Bol. Soc. Paran. Mat. (3s.) **v. 39** 3 (2021): 53–66.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.40065

On The Existence of Positive Solutions for a Local Fractional Boundary Value Problem with an Integral Boundary Condition

Asghar Ahmadkhanlu

ABSTRACT: In this work, we are concerned with the fractional differential equation

$$D^{\alpha}_{0+}u(t) + f(t, u(s)) = 0, \quad 1 < \alpha \leq 2$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, subject to the local boundary conditions

$$u(0) = 0, \quad u(1) + \int_0^{\eta} u(t)dt = 0, \quad 0 \leq \eta < 1.$$

We try to obtain the existence of positive solutions by using some fixed point theorems.

Key Words: Fractional Derivatives, Differential Equations, Boundary Value Problems, Fixed point Theorem.

Contents

1 Introduction **53**

2 Preliminaries **55**

3 Green Function **56**

4 Main Results **59**

5 Examples **62**

6 Conclusion **64**

1. Introduction

Fractional derivative, as an extension of ordinary derivative, is a suitable tool for modeling of various physical phenomena, chemical processes and engineering. Indeed, we can obtain numerous applications in viscoelasticity [3,37,38], dynamical processes in self-similar structures [22], bioscience and modeling of neurons in biology [33,23], Earth system dynamics [41] diffusion processes [24], electrochemistry [29], signal processing [30], system control theory [39], and linear time-invariant systems of any order with internal point delays [34]. Furthermore, fractional calculus has been found many applications in classical mechanics [32], and the calculus

2010 *Mathematics Subject Classification:* 34A08, 34B18, 26A33.
Submitted October 18, 2017. Published April 07, 2018

of variations [1], and is a very useful means for obtaining solutions of nonhomogenous linear ordinary and partial differential equations [28].
There are some numerical methods for solving fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [35], Optimal perturbation iteration method [7,36], He's variational iteration method [15,25] , homotopy perturbation method [26,27] and other methods. Since default assumption of all these methods is the existence of solution, the study of the existence and uniqueness of solution or multiplicity of solutions of initial and boundary value problem, including fractional differential equations, is important too in theoretical arguments. Remarkable researches in concern with the existence and multiplicity of positive solutions for nonlinear fractional boundary value problems have been done using fixed point theorems up to now (see [2,11,12,13,16,17,21,24,34,40,42]).

Bai and Lu [4] and Zhang [42], investigated the existence of positive solutions for equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1.1}$$

with one of the boundary conditions

$$u(0) = u(1) = 0 \tag{1.2}$$

$$u(0) + u'(0) = u(1) + u'(1) = 0, \tag{1.3}$$

respectively. Bai [5] obtained existence results of positive solution for the following nonlinear fractional boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1.4}$$

$$u(0) = 0, \quad \beta u(\eta) = u(1), \tag{1.5}$$

by the use of fixed point index theory. Wang, et al., [40], by using the lower and upper solutions method and fixed point theorem on cone, investigated the existence and uniqueness of solution for the problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1.6}$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s) ds. \tag{1.7}$$

Motivated by the above works, we study the existence and multiplicity of positive solutions of the following boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1.8}$$

$$u(0) = 0, \quad u(1) + \int_0^{\eta} u(t) dt = 0, \quad 0 < \eta \leq 1 \tag{1.9}$$

where $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. We obtain necessary and sufficient conditions for existence of positive solutions of the problem. The existence of integral

term in the boundary condition makes it more complicated to treat with the problem. Indeed calculating of the Green function and its properties have encountered complexity.
We organize the rest of this paper as follows: In Section 2, we present some basic definitions and conventions. Several preliminary facts and properties of fractional calculus are also presented there. In Section 3, we derive the corresponding Green function named by fractional Green function and we give some properties of the fractional Green function. In section 4, by using some fixed point theorems on cones, existence and multiplicity of positive solutions are obtained. Three examples demonstrate the applications of our results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by (the Abel-integral operator)*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \tag{2.1}$$

provided that the integral exist.

Definition 2.2. *The fractional derivative (in the sense of Reimann-Liouville) of order $0 < \alpha < 1$ of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as the left inverse of the fractional integral of f*

$$D^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t) \tag{2.2}$$

That is

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \tag{2.3}$$

provided that the right side exists.

Lemma 2.3. *Assume $u \in C(0,1) \cap L((0,1))$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L((0,1))$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) dt = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, n$. where $n = [\alpha] + 1$.

For the theory and applications of fractional integrals and fractional derivatives we refer the reader to [18,31].

