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The Generators of 3-class Group of Some Fields of Degree 6 Over $\mathbb{Q}$

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abstract:
Let $\mathrm{k} = \mathbb{Q}\left(\sqrt[3]{p}, \zeta_3\right)$, where p is a prime number such that $p \equiv 1 \pmod 9$, and let $C_{\mathrm{k},3}$ be the 3-component of the class group of k. In [6], Frank Gerth III proves a conjecture made by Calegari and Emerton [2] which gives necessary and sufficient conditions for $C_{\mathrm{k},3}$ to be of rank two. The purpose of the present work is to determine generators of $C_{\mathrm{k},3}$, whenever it is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Key Words: Pure cubic fields, 3-class groups, Generators.

Contents

1. Introduction

Let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ be a pure cubic field, where p is a prime number such that $p \equiv 1 \pmod 9$. We denote by $\zeta_3 = -1/2 + i\sqrt{3}/2$ the normalized primitive third roots of unity, $\mathrm{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ the normal closure of Γ and $C_{\mathrm{k},3}$ the 3-component of the class group of k.
Assuming 9 divides exactly the 3-class number of Γ. Then $C_{\mathrm{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ if and only if $u = 1$, where u is an index of units that will be defined in the notations below. In this paper, we will determine the generators of $C_{\mathrm{k},3}$ when $C_{\mathrm{k},3}$ is of type $(9, 3)$ and 3 is not a cubic residue modulo p. We spot that Calegari and Emerton ([2, Lemma 5.11]) proved that the rank of the 3-class group of $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, with $p \equiv 1 \pmod 9$, is equal to two if 9 divides the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$. Moreover, in his work [6, Theorem 1, p.471], Frank Gerth III proves that the converse to Calegari-Emerton's result is also true. The present work can be viewed as a continuation of the works [2] and [6] .
After reviewing some basic properties of the norm residue symbols and prime factorization in the normal closure of a pure cubic field that will be needed later, we will establish in section 3 some preliminary results of the 3-class group $C_{\mathrm{k},3}$. Using

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this, we arrive to determine the generators of 3-class groups $C_{\mathrm{k},3}$ of type $(9,3)$. All the study cases are illustrated by numerical examples and summarized in tables in section 4. The usual notations on which the work is based is as follows:

- $\Gamma = \mathbb{Q}(\sqrt[3]{d})$: a pure cubic field, where d is a cube-free natural number;
- $\mathrm{k}_0 = \mathbb{Q}(\zeta_3)$: the third cyclotomic field ;
- $\mathrm{k} = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$: the normal closure of the pure cubic field Γ;
- $u = [E_\mathrm{k} : E_0]$: the index of the sub-group E_0 generated by the units of intermediate fields of the extension $\mathrm{k}/\mathbb{Q}$ in E_k the group of units of k;
- $\lambda = 1 - \zeta_3$ prime integer of k_0;
- $\langle\tau\rangle = \mathrm{Gal}\,(\mathrm{k}/\Gamma)$, $\tau^2 = id, \tau(\zeta_3) = \zeta_3^2$ and $\tau(\sqrt[3]{d}) = \sqrt[3]{d}$;
- $\langle\sigma\rangle = \mathrm{Gal}\,(\mathrm{k}/\mathbb{Q}(\zeta_3))$, $\sigma^3 = id, \sigma(\zeta_3) = \zeta_3$ and $\sigma(\sqrt[3]{d}) = \zeta_3\sqrt[3]{d}$;
- For an algebraic number field L:
  - $\mathcal{O}_L$: the ring of integers of L;
  - E_L : the group of units of L;
  - $\mathcal{D}_L$: the discriminant of L;
  - h_L : the class number of L;
  - $h_{L,3}$: the 3-class number of L;
  - $C_{L,3}$: the 3-class group of L;
  - $L_3^{(1)}$: the Hilbert 3-class field of L;
  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$ in the class group of L;
- $\left(\dfrac{c}{p}\right)_3 = 1 \Leftrightarrow X^3 \equiv c (mod\ p)$ resolved on $\mathbb{Z} \Leftrightarrow c^{(p-1)/3} \equiv 1 \pmod{p}$, where $c \in \mathbb{Z}$ and p is a prime number congruent to $1 (mod\ 3)$.

2. Norm residue symbol and ideal factorization theory

2.1. The norm residue symbol

Let L/K an abelian extension of number fields with conductor f. For each finite or infinite prime ideal $\mathcal{P}$ of K, we note by $f_\mathcal{P}$ the largest power of $\mathcal{P}$ that divides f. Let $a \in K^*$, we determine an auxiliary number a_0 by the two conditions $a_0 \equiv a\ (mod\ f_\mathcal{P})$ and $a_0 \equiv 1\ (mod\ \frac{f}{f_\mathcal{P}})$. Let $\mathcal{Q}$ an ideal co-prime with $\mathcal{P}$ such that $(a_0) = \mathcal{P}^e\mathcal{Q}$ ($b = 0$ if $\mathcal{P}$ is infinite). We note by

$$\left(\frac{a, L}{\mathcal{P}}\right) = \left(\frac{L/K}{\mathcal{Q}}\right)$$

the Artin map in L/K applied to $\mathcal{Q}$.

