(\mathfrak{p}^j\omega)|^2\hat{\nu}^s(\omega)d\omega+T_2 \tag{3.4}$$

holds for all $g\in\mathscr{S}(\mathbb{F})$. We have

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2=\sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}\hat{\nu}^{2s}(\omega)|\hat{g}(\omega)|^2|\bar{\hat{\psi}}(\mathfrak{p}^j\omega)|^2d\omega+T_2, \tag{3.5}$$

where

$$\begin{aligned}T_2 &= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)\hat{g}(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(l))\\ &\qquad\times\hat{\psi}(\mathfrak{p}^j\omega+v(l))d\omega. \end{aligned}\tag{3.6}$$

By using the condition $sup\{\hat{\nu^s}(\omega)\sum_{j\in\mathbb{Z}}|\bar{\hat{\psi}}(\mathfrak{p}^j\omega)|^2:\omega\in\mathfrak{P}^{-1}\backslash\mathfrak{D}\}<+\infty$
and Levi's Lemma for integral, we get

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2=\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}|\bar{\hat{\psi}}(\mathfrak{p}^j\omega)|^2d\omega+T_2. \tag{3.7}$$

Now, we show that series (3.6) is absolutely convergent.

$$\begin{aligned}|T_2|&\le|\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)\hat{g}(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(l))\hat{\psi}(\mathfrak{p}^j\omega+v(l))|d\omega\\ &\le\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}}\hat{\nu}^{\frac{s}{2}}(\omega)|\hat{g}(\omega)|\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(l))|\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(l))|\frac{1}{2}[\hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2\\ &\qquad+\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(l))|\hat{\psi}(\mathfrak{p}^j\omega+v(l))|^2]d\omega.\end{aligned}$$

$$\begin{aligned}|T_2|\le|\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}}&\hat{\nu}^{\frac{s}{2}}(\mathfrak{p}^{-j}\omega)|\hat{g}(\mathfrak{p}^{-j}\omega)|\hat{\nu}^{\frac{s}{2}}(\mathfrak{p}^{-j}\omega+\mathfrak{p}^{-j}v(l))|\hat{g}(\mathfrak{p}^{-j}\omega+\mathfrak{p}^{-j}v(l))|\\ &\times\hat{\nu}^s(\mathfrak{p}^{-j}\omega)|\hat{\psi}(\omega)|^2d\omega.\end{aligned}\tag{3.8}$$

Since $g\in\mathscr{S}(\mathbb{F})$, there exist a constant $J>0$ such that for all $|j|>J$

$$\hat{g}(\mathfrak{p}^{-j}\omega)\hat{g}(\mathfrak{p}^{-j}\omega+\mathfrak{p}^{-j}v(l))=0. \tag{3.9}$$

On the other hand, for each $|j|>J$, there exist a constant L such that for all $l\ge L$

$$\hat{g}(\mathfrak{p}^{-j}\omega+\mathfrak{p}^{-j}v(l))=0. \tag{3.10}$$

Therefore only finite number of terms of the iterated series in (3.8) are nonzero .

$$|T_2|\le C\|\hat{\nu}^s(.)\hat{g}(.)\|_\infty^2\|\psi\|_{H^s(\mathbb{F})}. \tag{3.11}$$

Hence the T_2 is absolutely convergent. The proof is complete. $\square$

Now using above lemma, we establish sufficient condition of frame for $H^s(\mathbb{F})$. Let

$$\Delta_1 = \text{ess sup}\{\hat{\nu^s}(\omega)\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\omega)|^2 : \omega \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}, \quad (3.12)$$

and

$$\Delta_2 = \text{ess inf}\{\hat{\nu^s}(\omega)\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\omega)|^2 : \omega \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}. \quad (3.13)$$

We set

$$\beta_\psi(v(l)) = Sup\{\sum_{j\in\mathbb{Z}}|h_\psi(v(l),\mathfrak{p}^j\omega)| : \omega \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}, \quad (3.14)$$

where

$$h_\psi(v(l),\omega) = \sum_{l\in\mathbb{N}_0}\hat{\nu}^s(\omega)\hat{\psi}(\mathfrak{p}^{-j}\omega)\overline{\hat{\psi}(\mathfrak{p}^{-j}\omega + v(l))}. \quad (3.15)$$

Suppose that $Q = \{1, 2, 3, 4, ...q - 1\}$ and $q\mathbb{N}_0 = \{qk : k = 0, 1, 2, 3, ...\}$.