Theorem 2.4. *([20]) Let $\mathcal{B}$ be a Banach space and let $P \subset \mathcal{B}$ be cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. Let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$

Then T *has a fixed point in* $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.5. *([21]) Let* P *be a cone in a real Banach space* $\mathcal{B}$, $P_c = \{x \in P : \|x\| < C\}, \theta$ *is a nonnegative continuous concave functional on* P *such that* $\theta(x) \leq \|x\|$, *for all* $C \in \overline{P}_c$, *and* $P(\theta, b, d) = \{x \in P : b \leq \theta(x), \|x\| \leq d\}$ *Suppose that* $T : \overline{P}_c \to \overline{P}_c$ *is completely continuous and there exists positive constant* $0 < a < b < d \leq c$ *such that*

(i) $\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset$ *and* $\theta(Tx) > b$ *for* $x \in P(\theta, b, d)$,

(ii) $\|Tx\| < a$ *for* $x \in \overline{P}_a$,

(iii) $\theta(Tx) > b$ *for* $x \in P(\theta, b, c)$ *with* $\|Tx| > d$.

Then T *has at least three fixed points* x_1, x_2 *and* x_3 *with*
$\|x_1\| < a, \qquad b < \theta(x_2), \qquad a < \|x_3\|$ *with* $\theta(x_3) < b$.

3. Green Function

Lemma 3.1. *Let* $g(t) \in L([0,1])$ *and* $1 < \alpha \leq 2$, *the uniqe solution of*

$$\begin{array}{ll} D_{0+}^{\alpha} u(t) + g(t) = 0, & 0 < t < 1 \\ u(0) = 0, \quad u(1) + \int_0^{\eta} u(t)dt = 0, & 0 \leq \eta \leq 1 \end{array} \tag{3.1}$$

is

$$u(t) = \int_0^1 G(t,s)g(s)ds$$

where

$$G(t,s) = \begin{cases} \dfrac{\alpha t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} + t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & 0 \leq s \leq \eta \leq t \leq 1 \\ \dfrac{\alpha t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & 0 \leq \eta \leq t \leq s \leq 1 \\ \dfrac{\alpha t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} + t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & 0 \leq s \leq t \leq \eta \leq 1 \\ \dfrac{\alpha t^{\alpha-1}(1-s)^{\alpha-1} + t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & 0 \leq t \leq s \leq \eta \leq 1. \end{cases} \tag{3.2}$$

Proof. In view of Lemma 2.3, the fractional differential equation in (3.1) is equivalent to the integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}, \tag{3.3}$$

for some $C_1, C_2 \in \mathbb{R}$. From the boundary condition $u(0) = 0$, we have $C_2 = 0$. In view of the second boundary condition, we conclude that

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds + C_1 = -\int_0^\eta u(t) dt. \tag{3.4}$$

So

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds - \int_0^\eta u(t) dt. \tag{3.5}$$

Thus

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds - t^{\alpha-1} \int_0^\eta u(s) ds. \tag{3.6}$$

By integrating from both side of relation above in $[0, \eta]$, we have

$$\begin{aligned} \int_0^\eta u(s) ds \quad = \quad & -\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^\tau (\tau - s)^{\alpha-1} g(s) ds d\tau \\ & + \frac{1}{\Gamma(\alpha)} \int_0^\eta \tau^{\alpha-1} \int_0^1 (1-s)^{\alpha-1} g(s) ds d\tau \\ & - \int_0^\eta \tau^{\alpha-1} \int_0^\eta u(s) ds d\tau \end{aligned} \tag{3.7}$$

By use of the Fubini's theorem over the first integral we have

$$\begin{aligned} \int_0^\eta u(s) ds \quad = \quad & -\frac{1}{\Gamma(\alpha)} \int_0^\eta \int_s^\eta (\tau - s)^{\alpha-1} g(s) d\tau ds \\ & + \frac{\eta^\alpha}{\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds d\tau \\ & - \frac{\eta^\alpha}{\alpha} \int_0^\eta u(s) ds d\tau \\ = \quad & -\frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - s)^\alpha g(s) ds + \frac{\eta^\alpha}{\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds d\tau \\ & - \frac{\eta^\alpha}{\alpha} \int_0^\eta u(s) ds. \end{aligned} \tag{3.8}$$

So

$$\int_0^\eta u(s) ds = \frac{-1}{(\alpha + \eta^\alpha)\Gamma(\alpha)} \int_0^\eta (\eta - s)^\alpha g(s) ds + \frac{\eta^\alpha}{(\alpha + \eta^\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds. \tag{3.9}$$