Definition 2.1. *Let K be a number field containing the l^{th}-roots of units, where $l \in \mathbb{N}$, then for each $a, b \in K^*$ and prime ideal $\mathcal{P}$ of K, we define the* ***norm residue symbol*** *by:*

$$\left(\frac{a,b}{\mathcal{P}}\right)_l = \frac{\left(\frac{a, K(\sqrt[l]{b})}{\mathcal{P}}\right)\sqrt[l]{b}}{\sqrt[l]{b}}.$$

Therefore, if the prime ideal $\mathcal{P}$ is unramified in the field $K(\sqrt[l]{b})$, then we write

$$\left(\frac{b}{\mathcal{P}}\right)_l = \frac{\left(\frac{K(\sqrt[l]{b})}{\mathcal{P}}\right)\sqrt[l]{b}}{\sqrt[l]{b}}.$$

Remark 2.2. *Notice that $\left(\frac{a,b}{\mathcal{P}}\right)_l$ and $\left(\frac{b}{\mathcal{P}}\right)_l$ are two l^{th}-roots of units.*

Following [9], the principal properties of the norm residue symbol are given as follows:

Properties

1. The product formula:
   - $\left(\frac{a_1a_2,b}{\mathcal{P}}\right)_l = \left(\frac{a_1,b}{\mathcal{P}}\right)_l \left(\frac{a_2,b}{\mathcal{P}}\right)_l$;
   - $\left(\frac{a,b_1b_2}{\mathcal{P}}\right)_l = \left(\frac{a,b_1}{\mathcal{P}}\right)_l \left(\frac{a,b_2}{\mathcal{P}}\right)_l$;
2. The inverse formula: $\left(\frac{a,b}{\mathcal{P}}\right)_l = \left(\frac{b,a}{\mathcal{P}}\right)_l^{-1}$;
3. $\left(\frac{a,b}{\mathcal{P}}\right)_l = 1 \Leftrightarrow a$ is norm residue of $K(\sqrt[l]{b})$ modulo f_b;
4. $\left(\frac{\sigma a,\sigma b}{\sigma\mathcal{P}}\right)_l = \sigma\left(\frac{a,b}{\mathcal{P}}\right)_l$, for each automorphism σ of K;
5. If $\mathcal{P}$ is not divisible by the conductor f_b of $K(\sqrt[l]{b})$ and appears in (a) with the exponent e, then:
   - $\left(\frac{a,b}{\mathcal{P}}\right)_l = \left(\frac{b}{\mathcal{P}}\right)_l^{-e}$;
   - $\mathcal{P}$ is infinite $(e = 0) \Rightarrow \left(\frac{a,b}{\mathcal{P}}\right)_l = 1$;

6. The classical reciprocity law: let $a, b \in K^*$, and the conductors f_a and f_b of respectively $K(\sqrt[l]{a})$ and $K(\sqrt[l]{b})$ are co-prime, then:

$$\left(\frac{a}{(b)}\right)_l = \left(\frac{b}{(a)}\right)_l;$$

7. $\prod_{\mathcal{P}} \left(\frac{a,b}{\mathcal{P}}\right)_l = 1$, where the product is taken on the finite and infinite prime ideals;

8. Let L is a finite extension of K, $a \in L$ and $b \in K^*$, then:

$$\prod_{\mathfrak{P}|\mathcal{P}} \left(\frac{a,b}{\mathfrak{P}}\right)_l = \left(\frac{\mathcal{N}_{L/K}(a), b}{\mathcal{P}}\right)_l.$$

Remark 2.3. *From property* (3)*, we have:*

$$a \text{ is a norm in } K(\sqrt[l]{b}) \Rightarrow \left(\frac{a,b}{\mathcal{P}}\right)_l = 1,$$

for each prime ideal $\mathcal{P}$ *of* K*.*

For more basic properties of the norm residue symbol in the number fields, we refer the reader to the papers [3], [8] and [9]. Notice that in section 3, we will use the norm cubic residue symbols ($l = 3$). As the ring of integer $\mathcal{O}_{\mathrm{k}_0}$ is principal, $h_{\mathrm{k}_0} = 1$, we will write the norm cubic residue symbol as follows:

$$\left(\frac{a,b}{(\pi)}\right)_3 = \left(\frac{a,b}{\pi}\right)_3 \text{ and } \left(\frac{a}{(\pi)}\right)_3 = \left(\frac{a}{\pi}\right)_3$$

where $a, b \in \mathrm{k}_0^*$ and π is a prime integer of $\mathcal{O}_{\mathrm{k}_0}$.

2.2. Prime factorization in a pure cubic field and in its normal closure

Let be $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ a pure cubic field, and $\mathcal{O}_\Gamma$ the ring of integers of Γ. We write the natural integer d in form $d = ab^2$, where a and b are cube-free and co-prime positive integers. In his paper [3], Dedekind has defined two different types of pure cubic fields as follows:

Definition 2.4. *Using the same notations as above:*

1. *We say that* $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ *is of the* ***first kind*** *if* $3\mathcal{O}_\Gamma = \mathcal{P}^3$*, where* $\mathcal{P}$ *is a prime ideal of* $\mathcal{O}_\Gamma$*, in this case,* $a^2 - b^2 \not\equiv 0 \pmod 9$*.*

2. *We say that* $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ *is of the* ***second kind*** *if* $3\mathcal{O}_\Gamma = \mathcal{P}^2\mathcal{P}_1$*, where* $\mathcal{P} \neq \mathcal{P}_1$ *are two primes of* $\mathcal{O}_\Gamma$*, in this case,* $a^2 - b^2 \equiv 0 \pmod 9$*.*

Now, let p be a prime number. In the following Proposition, we give the decomposition of the prime p in the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{ab^2})$. We denote by $\mathcal{P}, \mathcal{P}_i$ prime ideals of Γ, and by $\mathcal{N}$ the absolute norm $\mathcal{N}_{\Gamma/\mathbb{Q}}$.

Proposition 2.5.
Let p a prime number such that $p \neq 3$, then:

1. *If p divides ab and $p \neq 3$, then $p\mathcal{O}_\Gamma = \mathcal{P}^3$, $\mathcal{N}(\mathcal{P}) = p$.*
2. *If $p \nmid 3ab$ and $p \equiv -1 \pmod 3$, then $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1$, with $\mathcal{N}(\mathcal{P}) = p$ and $\mathcal{N}(\mathcal{P}_1) = p^2$.*
3. *If $p \nmid 3ab$ and $p \equiv 1 \pmod 3$, then:*
   - *(a) $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1\mathcal{P}_2$ with $\mathcal{N}(\mathcal{P}) = \mathcal{N}(\mathcal{P}_1) = \mathcal{N}(\mathcal{P}_2)$, if ab^2 is a cubic residue modulo p;*
   - *(b) $p\mathcal{O}_\Gamma = \mathcal{P}$ with $\mathcal{N}(\mathcal{P}) = p^3$, if ab^2 is not a cubic residue modulo p.*