Theorem 3.1. *Suppose $\psi \in H^s(\mathbb{F})$ such that*

$$\rho_1(\psi) = \Delta_2 - \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} > 0,$$

$$\rho_2(\psi) = \Delta_1 + \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} < +\infty.$$

Then $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is wavelet frame for $H^s(\mathbb{F})$ with bounds $\rho_1(\psi)$ and $\rho_2(\psi)$.

Proposition 3.2. *For a given $l \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and unique $m \in q\mathbb{N}_0 + Q$ such that $l = q^k m$. Thus we have $\{v(l)\}_{l\in\mathbb{N}} = \{\mathfrak{p}^{-k}v(m)\}_{(k,m)\in\mathbb{N}_0\times\{q\mathbb{N}_0+Q\}}$. Since the last series in equation (3.2) is absolutely convergent.Therefore equation (3.2) become*

$$\begin{aligned} T_2 &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)\hat{g}(\omega)\bar{\hat{\psi}}(\mathfrak{p}^j\omega)[\sum_{l\in\mathbb{N}}\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(l))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(l)) \\ &\qquad \times\hat{\psi}(\mathfrak{p}^j\omega + v(l))]d\omega \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)\hat{g}(\omega)[\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+Q}\bar{\hat{\psi}}(\mathfrak{p}^j\omega)\hat{\nu}^s(\omega+\mathfrak{p}^{-j-k}v(m))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j-k}v(m)) \\
&\qquad \times\bar{\hat{\psi}}(\mathfrak{p}^j\omega+\mathfrak{p}^{-k}v(m)]d\omega \\
&= \int_{\mathbb{F}}\hat{\nu}^s(\omega)\hat{g}(\omega)[\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+Q}\sum_{j\in\mathbb{Z}}\bar{\hat{\psi}}(\mathfrak{p}^{j-k}\omega)\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(m))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(m)) \\
&\qquad \times\hat{\psi}(\mathfrak{p}^{j-k}\omega+\mathfrak{p}^{-k}v(m))]d\omega \\
&= \int_{\mathbb{F}}\hat{\nu}^{\frac{s}{2}}(\omega)\hat{g}(\omega)[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(m))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(m)) \\
&\quad \times\sum_{k\in\mathbb{N}_0}\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(m))\bar{\hat{\psi}}(\mathfrak{p}^{j-k}\omega)\hat{\nu}^{\frac{s}{2}}(\omega)\hat{\psi}(\mathfrak{p}^{-k}(\mathfrak{p}^j\omega+v(m)))]d\omega \\
&= \int_{\mathbb{F}}\hat{\nu}^{\frac{s}{2}}(\omega)\hat{g}(\omega)[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(m))h_\psi(v(m),\mathfrak{p}^j\omega) \\
&\qquad \times\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(m))]d\omega \\
&= \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}\int_{\mathbb{F}}\hat{\nu}^{\frac{s}{2}}(\omega)\hat{g}(\omega)\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(m))\bar{\hat{g}}(\omega+\mathfrak{p}^{-j}v(m)) \\
&\qquad \times h_\psi(v(m),\mathfrak{p}^j\omega)d\omega.
\end{aligned}
$$

We derive further that

$$
\begin{aligned}
|T_2| &\le \int_{\mathbb{F}}\hat{\nu}^{\frac{s}{2}}(\omega)|\hat{g}(\omega)|[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}\hat{\nu}^{\frac{s}{2}}(\omega+\mathfrak{p}^{-j}v(m))|\hat{g}(\omega+\mathfrak{p}^{-j}v(m))| \\
&\qquad \times|h_\psi(v(m),\mathfrak{p}^j\omega)|]d\omega \\
&\le \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}[\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2|h_\psi(v(m),\mathfrak{p}^j\omega)|d\omega]^{\frac{1}{2}} \\
&\qquad \times[\int_{\mathbb{F}}\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(m))|\hat{g}(\omega+\mathfrak{p}^{-j}v(m))|^2|h_\psi(v(m),\mathfrak{p}^j\omega)|d\omega]^{\frac{1}{2}} \\
&\le \sum_{m\in q\mathbb{N}_0+Q}[\sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2|h_\psi(v(m),\mathfrak{p}^j\omega)|d\omega]^{\frac{1}{2}} \\
&\qquad \times[\sum_{j\in\mathbb{Z}}\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2|h_\psi(-v(m),\mathfrak{p}^j\omega)|d\omega]^{\frac{1}{2}} \\
&\le \sum_{m\in q\mathbb{N}_0+Q}[\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\beta_\psi(v(m))d\omega]^{\frac{1}{2}}[\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2|\beta_\psi(-v(m))d\omega]^{\frac{1}{2}}
\end{aligned}
$$

$$= \int_{\mathbb{F}} \hat{\nu}^s(\omega)|\hat{g}(\omega)|^2 d\omega \sum_{m\in q\mathbb{N}_0+Q} [\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}.$$