Now

$$\begin{aligned} u(t) \quad = \quad & \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds + \frac{\alpha t^{\alpha-1}}{(\alpha + \eta^\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds \\ & + \frac{t^{\alpha-1}}{(\alpha + \eta^\alpha)\Gamma(\alpha)} \int_0^\eta (\eta - s)^\alpha g(s) ds. \end{aligned} \tag{3.10}$$

This complete the proof. □

Lemma 3.2. *The function $G(t,s)$ in Lemma 3.1 satisfies the following conditions.*

(i) $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$,

(ii) $G(t,s)>0$, *for any* $s,t\in(0,1)$,

(iii) $G(t,s)\le G(s,s)$ *for* $s,t\in[0,1]$,

(iv) there exists a positive $\gamma(s)\in C(0,1)$, *such that*

$$\min_{\eta\le t\le 1} G(t,s)\ge \gamma(s)\max_{0\le t\le 1} G(t,s)=\gamma(s)G(s,s)\quad \forall\quad 0<s<1$$

Proof. Let us assume

$$\begin{aligned}
g_1(t,s)&=\frac{\alpha t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}+t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, && 0\le s\le\eta\le t\le 1\\
g_2(t,s)&=\frac{\alpha t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, && 0\le \eta\le t\le s\le 1\\
g_3(t,s)&=\frac{\alpha t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}+t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, && 0\le s\le t\le \eta\le 1\\
g_4(t,s)&=\frac{\alpha t^{\alpha-1}(1-s)^{\alpha-1}+t^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, && 0\le t\le s\le \eta\le 1.
\end{aligned}\tag{3.11}$$

One can check easily that (i) is true. So we prove that (ii) holds. It is clear that for all $0\le s,t\le 1$, g_2 and g_4 are positive. We show that $g_1(t,s)>0, 0\le s\le\eta\le t\le 1$. The same argument can be used for $g_3(t,s)$. Let $h(t,s)=\alpha(1-s)^{\alpha-1}-(1-\frac{s}{t})^{\alpha-1}+(\eta-s)^{\alpha}$. Then $g_1(t,s)=\frac{t^{\alpha-1}h(t,s)}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}$. Since $\frac{\partial h(t,s)}{\partial t}=-(\alpha-1)(\frac{s}{t^2})(1-\frac{s}{t})^{\alpha-2}\le 0$, $h(t,s)$ is decreasing on $[s,1]$ with respect to t. It is enough to show that $h(1,s)>0$. By simple calculation we can deduce that $h(1,s)=(\alpha-1)(1-s)^{\alpha-1}+(\eta-s)^{\alpha}$ is positive. That is $g_1(t,s)$ is positive for $s,t\in(0,1)$.
Next we show that (iii) holds. It is clear that $g_2(t,s)$ and $g_4(t,s)$ are increasing with respect to t on $[0,s]$ and by the same argument in (ii) one can show that $g_1(t,s)$ and $g_3(t,s)$ are decreasing in $[s,1]$. Thus $G(t,s)$ is increasing with respect to t for $t\le s$ and $G(t,s)$ is decreasing with respect to t for $t\ge s$. Hence $G(t,s)\le G(s,s)$ for $s,t\in[0,1]$.
Finally we show that (iv) holds. In view of $G(t,s)$ in (3.2), we have

$$\begin{aligned}
\min_{\eta\le t\le 1} G(t,s) &= \begin{cases}\min_{\eta\le t\le 1}\{g_1(t,s),g_3(t,s)\}, & 0\le s\le\eta,\\ \min_{\eta\le t\le 1}\{g_2(t,s),g_4(t,s)\}, & \eta\le s\le 1\end{cases}\\
&= \begin{cases} g_1(\eta,s), & 0\le s\le\eta,\\ g_2(\eta,s), & \eta\le s\le 1\end{cases}
\end{aligned}\tag{3.12}$$

Let

$$\gamma(s)=\begin{cases}\frac{g_1(\eta,s)}{G(s,s)}, & 0\le s\le\eta,\\ \frac{g_2(\eta,s)}{G(s,s)}, & \eta\le s\le 1,\end{cases}\tag{3.13}$$

where

$$G(s,s)=\begin{cases}\frac{\alpha s^{\alpha-1}(1-s)^{\alpha-1}+s^{\alpha-1}(\eta-s)^{\alpha}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & 0\le s\le \eta,\\ \frac{\alpha s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(\alpha+\eta^{\alpha})}, & \eta\le s\le 1.\end{cases} \tag{3.14}$$