Proof: See [3]. □

The ramification of the prime 3 need a particular treatment, it is the purpose of the following Proposition:

Proposition 2.6.
The decomposition into prime factors of 3 is:

$$3\mathcal{O}_\Gamma = \begin{cases} \mathcal{P}^3, & if \quad a^2 \not\equiv b^2 \pmod 9, \\ \mathcal{P}^2\mathcal{P}_1, & if \quad a^2 \equiv b^2 \pmod 9. \end{cases}$$

Proof: See [3]. □

The ideal factorization rules for the 3rd cyclotomic field k_0 (see [11]) is as follows:

(*i*) $3\mathcal{O}_{\mathrm{k}_0} = \lambda^2 = (1-\zeta_3)^2$;

(*ii*) $p\mathcal{O}_{\mathrm{k}_0} = \pi_1\pi_2$ in k_0 if $p \equiv 1 \pmod 3$;

(*iii*) $q\mathcal{O}_{\mathrm{k}_0} = \mathsf{q}$ in k_0 if $q \equiv -1 \pmod 3$.

Next, let k be the normal closure of Γ. We note by $\mathcal{O}_\mathrm{k}$ the ring of integers of k, $\mathfrak{P}$ and $\mathfrak{P}_s$ are prime ideals of k, $\mathcal{N} = \mathcal{N}_{k/\mathbb{Q}}$ the norm of k on $\mathbb{Q}$. Combining the ideal factorization rules for Γ with those of the field k_0. The decomposition of the prime 3 in k is the purpose of the following Theorem:

Proposition 2.7.
The prime 3 decomposes in k as follows:

$$3\mathcal{O}_\mathrm{k} = \begin{cases} \mathfrak{P}^6, & si \quad a^2 \not\equiv b^2 \pmod 9, \\ \mathfrak{P}_1^2\mathfrak{P}_2^2\mathfrak{P}_3^2, & si \quad a^2 \equiv b^2 \pmod 9. \end{cases}$$

Proof:
We have 3 ramifies in the quadratic field $\mathrm{k}_0 = \mathbb{Q}(\zeta_3)$.

1) Suppose that Γ is the first kind, then by Proposition 2.6 we have $3\mathcal{O}_\Gamma = \mathcal{P}^3$. Hence, $3\mathcal{O}_k = \mathfrak{P}^6$.

2) Conversely, suppose that Γ is of second kind, then $3\mathcal{O}_\Gamma = \mathcal{P}^2\mathcal{P}_1$. It follows that $3\mathcal{O}_k = \mathfrak{P}_1^2\mathfrak{P}_2^2\mathfrak{P}_3^2$.

□

However, we have the following Proposition in which we characterize the decomposition of prime ideals of $p \neq 3$ in k.

Proposition 2.8.
Let p a prime number such that $p \neq 3$, then:

1. *If p divides $\mathcal{D}_\Gamma$, then:*
   (a) *$p\mathcal{O}_\mathrm{k} = \mathfrak{P}_1^3\mathfrak{P}_2^3$, with $\mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = p$, if and only if -3 is a quadratic residue modulo p.*
   (b) *$p\mathcal{O}_\mathrm{k} = \mathfrak{P}^3$, with $\mathcal{N}(\mathfrak{P}) = p^2$, if and only if -3 is not a quadratic residue modulo p.*
2. *If p does not divides $\mathcal{D}_\Gamma$ and $p \equiv 1 \pmod 3$, then:*
   (a) *p decomposes completely in k if and only if $\mathcal{D}_\Gamma$ is a cubic residue modulo p.*
   (b) *$p\mathcal{O}_\mathrm{k} = \mathfrak{P}_1\mathfrak{P}_2$, with $\mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = p^3$, if and only if $\mathcal{D}_\Gamma$ is not a cubic residue modulo p.*
3. *If p does not divides $\mathcal{D}_\Gamma$ and $p \equiv -1 \pmod 3$, then: $p\mathcal{O}_\mathrm{k} = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3$, with $\mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = \mathcal{N}(\mathfrak{P}_3) = p^2$, if and only if -3 is not a quadratic residue modulo p.*

Proof:

1. We use Proposition 2.5 and the decomposition of prime ideals in the quadratic fields $\mathrm{k}_0 = \mathbb{Q}(\zeta_3)$.
2. Suppose that p does not divide $\mathcal{D}_\Gamma$ and $p \equiv 1 \pmod 3$, then -3 is a quadratic residue modulo p, then the multiplication formula gives

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right).$$

On the one hand, by the Euler's Theorem we have

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2},$$

On the other hand, the quadratic reciprocity law gives

$$\left(\frac{p}{3}\right)\left(\frac{3}{p}\right) = (-1)^{(p-1)/2},$$

since $p \equiv 1$ $(mod\ 3)$, then p is a square modulo 3, which gives $\left(\frac{p}{3}\right) = 1$, so

$$\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}.$$

Then

$$\left(\frac{-3}{p}\right) = ((-1)^{(p-1)/2})^2 = (-1)^{p-1} = 1.$$

Thus, p decomposes completely in k_0.

(a) If $\mathcal{D}_\Gamma$ is a cubic residue modulo p, then by Proposition 2.5 we have p split completely in Γ. Hence p split completely in k.

(b) If $\mathcal{D}_\Gamma$ is not a cubic residue modulo p, we have p remains prime in Γ. Hence $p\mathcal{O}_\mathrm{k} = \mathfrak{P}_1\mathfrak{P}_2$.

3. We have $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1$, and p remains inert in k_0, hence the result.

□

Remark 2.9. *In the preceding Proposition 2.8, the situation $p\mathcal{O}_\mathrm{k} = \mathfrak{P}_1\mathfrak{P}_2$ is never happens because if $p \equiv -1$ $(mod\ 3)$, we have always* $\left(\frac{-3}{p}\right) = -1$.