Now it follows from equation (3.1) in Lemma 3.1 that

$$\int_{\mathbb{F}} \hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\{\sum_{j\in\mathbb{Z}} \hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2 - \sum_{m\in q\mathbb{N}_0+Q} [\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}\}d\omega \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2, \quad (3.16)$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{j,k}\rangle_{H^s(\mathbb{F})}|^2 \leq \int_{\mathbb{F}} \hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\{\sum_{j\in\mathbb{Z}} \hat{\nu}^s(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)|^2 + \sum_{m\in q\mathbb{N}_0+Q} [\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}\}d\omega. \quad (3.17)$$

Taking infimum and suprimum in above two inequality respectively, we get

$$\rho_2(\psi)\|g\|_{H^s(\mathbb{F})} \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{j,k}\rangle|^2 \leq \rho_1(\psi)\|g\|_{H^s(\mathbb{F})}. \quad (3.18)$$

The proof of theorem 3.1 is complete.

Theorem 3.3. *Suppose $\psi \in H^s(\mathbb{F})$ such that*

$$\Delta_3(\psi) = ess\ inf_{\,\omega\in\mathfrak{P}^{-1}\setminus\mathfrak{D}}\{\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathfrak{p}^j\omega)|^2 - \hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} |\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega + v(l))}|\} > 0, \quad (3.19)$$

$$\Delta_4(\psi) = ess\ sup_{\,\omega\in\mathfrak{P}^{-1}\setminus\mathfrak{D}}\{\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega + v(k))}|\} < +\infty. \quad (3.20)$$

Then $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is a wavelet frame for $H^s(\mathbb{F})$ with bounds $\Delta_3(\psi)$ and $\Delta_4(\psi)$.

Proof. We use Lemma 3.1 to calculate T_2 in (3.2) for $g \in \mathscr{S}(\mathbb{F})$ with another way. We first deduce that

$$
\begin{aligned}
|T_2| &= |\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}} \hat{\nu}^s(\omega)\overline{\hat{g}(\omega)}\hat{\psi}(\mathfrak{p}^j\omega)\hat{\nu}^s(\omega+\mathfrak{p}^{-j}v(l))\hat{g}(\omega+\mathfrak{p}^{-j}v(l)) \\
&\qquad \times\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}d\omega| \\
&\leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\{\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^{2s}(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}|d\omega\}^{\frac{1}{2}} \\
&\qquad \times\{\int_{\mathbb{F}}|\hat{g}(\omega+\mathfrak{p}^{-j}v(l))|^2\hat{\nu}^{2s}(\omega+\mathfrak{p}^{-j}v(l)))|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}|d\omega\}^{\frac{1}{2}} \\
&= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\{\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^{2s}(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}|d\omega\}^{\frac{1}{2}} \\
&\qquad \times\{\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^{2s}(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega-v(l))}|d\omega\}^{\frac{1}{2}}.
\end{aligned}
$$

Since $\{v(k):k\in\mathbb{N}\}=\{-v(k):k\in\mathbb{N}_0\}$, we have

$$
|T_2|\leq\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^{2s}(\omega)|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega-v(l))}|d\omega. \tag{3.21}
$$

By Levi Lemma we obtain,

$$
|T_2|\leq\int_{\mathbb{F}}|\hat{g}(\omega)|^2\hat{\nu}^{2s}(\omega)\left\{\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega-v(l))}|\right\}d\omega. \tag{3.22}
$$

Using equation (3.1), we get

$$
\begin{aligned}
&\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\{\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}|\hat{\psi}(\mathfrak{p}^j\omega)|^2-\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}|\}d\omega \\
&\leq\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle|^2,
\end{aligned} \tag{3.23}
$$

and

$$
\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle|^2\leq\int_{\mathbb{F}}\hat{\nu}^s(\omega)|\hat{g}(\omega)|^2\{\hat{\nu}^s(\omega)\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}|\hat{\psi}(\mathfrak{p}^j\omega)\overline{\hat{\psi}(\mathfrak{p}^j\omega+v(l))}|\}d\omega. \tag{3.24}
$$

Taking infimum in (3.23) and supremum in (3.24) , we obtain that

$$
\Delta_3(\psi)\|g\|^2_{H^s(\mathbb{F})}\leq\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0}|\langle g,\psi_{j,k}\rangle|^2\leq\Delta_4(\psi)\|g\|^2_{H^s(\mathbb{F})} \tag{3.25}
$$

hold for all $g \in \mathscr{S}(\mathbb{F})$. The proof of theorem 3.3 is complete. □

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