Then

$$\min_{\eta\le t\le 1} G(t,s)\ge \gamma(s)\max_{0\le t\le 1} G(t,s)=\gamma(s)G(s,s), \qquad \text{for} \quad 0<s<1. \tag{3.15}$$

This completes the proof. □

4. Main Results

In this section, we give some existence and multiplicity theorem for the problem (1.8)-(1.9). For this, we impose some conditions on f which allow us to apply Theorem 2.4 and 2.5. Let $E=C[0,1]$ be endowed with the maximum norm, $|u|=\max_{0\le t\le 1}|u(t)|$ and ordering $u\le v$ if $u(t)\le v(t)$ for all $t\in[0,1]$. Also let $P\subset E$ be $u\in E$ such that $u(t)\ge 0$. We define the nonnegative continuous concave functional θ on the cone P by

$$\theta(u)=\min_{\eta\le t\le 1}|u(t)|.$$

Throughout this section, we may use the following conditions.

(H1) (i) $f(t,u)$ is Lebesque measurable with respect to t on $[0,1]$.
(ii) $f(t,u)$ is continuous with respect to u on $[0,\infty)$

(H2) $f(t,u)$ is continuous on $[0,1]\times[0,\infty)$.

Theorem 4.1. *Assume that the condition (H1) holds and there exists a real-valued function $h(t)\in[0,1]$ such that for almost every $t\in[0,1]$ and all $u,v\in[0,\infty)$, we have $|f(t,u)-f(t,v)|\le h(t)|u-v|$. If $0<\int_0^1 G(s,s)h(s)ds<1$, then there exists a unique solution of FBVP (1.8)-(1.9) on $[0,1]$.*

Proof. Consider the operator $T:P\to P$ defined as

$$Tu(t):=\int_0^1 G(t,s)f(s,u(s))ds. \tag{4.1}$$

We show that T is a contraction mapping. In fact, for any $u,v\in P$, we have

$$\begin{aligned}|Tu(t)-Tv(t)| &= \left|\int_0^1 G(t,s)[f(s,u(s))-f(s,v(s))]ds\right|\\ &\le \int_0^1 G(t,s)|f(s,u(s))-f(s,v(s))|ds\\ &\le \int_0^1 G(t,s)h(s)|u(s)-v(s)|ds\\ &\le \int_0^1 G(t,s)h(s)ds\|u-v\|,\end{aligned}$$

That is

$$\|Tu - Tv\| \leq k\|u - v\|, \tag{4.2}$$

where $k = \int_0^1 G(t,s)h(s)ds \in (0,1)$. By the Banach contraction mapping principle, T has unique fixed point which is a solution of FBVP (1.8)-(1.9). This complete the proof. □

Lemma 4.2. *Assume that (H2) holds. Let $T : P \to E$ be the operator defined by*

$$Tu(t) := \int_0^1 G(t,s)f(s,u(s))ds, \tag{4.3}$$

Then $T : P \to P$ is completely continuous

Proof. Since $f(t,u)$ and $G(t,s)$ are continuous and nonnegative, we deduce that $T : P \to P$ is continuous. Now let $\Omega \subset P$ be bounded, that's mean there exists a positive constant K such that $\|u\| \leq K$, for all $u \in \Omega$. Let $M = \max_{0\leq t\leq 1, 0\leq u\leq K} |f(t,u)| + 1$. Then for all $u \in \Omega$, we have

$$|Tu(t)| \leq \int_0^1 G(t,s)f(s,u(s))ds \leq M\int_0^1 G(s,s)ds. \tag{4.4}$$

Hence $T(\Omega)$ is bounded.
For each $u \in \Omega$ and for all $t_1, t_2 \in [0,1]$ satisfy $t_1 < t_2$, we have

$$\begin{aligned}
& |Tu(t_2) - Tu(t_1)| \\
= & \left|\int_0^1 G(t_2,s)f(s,u(s))ds - \int_0^1 G(t_1,s)f(s,u(s))ds\right| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]f(s,u(s))ds \\
& + \frac{\alpha(t_2^{\alpha-1} - t_1^{\alpha-1})}{(\alpha+\eta^\alpha)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f(s,u(s))ds \\
& + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{(\alpha+\eta^\alpha)\Gamma(\alpha)}\int_0^\eta (\eta - s)^{\alpha}f(s,u(s))ds \\
\leq & \frac{M(\alpha+1+\eta^{\alpha+1})}{(\alpha+\eta^\alpha)\Gamma(\alpha)(\alpha+1)}(t_2^{\alpha-1} - t_1^{\alpha-1})
\end{aligned}$$