3. The generators of $C_{\mathrm{k},3}$

First, we let $C_{\mathrm{k},3}^{(\sigma)} = \{\mathcal{A} \in C_{\mathrm{k},3} \mid \mathcal{A}^\sigma = \mathcal{A}\}$ be the group of ambiguous ideal classes of k/k_0, where σ is a generator of $\mathrm{Gal}\,(\mathrm{k}/\mathrm{k}_0)$, and put $q^* = 0$ or 1 according to ζ_3 is not norm or norm of an element of $\mathrm{k}\backslash\{0\}$. Let t be the number of primes ramifies in k/k_0. Then according to [4], we have

$$|C_{\mathrm{k},3}^{(\sigma)}| = 3^{t-2+q^*}.$$

If we denote by $C_{\mathrm{k}_0,3}$ the Sylow 3-subgroup of the ideal class group of k_0, $C_{\mathrm{k}_0,3} = \{1\}$. Let be $C_{\mathrm{k},3}^{(1-\sigma)} = \{\mathcal{A}^{(1-\sigma)} \mid \mathcal{A} \in C_{\mathrm{k},3}\}$. By the exact sequence :

$$1 \longrightarrow C_{\mathrm{k},3}^{(\sigma)} \longrightarrow C_{\mathrm{k},3} \xrightarrow{1-\sigma} C_{\mathrm{k},3} \longrightarrow C_{\mathrm{k},3}/C_{\mathrm{k},3}^{1-\sigma} \longrightarrow 1$$

we deduce that

$$|C_{\mathrm{k},3}^{(\sigma)}| = |C_{\mathrm{k},3}/C_{\mathrm{k},3}^{1-\sigma}|.$$

The fact that $C_{\mathrm{k},3}^{(\sigma)}$ and $C_{\mathrm{k},3}/C_{\mathrm{k},3}^{1-\sigma}$ are elementary abelian 3-groups imply that:

$$\mathrm{rank}\ C_{\mathrm{k},3}^{(\sigma)} = \mathrm{rank}\,(C_{\mathrm{k},3}/C_{\mathrm{k},3}^{1-\sigma}).$$

Define the 3-group $C_{\mathrm{k},3}^{(1-\sigma)^i}$ for each $i \in \mathbb{N}$ by

$$C_{\mathrm{k},3}^{(1-\sigma)^i} = \{\mathcal{A}^{(1-\sigma)^i} \mid \mathcal{A} \in C_{\mathrm{k},3}\},$$

and let s be the positive integer such that $C_{\mathrm{k},3}^{(\sigma)} \subseteq C_{\mathrm{k},3}^{(1-\sigma)^{s-1}}$ and $C_{\mathrm{k},3}^{(\sigma)} \not\subseteq C_{\mathrm{k},3}^{(1-\sigma)^{s}}$.

The following Proposition gives the structure of the 3-class group $C_{\mathrm{k},3}$ when 27 divides exactly the class number of k:

Proposition 3.1. *Let be Γ a pure cubic field,* k *its normal closure and u the index of units defined as above, then:*

1) $C_{\mathrm{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \Leftrightarrow [C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$ *and* $u = 1]$*;*

2) $C_{\mathrm{k},3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \Leftrightarrow [C_{\Gamma,3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *and* $u = 1]$.

Proof:

1) Assume that $C_{\mathrm{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, then $h_{\mathrm{k},3} = 27$. According to Theorem 14.1 of [1], we have $27 = \frac{u}{3} \cdot h_{\Gamma,3}^2$, then $u = 1$ because otherwise 27 will be a square in $\mathbb{N}$, which is a contradiction. Then $h_{\Gamma,3}^2 = 3^4$ and $h_{\Gamma,3} = 9$.
   On the other hand, by Lemma 2.1 and Lemma 2.2 of [5] we have $C_{\mathrm{k},3} \simeq C_{\Gamma,3} \times C_{\mathrm{k},3}^-$, then $|C_{\mathrm{k},3}^-| = 3$. Since $C_{\mathrm{k},3}$ is of type $(9,3)$, we deduce that $C_{\mathrm{k},3}^-$ is a cyclic 3-group of order 3 and $C_{\mathrm{k},3}^+$ is a cyclic 3-group of order 9. Therefore $u = 1$ and $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$. Reciprocally, assume that $u = 1$ and $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$. By Theorem 14.1 of [1], we deduce that $|C_{\mathrm{k},3}| = \frac{1}{3} \cdot |C_{\Gamma,3}|^2$, then $|C_{\mathrm{k},3}| = 27$ and $|C_{\mathrm{k},3}^-| = 3$. Thus:

$$C_{\mathrm{k},3} \simeq C_{\Gamma,3} \times C_{\mathrm{k},3}^- \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

2) We have the same proof as above.

□

Lemma 3.2. *Let* $\mathrm{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$*, where p is a prime number such that* $p \equiv 1 \pmod 3$*. Let $C_{\mathrm{k},3}^{(\sigma)}$ be the ambiguous ideal class group of* $\mathrm{k}/\mathbb{Q}(\zeta_3)$*, where σ is a generator of* $\mathrm{Gal}(\mathrm{k}/\mathbb{Q}(\zeta_3))$*. Then* $|C_{\mathrm{k},3}^{(\sigma)}| = 3$.

Proof: Since $p \equiv 1 \pmod 3$, then according to section 2.2, we have $p = \pi_1\pi_2$, where π_1 and π_2 are two primes of k_0 such that $\pi_2 = \pi_1^\tau$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{\mathrm{k}_0}}$. We study all cases depending on the congruence class of p modulo 9, then:

- If $p \equiv 4$ or $7 \pmod 9$, then according to section 2.2, the prime 3 is ramified in the field L, so the prime ideal $(1 - \zeta_3)$ is ramified in k/k_0. Also π_1 and π_2 are totally ramified in k. So $t = 3$. In addition, the fact that $p \equiv 4$ or 7 (mod 9) imply that $\pi_i \not\equiv 1 \pmod{(1-\zeta_3)^3}$ for $i = \{1, 2\}$, then according to section 5 of [4] we obtain

$$\left(\frac{\zeta_3, p}{p}\right)_3 \neq 1$$

where the symbol $(\frac{\cdot,\cdot}{})_3$ is the cubic Hilbert symbol. We deduce that ζ_3 is not a norm in the extension k/k_0, so $q^* = 0$. Hence rank $C_{\mathrm{k},3}^{(\sigma)} = 1$ and then $|C_{\mathrm{k},3}^{(\sigma)}| = 3$.

- If $p \equiv 1 \pmod 9$, the prime ideals which ramified in k/k_0 are π_1 and π_2, so $t = 2$. Moreover, $\pi_1 \equiv \pi_2 \equiv 1 \pmod{(1-\zeta_3)^3}$, then according to [4], the cubic Hilbert symbol:

$$\left(\frac{\zeta_3, p}{\pi_1}\right)_3 = \left(\frac{\zeta_3, p}{\pi_2}\right)_3 = 1,$$

We conclude that ζ_3 is a norm in the extension k/k_0, then $q^* = 1$, so rank $C_{\mathrm{k},3}^{(\sigma)} = 1$ and $|C_{\mathrm{k},3}^{(\sigma)}| = 3$.

□

The basic result for determining the generators of the 3-class group of $\mathrm{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ when the 3-class number of k is divisible by 27 exactly, where p is a prime number such that $p \equiv 1 \ (mod\ 9)$, is summarized in the following Theorem:

Theorem 3.3. *Let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, where p is a prime number such that $p \equiv 1 \ (mod\ 9)$, $\mathrm{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ its normal closure and $C_{\mathrm{k},3}$ the 3-class group of k. Assuming 9 divides the 3-class number of Γ exactly, then:*
The 3-class group $C_{\mathrm{k},3}$ is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ if and only if $u = 1$.

Proof:

$\Rightarrow$) By Proposition 3.1, it is clear that if $C_{\mathrm{k},3}$ is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ then $u = 1$.

$\Leftarrow$) Assume that $u = 1$, then according to Theorem 14.1 of [1], $h_{\mathrm{k},3} = 27$. Since 9 divides the 3-class number of Γ, then by Lemma 5.11 of [2] we have rank $C_{\mathrm{k},3} = 2$. Hence $C_{\mathrm{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

□

Proposition 3.4. *Let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, where p is a prime number such that $p \equiv 1 \ (mod\ 9)$, $\mathrm{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ it's normal closure, and $C_{\mathrm{k},3}$ be the 3-class group of k. Assume that 9 divides the 3-class number of Γ exactly and $u = 1$. Put $\langle \mathcal{A} \rangle = C_{\mathrm{k},3}^{+}$, where $\mathcal{A} \in C_{\mathrm{k},3}$ such that $\mathcal{A}^9 = 1$ and $\mathcal{A}^3 \neq 1$. Let $C_{\mathrm{k},3}^{(\sigma)}$ be the 3-group of ambiguous ideal classes of k/k_0 and $C_{\mathrm{k},3}^{1-\sigma} = \{\mathcal{A}^{1-\sigma} \mid \mathcal{A} \in C_{\mathrm{k},3}\}$ be the principal genus of $C_{\mathrm{k},3}$. Then:*

1. *$C_{k,3}^{(\sigma)}$ is a subgroup of $C_{k,3}^{+}$, $\mathcal{A} \notin C_{k,3}^{(\sigma)}$ and $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle = \langle \mathcal{B}^{1-\sigma} \rangle$, where $\mathcal{B} \in C_{k,3}$ such that $C_{k,3}^{-} = \langle \mathcal{B} \rangle$.*

2. *$C_{k,3}^{-} = \langle (\mathcal{A}^2)^{\sigma-1} \rangle$, and we have $C_{k,3}^{1-\sigma} = C_{k,3}^{-} \times C_{k,3}^{(\sigma)}$ is a 3-group of type $(3,3)$,*

where $C_{k,3}^{+}$ and $C_{k,3}^{-}$ are defined in Lemma 2.1 of [5].

Proof:

1. Since 9 divides the 3-class number of Γ exactly and $u = 1$, then according to Theorem 3.3, $C_{k,3}$ is of type $(9,3)$, this implies by [6] that the integer s defined above is equal 3, and according to Case 4 of [6], we conclude that$|\left(C_{k,3}^{(\sigma)}\right)^{+}| = 3$ and $|\left(C_{k,3}^{(\sigma)}\right)^{-}| = 1$, this implies that $C_{k,3}^{(\sigma)}$ is a subgroup of $C_{k,3}^{+}$. Therefore, $\langle \mathcal{A}^3 \rangle$ is the unique subgroup of order 3 of $C_{k,3}^{+}$ and $C_{k,3}^{(\sigma)}$ is cyclic of order 3, then $C_{k,3}^{(\sigma)} = \langle \mathcal{A}^3 \rangle$.
   Moreover, if $C_{k,3}^{-} = \langle \mathcal{B} \rangle$ where $\mathcal{B} \in C_{k,3}$, then $\mathcal{B} \notin C_{k,3}^{(\sigma)}$, so $\mathcal{B}^{\sigma} \neq \mathcal{B}$. Furthermore, $\mathcal{B}^{\sigma} \neq \mathcal{B}^2$ because otherwise we will have $\mathcal{B}^{\sigma^2} = (\mathcal{B}^2)^{\sigma} = (\mathcal{B}^{\sigma})^2 = \mathcal{B}^4$, as $\mathcal{B} \in C_{k,3}^{-}$, then $\mathcal{B}^3 = 1$. Therefore, $\mathcal{B}^{\sigma^2} = \mathcal{B}$, so $\mathcal{B}^{\sigma^3} = \mathcal{B}^{\sigma}$, since $\sigma^3 = 1$, then $\mathcal{B}^{\sigma} = \mathcal{B}$. This is impossible because $\mathcal{B}^{\sigma} \neq \mathcal{B}$. As $\mathcal{B}^3 = 1$ and $\mathcal{B}^{1+\sigma+\sigma^2} = 1$, then $\mathcal{B}^{\sigma^2} = \mathcal{B}^{2+2\sigma}$. This equality makes it possible to show that $\mathcal{B}^{1-\sigma}$ is an ambiguous class. We conclude that $C_{k,3}^{(\sigma)} = \langle \mathcal{B}^{1-\sigma} \rangle$.