Since $t^{\alpha-1}$ is uniformly continuous when $t \in [0,1]$ and $1 < \alpha \leq 2$, it is easy to show that $T(\Omega)$ is equicontinuous. As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Using continuity of the Arzela-Ascoli Theorem we deduce that $\overline{T(\Omega)}$ is compact. That is, $T : P \to P$ is completely continuous. □

Our second result is based on Theorem 2.4. Let

$$M = \left(\int_0^1 G(s,s)ds\right)^{-1}, \qquad N = \left(\int_\eta^1 \gamma(s)G(s,s)ds\right)^{-1}$$

Theorem 4.3. *Let (H2) holds and assume there exists two positive constant $R > r > 0$ such that*

(A1) $f(t,u) \leq MR$, *for* $(t,u) \in [0,1] \times [0,R]$

(A2) $f(t,u) \geq Nr$, *for* $(t,u) \in [0,1] \times [0,r]$

Then FBVP (1.8)-(1.9) has at least one positive solution u such that $r_1 \leq \|u\| \leq R$.

Proof. By Lemma 4.2, $T : P \to P$ is completely continuous. We apply Theorem 2.4. Our proof will be continuoud in two steps.
Step 1. Let $\Omega_2 := \{u \in P : \|u\| < R\}$ For $u \in \partial\Omega_2$, we have $0 \leq u(t) \leq R$ for all $t \in [0,1]$. From (A1) for $t \in [0,1]$, we have

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)f(s,u(s))ds \leq MR \int_0^1 G(s,s)ds = R = \|u\|.$$

Step 2. Let $\Omega_1 := \{u \in P : \|u\| < r\}$. For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq r$ for all $t \in [0,1]$. By (A2) for $t \in [\eta,1]$, there is

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \geq \int_0^1 \gamma(s)G(s,s)f(s,u(s))ds \\ &\geq Nr \int_\eta^1 \gamma(s)G(s,s)ds = r = \|u\|. \end{aligned}$$

So for all $u \in \partial\Omega_1$ we have

$$\|Tu\| \geq \|u\|.$$

Therefore (ii) of Theorem 2.4 complete the proof. □

Theorem 4.4. *Let (H2) holds and there exists constants $0 < a < b < c$ such that the following assumptions hold*

(A'1) $f(t,u) < Ma$, *for* $(t,u) \in [0,1] \times [0,a]$;

(A'2) $f(t,u) \geq Nb$, *for* $(t,u) \in [\eta,1] \times [b,c]$;

(A'3) $f(t,u) < Mc$, *for* $(t,u) \in [0,1] \times [0,c]$;

Then the boundary value problem (1.8)-(1.9) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{0 \leq t \leq 1} |u_1(t)| < a, \quad b < \min_{\eta \leq t \leq 1} |u_2(t)|, \quad a < \max_{0 \leq t \leq 1} |u_3(t)|, \quad \min_{\eta \leq t \leq 1} |u_3(t)| < b.$$

Proof. We show that the conditions of Theorem 2.5 hold. If $u \in \bar{P}_c$, then $\|u\| \leq c$. From Assumption (A'3) we have $f(t,u(t)) \leq Mc$ for all $0 \leq t \leq 1$. Consequently

$$\begin{aligned} \|Tu\| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t,s)f(s,u(s))ds \right| \leq G(s,s)f(s,u(s))ds \\ &\leq \int_0^1 G(s,s)Mcds \leq c. \end{aligned}$$

Hence, $T : \bar{P}_c \to \bar{P}_c$. In the same way, if $u \in \bar{P}_a$, then assumption (A'1) $f(t, u(t)) < Ma, 0 \le t \le 1$. Therefore, condition (ii) of Theorem 2.5 is satisfied. To check condition (i) of Theorem 2.5, we choose $u(t) = (b+c)/2, 0 \le t \le 1$. it is easy to see that $u(t) = (b+c)/2 \in P(\theta, b, c), \theta(u) = \theta((b+c)/2) > b$, consequently, $\{u \in P(\theta, b, c) : \theta(u) > b\} \neq \emptyset$, then $b \le u(t) \le c$ for $\eta \le t \le 1$. Now from assumption (A'2), we have $f(t, u(t)) \ge Nb$ for $\eta \le t \le 1$. So

$$\begin{aligned} \theta(Tu) &= \min_{\eta \le t \le 1} |(Tu)(t)| \ge \int_0^1 \gamma(s) G(s,s) f(s, u(s)) ds \\ &> \int_\eta^1 \gamma(s) G(s,s) Nb ds = b, \end{aligned}$$

consequently

$$\theta(Tu) > b, \quad \text{for all } u \in P(\theta, b, c)$$

This shows that condition (i) of Theorem 2.5 is satisfied. Hence by Theorem 2.5, the FBVP (1.8)-(1.9) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{0 \le t \le 1} |u_1(t)| < a, \quad b < \min_{\eta \le t \le 1} |u_2(t)|, \quad a < \max_{0 \le t \le 1} |u_3(t)|, \quad \min_{\eta \le t \le 1} |u_3(t)| < b.$$

This complete the proof. □

5. Examples

Example 5.1.