2. We reason as in the assertion 1. Since $\mathcal{A}^2 \notin C_{k,3}^{-}$, we deduce that $C_{k,3}^{-} = \langle (\mathcal{A}^2)^{1-\sigma} \rangle$. then $C_{k,3}^{-}$ and $C_{k,3}^{(\sigma)}$ are contained in $C_{k,3}^{1-\sigma}$ which is of order 9, because $|C_{k,3}| = 27$ and $|C_{k,3}^{(\sigma)}| = 3$. Consequently, $C_{k,3}^{1-\sigma} = C_{k,3}^{(\sigma)} \times C_{k,3}^{-} = \langle \mathcal{A}^3, \mathcal{B} \rangle$.

□

Our principal result can be stated as follows:

Theorem 3.5. *Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where p is a prime number such that $p \equiv 1 \pmod 9$. The prime 3 decomposes in k as $3\mathcal{O}_k = \mathcal{P}^2\mathcal{Q}^2\mathcal{R}^2$, where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are prime ideals of k . Put $h = \frac{h_k}{27}$, where h_k is the class number of k. Assume that 9 divides exactly the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$ and $u = 1$. If 3 is not a cubic residue modulo p, then:*

1. *The class $[\mathcal{R}^h]$ generate $C_{k,3}^{+}$;*

2. *The 3-class group $C_{k,3}$ is generated by classes $[\mathcal{R}^h]$ and $[\mathcal{R}^h][\mathcal{P}^h]^2$, and we have:*
$$C_{k,3} = \langle [\mathcal{R}^h] \rangle \times \langle [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle = \langle [\mathcal{R}^h], [\mathcal{P}^h]^2 \rangle.$$

In Appendix of this paper, we illustrated this results by the numerical examples with the aid of *Pari* programming [12] and summarized in some tables in section 4.

Proof:
We start our proof by showing that $[\mathcal{R}^h]$ is of order 9:
Since the field $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ with $p \equiv 1 \pmod 9$ is of second kind, then by Proposition 2.6 we have $3\mathcal{O}_\Gamma = \mathcal{H}^2\mathcal{S}$, where $\mathcal{H}$ and $\mathcal{S}$ are prime of Γ, since $\mathcal{H}\mathcal{O}_k = \mathcal{P}\mathcal{Q}$ and $\mathcal{S}\mathcal{O}_k = \mathcal{R}^2$, then $3\mathcal{O}_{\mathrm{k}} = \mathcal{P}^2\mathcal{Q}^2\mathcal{R}^2$, where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are prime ideals of k. Moreover, the prime ideal $\mathcal{R}$ is invariant by τ, then $[\mathcal{R}] \in \{\chi \in C_{\mathrm{k},3} | \chi^\tau = \chi\}$.
If 9 divides the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$ exactly and $u = 1$, then by Theorem 3.3 we have $C_{\mathrm{k},3}$ is of type $(9, 3)$. According to Proposition 3.1, we have $C^+_{\mathrm{k},3}$ is cyclic of order 9, thus $[\mathcal{R}^h]^9 = 1$. Hence the class $[\mathcal{R}^h]$ is of order 9 if and only if $\mathcal{R}^h$ and $\mathcal{R}^{3h}$ are not principal.
We argue by the absurd: assume that $\mathcal{R}^h$ is principal, we have

$$\begin{aligned}
[\mathcal{R}^h] = 1 \quad &\Rightarrow \quad \exists \alpha \in \mathrm{k} \ | \ \mathcal{R}^h = \alpha\mathcal{O}_{\mathrm{k}},\\
&\Rightarrow \quad \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\mathcal{R}^h) = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha\mathcal{O}_{\mathrm{k}}),\\
&\Rightarrow \quad \lambda^h\mathcal{O}_{\mathrm{k}_0} = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha)\mathcal{O}_{\mathrm{k}_0}, \ \textit{where } \lambda = 1-\zeta_3,\\
&\Rightarrow \quad \exists \epsilon \in E_{\mathrm{k}_0} \ | \ \lambda^h = \epsilon \cdot \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha),\\
&\Rightarrow \quad \exists \beta \in \mathcal{O}_{\mathrm{k}} \ | \ \lambda^h = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\beta), \ \textit{because } E_{\mathrm{k}_0} \subseteq \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\mathrm{k}^*),
\end{aligned}$$

that is to say λ^h is a norm in $\mathrm{k} = \mathrm{k}_0(\sqrt[3]{p}) = \mathrm{k}_0(\sqrt[3]{\pi_1\pi_2})$, where π_1 and π_2 are two primes of k_0 such that $p = \pi_1\pi_2$. Hence, by property (5) we have:

$$(*) \qquad \left(\frac{\lambda^h, \pi_1\pi_2}{\mathcal{P}}\right)_3 = 1,$$

for all ideal $\mathcal{P}$ of k_0.
In particular, we calculate this symbol for $\mathcal{P} = \pi_1\mathcal{O}_{\mathrm{k}_0}$ or $\mathcal{P} = \pi_2\mathcal{O}_{\mathrm{k}_0}$.
For $\mathcal{P} = \pi_1\mathcal{O}_{\mathrm{k}_0}$, using the property (1) of the norm residue symbol, we have:

$$\left(\frac{\lambda^h, \pi_1\pi_2}{\mathcal{P}}\right)_3 = \left(\frac{\lambda^h, \pi_1\pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda^h, \pi_1}{\pi_1}\right)_3 \cdot \left(\frac{\lambda^h, \pi_2}{\pi_1}\right)_3$$

the properties (2) and (5) imply that:

$$\left(\frac{\lambda^h, \pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda, \pi_2}{\pi_1}\right)_1^h = \left(\frac{\lambda}{\pi_1}\right)_3^{0\times h} = 1.$$

and from the properties (1) and (6) we have

$$\left(\frac{\lambda^h, \pi_1}{\pi_1}\right)_3 = \left(\frac{\lambda, \pi_1}{\pi_1}\right)_3^h = \left(\frac{\lambda}{\pi_1}\right)_3^h$$

consequently

$$\left(\frac{\lambda^h, \pi_1\pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda}{\pi_1}\right)_3^h$$

.