Consider the fractional boundary value problem

$$\begin{aligned} & D_{0^+}^{\frac{3}{2}} u(t) + \frac{e^t u}{(1+e^t)(1+u)} + \sin^2 \pi t + 1 = 0, \quad 0 < t < 1 \\ & u(0) = 0, \quad u(1) + \int_0^{\frac{1}{2}} u(t) dt = 0. \end{aligned} \tag{5.1}$$

where $f(t, u) = \frac{e^t u}{(1+e^t)(1+u)} + \sin^2 \pi t + 1 \in [0, 1] \times [0, \infty), h(t) = \frac{e^t}{1+e^t}$. for all $u, v \in [0, \infty), t \in [0, 1]$, we have

$$\begin{aligned} |f(t, u) - f(t, v)| &\le \frac{e^t}{1+e^t} \left| \frac{u}{(1+u)} - \frac{v}{(1+v)} \right| \\ &\le \frac{e^t}{1+e^t} \frac{|u-v|}{(1+u)(1+v)} \le \frac{e^t}{1+e^t} |u-v|. \end{aligned}$$

Now by simple calculation, we have

$$\begin{aligned}\int_0^1 G(s,s)h(s)ds &\leq \frac{\frac{3}{2}}{\Gamma(\frac{3}{2})(\frac{3}{2}+(\frac{1}{2})^{\frac{3}{2}})}\left(\int_0^1 s^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\frac{e^t}{1+e^t}ds\right)\\ &+ \frac{\frac{3}{2}}{\Gamma(\frac{3}{2})(\frac{3}{2}+(\frac{1}{2})^{\frac{3}{2}})}\left(\int_0^{\frac{1}{2}} s^{\frac{1}{2}}(\frac{1}{2}-s)^{\frac{3}{2}}\frac{e^t}{1+e^t}ds\right)\\ &\leq \frac{\frac{3}{2}}{\Gamma(\frac{3}{2})(\frac{3}{2}+(\frac{1}{2})^{\frac{3}{2}})}\left(\int_0^1 s^{\frac{1}{2}}(1-s)^{\frac{1}{2}}ds+\int_0^{\frac{1}{2}} s^{\frac{1}{2}}(\frac{1}{2}-s)^{\frac{3}{2}}ds\right)\\ &\cong 0.6637.\end{aligned}$$

Thus all the assumption in Theorem 4.1 are satisfied, our results can be applied to the problem (5.1)

Example 5.2.

Consider the problem

$$\begin{aligned}&D_{0+}^{\frac{3}{2}}u(t)+\frac{1}{4}(u^2(t)+\sin^2\pi t)+1=0\\ &u(0)=0,\quad u(1)+\int_0^{\frac{1}{2}}u(t)dt=0.\end{aligned} \tag{5.2}$$

Let $f(t,u)=\frac{1}{4}(u^2(t)+\sin^2 t)+1$, $(t,u)\in[0,1]\times[0,\infty)$. In view of Example 5.1, $\int_0^1 G(s,s)ds \cong 0.6637$ and

$$\begin{aligned}\int_\eta^1 \gamma(s)G(s,s)ds &= \int_{\frac{1}{2}}^1 g_2(\frac{1}{2},s)ds = \frac{\frac{3}{2}}{\Gamma(\frac{3}{2})(\frac{3}{2}+(\frac{1}{2})^{\frac{3}{2}})}\int_{\frac{1}{2}}^1(\frac{1}{2})^{\frac{1}{2}}(1-s)^{\frac{1}{2}}ds\\ &= \frac{12}{36+6\sqrt{2}}.\end{aligned}$$

Hence $M=1.5067....$ and $N=3.7071.....$ Now by choosing $r=0.25$ and $R=1$, we have

$$\begin{aligned}f(t,u)&=\frac{1}{4}(u^2(t)+\sin^2 t)+1\leq 1.5\leq MR,\quad (t,u)\in[0,1]\times[0,R],\\ f(t,u)&=\frac{1}{4}(u^2(t)+\sin^2 t)+1\geq 1\geq Nr,\quad (t,u)\in[0,1]\times[0,r]\end{aligned}$$

Hence all conditions of Theorem 4.3 are satisfied consequently FBVP (5.2) has at least one positive solution u such that $\frac{1}{4}\leq\|u\|\leq 1$.

Example 5.3.

Consider the problem

$$\begin{aligned}&D_{0+}^{\frac{3}{2}}u(t)+f(t,u(t))=0\\ &u(0)=0,\quad u(1)+\int_0^{\frac{1}{2}}u(t)dt=0,\end{aligned} \tag{5.3}$$

where

$$f(t,u) = \begin{cases} 4u^2 + \frac{\sin^2 \pi t}{10}, & (t,u) \in [0,1] \times [0,1], \\ u + 3 + \frac{\sin^2 \pi t}{10}, & (t,u) \in [0,1] \times (1,\infty). \end{cases}$$

By considering M and N as in Example 5.2 and choosing $a = \frac{1}{4}, b = 1, c = 8$, we have

$$f(t,u) = 4u^2 + \frac{\sin^2 \pi t}{10} \le 0.35 \le Ma = 0.3767...., \quad (t,u) \in [0,1] \times [0,a],$$

$$f(t,u) = u + 3 + \frac{\sin^2 \pi t}{10} \le 11.1 \le Mc = 12.0536...., \quad (t,u) \in [0,1] \times [0,8],$$

$$f(t,u) = u + 3 + \frac{\sin^2 \pi t}{10} \ge 4 \ge Nb = 3.7071...., \quad (t,u) \in [\frac{1}{2},1] \times [1,8]$$

Hence all conditions of Theorem 4.4 are satisfied consequently, FBVP (5.3) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < \frac{1}{4}, \quad 1 < \theta(u_2), \quad \frac{1}{4} < \|u_3\|.$$

6. Conclusion

In this paper, by adding an integral term containing a parameter to one of boundary conditions a new problem was defined. Using fixed point theory, enabled us to prove the existence of positive solutions for this problem. Findings were applied on some illustrative examples.

References

1. Agrawal, O. P., *Formulation of Euler-Lagrange equations for fractional variational problems*, Journal of Mathematical Analysis and Applications, 272(1), 368-379, (2002).

2. Ahmadkhanlu A, Jahanshahi M., *On the Existence and Uniqueness of Solution of Initial Value problem for Fractional order Differential Equations on Time scales*, Bull. Ira. Math. Soc. 38(1), 241-252, (2012).

3. Bagley, R. I., *A theoretical basis for the application of fractional calculus to viscoelasticity*, Journal of Rheology, 27(3), 201-210, (1983).

4. Bai, Z., Lu, H.S., *Positive solutions of boundary value problems of nonlinear fractional differential equation*, J. Math. Anal. Appl.,311, 495-505, (2005).

5. Bai, Z., On positive solutions of a nonlocal fractional boundary value problem, Nonlinear Anal., 72, 916-924, (2010).

6. Bayin, S., *Mathematical Methods in Science and Engineering*. United States: John Wiley & Sons, Inc., (2006).

7. Bildik, N. Deniz, S..*A new efficient method for solving delay differential equations and a comparison with other methods* Eur. Phys. J. Plus 132(51), (2017).

8. Diethelm, .K, Ford, N. *Analysis of fractional differential equations*, J. Math. Anal. Appl., 265, 229-248, (2002).

9. Douglas, J. F., *Some applications of fractional calculus to polymer science*, Advances in chemical physics, John Wiley & Sons, Inc., 102, 121-191, (1997).

10. Fellah, Z. E. A, Depollier, C., *Application of fractional calculus to the sound waves propagation in rigid porous materials:Validation via ultrasonic measurement.*, Acta Acustica, 88, 34-39,(2002).

11. Furati, K. M, Tata,r N., *An Existence Result for a Nonlocal Fractional Differential Problem,* J. Frac. Calculus., 26, 43-51, (2004).

12. Hadid S. B. Local and global exitence theorems on differential equations of non-integer order., *J. Fract. Calculus*, **7**: 101-105, (1995).

13. Hadid, S. B., Masaedeh, B., Momani, S., *On the exitence of maximal and minimal solutions of differential equations of non-integer order.*, J. Fract. Calculus, 9, 41-44, (1996).

14. Hilfer, R., *Applications of Fractional Calculus in Physics*, WorldScientific Publishing Co., (2000).

15. Inc, M., *The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method*, J.Math. Anal. Appl, 345, 476-484, (2008).

16. Jahanshahi, M., A. Ahmadkhanlu, *On Well-Posed of Boundary Value Problems Including Fractional Order Differential Equation*, Bull. South. Asi. Bull. Math. Soc., 38, 53-59, (2014).