Since the two primes π_1 and π_2 play symmetric roles, then we obtain a similar relation when $\mathcal{P} = \pi_2$:

$$\left(\frac{\lambda^h, \pi_1\pi_2}{\pi_2}\right)_3 = \left(\frac{\lambda}{\pi_2}\right)_3^h.$$

The equation $(*)$ imply that

$$\left(\frac{\lambda}{\pi_1}\right)_3^h = \left(\frac{\lambda}{\pi_2}\right)_3^h = 1.$$

The fact that 3 is not a cubic residue modulo p imply that

$$\left(\frac{\lambda}{\pi_1\pi_2}\right)_3 \neq 1$$

then

$$\left(\frac{\lambda}{\pi_1}\right)_3 \neq 1 \text{ or } \left(\frac{\lambda}{\pi_2}\right)_3 \neq 1.$$

Since 3 does not divide h, then

$$\left(\frac{\lambda}{\pi_1}\right)_3^h \neq 1 \text{ or } \left(\frac{\lambda}{\pi_2}\right)_3^h \neq 1.$$

which is a contradiction. Consequently, the ideal $\mathcal{R}^h$ is not principal.

Since the class $[\mathcal{R}^h]$ is invariant by τ, we deduce that the ideal $\mathcal{R}^{3h}$ is principal if and only if $\langle[\mathcal{R}^h]\rangle = C_{\mathrm{k},3}^{(\sigma)}$.
Since 9 divides exactly the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$ and $u = 1$, then by we get $|C_{\mathrm{k},3}| = 27$, so the positive integer s defined above is equal 3, then $C_{k,3}^{(1-\sigma)^3} = 1$, this implies that $C_{k,3}^{(\sigma)} = C_{k,3}^{(1-\sigma)^2}$. Suppose that $[\mathcal{R}^h] \in C_{\mathrm{k},3}^{(\sigma)}$, then $[\mathcal{R}^h] = [\mathcal{L}^{(1-\sigma)^2}]$ with $\mathcal{L}$ is prime ideal of k, then there exist $\alpha \in \mathrm{k}^*$ such that $\mathcal{R}^h = (\alpha)\cdot\mathcal{L}^{(1-\sigma)^2}$, so $\mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\mathcal{R}^h) = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha.\mathcal{L}^{(1-\sigma)^2})$, since $\mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\mathcal{L}^{(1-\sigma)^2}) = \mathcal{L}^{(1-\sigma)(1-\sigma^3)} = 1$, then $\lambda^h\mathcal{O}_{\mathrm{k}} = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha)\mathcal{O}_{\mathrm{k}}$, where $\lambda = 1-\zeta_3$, so there exist $\varepsilon \in E_{\mathrm{k}}$ such that $\lambda^h = \varepsilon\cdot\mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha)$, as λ^h and $\mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha)$ are in k_0 then $\varepsilon \in E_{\mathrm{k}_0}$, since $E_{\mathrm{k}_0} \subseteq \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\mathrm{k}^*)$ then $\lambda^h = \mathcal{N}_{\mathrm{k}|\mathrm{k}_0}(\alpha_1)$ where $\alpha_1 \in \mathcal{O}_{\mathrm{k}}$, that means λ^h is a norm in $\mathrm{k} = \mathrm{k}_0(\sqrt[3]{p})$ which is impossible. Finally, $[\mathcal{R}^h]$ is of order 9. This completes the proof of the first statement.

The second step in the proof is showing that the class $[\mathcal{R}^h][\mathcal{P}^h]^2$ is of order 3. We know that $(\mathcal{R}^h)^\tau = \mathcal{R}^h$ and $(\mathcal{P}^h)^\tau = \mathcal{Q}^h$, then:

$$\begin{aligned}\left(\mathcal{R}^h \cdot (\mathcal{P}^h)^2\right)^{1+\tau} &= \left(\mathcal{R}^h\right)^{1+\tau} \cdot \left((\mathcal{P}^h)^2\right)^{1+\tau} \\ &= (\mathcal{P}^h)^2 \cdot (\mathcal{R}^h)^2 \cdot (\mathcal{Q}^h)^2 \\ &= 3^h \mathcal{O}_{\mathrm{k}},\end{aligned}$$

which imply that $[\mathcal{R}^h \cdot (\mathcal{P}^h)^2]^{1+\tau} = 1$. Hence $[\mathcal{R}^h \cdot (\mathcal{P}^h)^2] \in C^-_{\mathrm{k},3}$.
On the other hand $\mathcal{R}^h \cdot (\mathcal{P}^h)^2$ is not principal, because otherwise we have $[\mathcal{R}^h] = [\mathcal{P}^h]^7$, the fact that $[(\mathcal{R}^h)^2 \cdot (\mathcal{P}^h)^2 \cdot (\mathcal{Q}^h)^2] = 1$ imply that $[(\mathcal{Q}^h)^2] = 1$, which is a contradiction because the class $[\mathcal{Q}^h]$ is of order 9 (reasoning as $\mathcal{R}^h$). Hence $[\mathcal{R}^h][\mathcal{P}^h]^2$ is of order 3 and generate the group $C^-_{\mathrm{k},3}$.
Since $[\mathcal{R}^h]$ is a generator of $C^+_{\mathrm{k},3}$, we deduce that

$$C_{\mathrm{k},3} = \langle [\mathcal{R}^h], [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle.$$

□

Corollary 3.6. *Using the same notation as above, we have the following properties:*

1. $\mathcal{P}^\sigma = \mathcal{Q}$, $\mathcal{Q}^\sigma = \mathcal{R}$;
2. $\mathcal{R}^\tau = \mathcal{R}$ *and* $\langle [\mathcal{R}] \rangle = \{\chi \in C_{\mathrm{k},3} | \chi^\tau = \chi\}$;
3. $\mathcal{P}^{\tau\sigma} = \mathcal{P}$ *and* $\langle [\mathcal{P}] \rangle = \{\chi \in C_{\mathrm{k},3} | \chi^{\tau\sigma} = \chi\}$;
4. $\mathcal{Q}^{\tau\sigma^2} = \mathcal{Q}$ *and* $\langle [\mathcal{Q}] \rangle = \{\chi \in C_{\mathrm{k},3} | \chi^{\tau\sigma^2} = \chi\}$;
5. *The* 3*-class group can be generated also by:*
$$C_{\mathrm{k},3} = \langle [\mathcal{P}^h], [\mathcal{P}^h][\mathcal{Q}^h]^2 \rangle = \langle [\mathcal{Q}^h], [\mathcal{Q}^h][\mathcal{R}^h]^2 \rangle.$$
6. *The* 3*-group* $C^{(\sigma)}_{\mathrm{k},3}$ *of ambiguous ideal classes is given by:*
$$C^{(\sigma)}_{\mathrm{k},3} = \langle [\mathcal{R}^{3h}] \rangle = \langle [\mathcal{P}^{3h}] \rangle = \langle [\mathcal{Q}^{3h}] \rangle.$$
7. *The principal genus* $C^{1-\sigma}_{\mathrm{k},3} = \{\mathcal{A}^{1-\sigma} \,|\, \mathcal{A} \in C_{\mathrm{k},3}\}$ *is of type* $(3,3)$ *and generated by:*
$$C^{1-\sigma}_{\mathrm{k},3} = \langle [\mathcal{R}^{3h}], [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle.$$

Proof:
The fact that the ideals $\mathcal{P}^h$, $\mathcal{Q}^h$ and $\mathcal{R}^h$ are not principals, we prove the assertions $(1), (2), (3)$ and (4) by applying the decomposition of 3 in the normal closure k.
For the assertion (5), since the ideals $\mathcal{P}^h$, $\mathcal{Q}^h$ and $\mathcal{R}^h$ are not principal, we obtain the result by the same reasoning above.
The assertions (6) and (7) follows by using Proposition 3.4. □

4. Appendix

Using the *Pari* programming [12], we illustrate the results of our main Theorem 3.5 by numerical examples. We have

$$C_{\mathrm{k},3} = \langle [\mathcal{R}^h], [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle$$

The following table verifies, for each prime number $p \equiv 1 \pmod 9$ such that $\left(\dfrac{3}{p}\right)_3 \neq 1$ and 9 divides the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$ exactly and $u = 1$, that the ideals $\mathcal{R}^h$ and $\mathcal{R}^{3h}$ are not principal. Therefore, the ideal $\mathcal{R}^{9h}$ is always principal.

Table 1

p	Type of $C_{\mathrm{k},3}$	Is principal $\mathcal{R}^h$	Is principal $\mathcal{R}^{3h}$	Is principal $\mathcal{R}^{9h}$
199	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
487	$[9,3]$	$[10,0]$	$[12,0]$	$[0,0]$
1297	$[9,3]$	$[16,0]$	$[12,0]$	$[0,0]$
1693	$[9,3]$	$[2,2]$	$[6,0]$	$[0,0]$
1747	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
1999	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
2017	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
2143	$[9,3]$	$[14,0]$	$[6,0]$	$[0,0]$
2377	$[9,3]$	$[7,0]$	$[3,0]$	$[0,0]$
2467	$[9,3]$	$[20,0]$	$[15,0]$	$[0,0]$
2593	$[9,3]$	$[4,2]$	$[3,0]$	$[0,0]$
2917	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
3511	$[9,3]$	$[10,0]$	$[12,0]$	$[0,0]$
3673	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
3727	$[9,3]$	$[5,0]$	$[6,0]$	$[0,0]$
4159	$[9,3]$	$[4,2]$	$[12,0]$	$[0,0]$
4519	$[9,3]$	$[4,4]$	$[12,0]$	$[0,0]$
4591	$[9,3]$	$[1,2]$	$[3,0]$	$[0,0]$
4789	$[9,3]$	$[25,5]$	$[30,0]$	$[0,0]$
5347	$[9,3]$	$[8,0]$	$[6,0]$	$[0,0]$
5437	$[9,3]$	$[77,0]$	$[33,0]$	$[0,0]$
6949	$[9,3]$	$[7,2]$	$[3,0]$	$[0,0]$
8209	$[9,3]$	$[2,2]$	$[6,0]$	$[0,0]$
8821	$[9,3]$	$[4,0]$	$[3,0]$	$[0,0]$
9631	$[9,3]$	$[2,1]$	$[6,0]$	$[0,0]$
9721	$[9,3]$	$[2,0]$	$[6,0]$	$[0,0]$

However, we verify in the following table that the ideal $\mathcal{R}^h\mathcal{P}^{2h}$ is not principal and $\mathcal{R}^h\mathcal{P}^{2h}$ is of order 3.

Table 2

p	Type of $C_{k,3}$	Is principal $\mathcal{R}^h\mathcal{P}^{2h}$	Is principal $(\mathcal{R}^h\mathcal{P}^{2h})^3$
199	$[9,3]$	$[0,1]$	$[0,0]$
487	$[9,3]$	$[0,2]$	$[0,0]$
1297	$[9,3]$	$[6,4]$	$[0,0]$
1693	$[9,3]$	$[6,2]$	$[0,0]$
1747	$[9,3]$	$[0,1]$	$[0,0]$
1999	$[9,3]$	$[0,2]$	$[0,0]$
2017	$[9,3]$	$[0,2]$	$[0,0]$
2143	$[9,3]$	$[0,4]$	$[0,0]$
2377	$[9,3]$	$[3,2]$	$[0,0]$
2467	$[9,3]$	$[0,10]$	$[0,0]$
2593	$[9,3]$	$[0,2]$	$[0,0]$
2917	$[9,3]$	$[0,1]$	$[0,0]$
3511	$[9,3]$	$[0,2]$	$[0,0]$
3673	$[9,3]$	$[0,1]$	$[0,0]$
3727	$[9,3]$	$[3,1]$	$[0,0]$
4159	$[9,3]$	$[6,2]$	$[0,0]$
4519	$[9,3]$	$[24,4]$	$[0,0]$

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