17. Kilbas, A. A., Marzan, A.,*Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, J. Diff. Equa., 41, 84-89, (2005).

18. Kilbas, A. A., Srivastava ,H. M., Trujillo, J. J.m *Theory And Applications of Fractional Differential Equations* North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, (2006).

19. Kilbas, A. A.,Marichev, O. I., Samko, S. G., *Fractional Integral And Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, (1993).

20. Krasnosel'skii, M. A., *Topological Methods in the Theory of Nonlinear Integral Equations* (A.H. Armstrong, Trans.), Pergamon, Elmsford, (1964).

21. Leggett, R.W., Williams, L.R., *Multiple positive fixed points of nonlinear operators on ordered Banach space*, Indiana Univ. Math. J., 28, 673-688, (1979).

22. Mainardi, F., *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, New York, NY, USA, (1997).

23. Magin, R., *Fractional calculus in bioengineering*, Critical Reviews in Biomedical Engineering 32(1), 1-104, (2004).

24. Metzler, R., Klafter, J., *Boundary value problems for fractional diffusion equations*, Physica A, 278(1-2), 107-125, (2000).

25. Momani, S., Odibat, Z., *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A, 355, 271-279, (2006).

26. Momani, S., Odibat, Z., *Homotopy perturbation method for nonlinear partial differential equations of fractional order*, Phys. Lett. A, 365, 345-350, (2007).

27. Odibat, Z., Momani, S., *Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order*, Chaos, Solitons and Fractals, 36, 167-174, (2008).

28. Nishimoto, S. Tu, K., Jaw, S., Lin, S. D., *Applications of fractional calculus to ordinary and partial differential equations of the second order,* Hiroshima Mathematical Journal, 23(1), 63-77, (1993).

29. Oldham, K. B. , *Fractional differential equations in electrochemistry*, Advances in Engineering Software, 41(1), 9-12, (2010).

30. Ortigueira, M., *Special issue on fractional signal processing and applications*, Signal Processing, 83(11), 2285-2480, (2003).

31. Podlubny, I., *Fractional Differential Equations* Academic Press, (1999).

32. Rabei,E. M., Alhalholy, T. S., Rousan, A., *Potentials of arbitrary forceswith fractional derivatives,* International Journal of Modern Physics A, 19(17-18), 3083-3092, (2004).

33. Robinson, D. A., *The use of control systems analysis in neurophysiology of eye movements*, Ann. Rev. Neurosci., 4, 462–503, (1981).

34. De la Sen, M., *Positivity and stability of the solutions of Caputo fractional linear time-invariant systems of any order with internal point delays*, Abstract and Applied Analysis, 2011, Article ID 161246, 25 pages, 2011.

35. Deniz, S., Bildik, N., *Application of Adomian decomposition method for singularly perturbed fourth order boundary value problems*, AIP Conf. Proc., 1738, 290017 (2016).

36. Deniz, S., Bildik, N.,*A new analytical technique for solving Lane - Emden type equations arising in astrophysics*, Bull. Belg. Math. Soc. Simon Stevin 24(2), (2017), 305-320.

37. Sorrentinos, G., *Fractional derivative linear models for describing the viscoelastic dynamic behavior of polymeric beams*, Proceedings of the IMAS Conference and Exposition on Structural Dynamics, St. Louis, Mo, USA, (2006).

38. Sorrentinos, G., *Analytic modeling and experimental identification of viscoelastic mechanical systems*, Advances in Fractional Calculus, Springer, (2007).

39. Vinagre, B. M.,Podlubny, I., Hernandez, A., Feliu, V. *Some approximations of fractional order operators used in control theory and applications*, Fractional Calculus & Applied Analysis, 3(3), 231-248, (2000).

40. Wang, G., Liu, S.,Agarwal, R. P., Zhang, L., *Positive solutions of integral boundary value problem involving Riemann-Liouville fractional derivative*, Journal of Fractional Calculus and Applications, 4(2), 312-321, (2013).

41. Zhang, Y., Sun, H. G., Stowell, H. H., Zayernouri, M., Hansen S. E., *A review of applications of fractional calculus in Earth system dynamics* Chaos, Solitons and Fractals, (in Appear).

42. Zhang, S.Q., *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Electr. J. Differential Equations 2006, 1-12, (2006).

Asghar ahmadkhanlu,
Department of Mathematics,
Azarbaijan Shahid Madani University,
Tabriz-Iran.
E-mail address: ahmadkhanlu@azaruniv.ac.ir