Bol. Soc. Paran. Mat. (3s.) **v. 39** 3 (2021): 27–35.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.40197

# Graded Semiprime Multiplication Modules

Rashid Abu-Dawwas

ABSTRACT: Let $M$ be a $G$-graded $R$-module. In this article, we introduce the concept of graded semiprime multiplication modules. A graded $R$-module $M$ is said to be graded semiprime multiplication if $M$ has no graded semiprime $R$-submodules or for every graded semiprime $R$-submodule $N$ of $M$, $N = IM$ for some graded ideal $I$ of $R$. We introduce several results concerning graded semiprime submodules and we investigate them to present several results on graded semiprime multiplication modules.

Key Words: Graded semiprime submodules, Graded semiprime multiplication modules, Graded multiplication modules, Graded modules.

## Contents



## 1. Introduction

Throughout this article, $R$ is assumed to be a commutative ring with a nonzero unity 1. Let $G$ be a group with identity $e$. A ring $R$ is said to be $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and $R_e$ (the identity component of $R$) is a subring of $R$ with $1 \in R_e$. For $x \in R$, $x$ can be written uniquely as $\sum_{g\in G} x_g$ where $x_g$ is the component of $x$ in $R_g$. Also, we write $h(R) = \bigcup_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. Let $M$ be a left $R$ -module. Then $M$ is a $G$-graded $R$-module if there exist additive subgroups $M_g$ of $M$ indexed by the elements $g \in G$ such that $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of $M_g$ are called homogeneous of degree $g$. If $x \in M$, then $x$ can be written uniquely as $\sum_{g\in G} x_g$, where $x_g$ is the component of $x$ in $M_g$. Clearly, $M_g$ is $R_e$-submodule of $M$ for all $g \in G$. Also, we write $h(M) = \bigcup_{g\in G} M_g$. and

2010 *Mathematics Subject Classification:* 16W50.
Submitted October 26, 2017. Published April 02, 2018

$supp(M,G) = \{g \in G : M_g \neq 0\}$. Let $R$ be a $G$-graded ring and $I$ be an ideal of $R$. Then $I$ is called $G$-graded ideal if $I = \bigoplus_{g\in G} \left(I \bigcap R_g\right)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. Not all ideals of a $G$-graded ring are $G$-graded.

Let $M$ be a $G$-graded $R$-module and $N$ be an $R$-submodule of $M$. Then $N$ is called $G$-graded $R$-submodule if $N = \bigoplus_{g\in G} \left(N \bigcap M_g\right)$, i.e., if $x \in N$ and $x = \sum_{g\in G} x_g$, then $x_g \in N$ for all $g \in G$. Not all $R$-submodules of a $G$-graded $R$-module are $G$-graded.

For more details in this terminology, see [15]. Moreover, the following lemma can be found in ( [9], Lemma 2.1).

**Lemma 1.1.** *Let $R$ be a $G$-graded ring and $M$ be a $G$-graded $R$-module.*

1. *If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \bigcap J$ are graded ideals of $R$.*
2. *If $N$ and $K$ are graded $R$-submodules of $M$, then $N + K$ and $N \bigcap K$ are graded $R$-submodules of $M$.*
3. *If $N$ is a graded $R$-submodule of $M$, $r \in h(R)$, $x \in h(M)$ and $I$ is a graded ideal of $R$, then $Rx$, $IN$ and $rN$ are graded $R$-submodules of $M$. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of $R$.*

Also, in [10], if $N$ is a graded $R$-submodule of $M$, then

$$Ann(N) = \{r \in R : rN = 0\}$$

is a graded ideal of $R$.

Let $M$ be a $G$-graded $R$-module and $N$ be an $R$-submodule of $M$. Then $M/N$ may be made into a graded module by putting $(M/N)_g = (M_g + N)/N$ for all $g \in G$ (see [15]). In fact, we prove the following.

**Lemma 1.2.** *Let $M$ be a graded $R$-module, $K$ and $N$ be $R$-submodules of $M$ such that $K \subseteq N$. Then $N$ is a graded $R$-submodule of $M$ if and only if $N/K$ is a graded $R$-submodule of $M/K$.*

**Proof:** Suppose that $N$ is a graded $R$-submodule of $M$. Clearly, $N/K$ is an $R$-submodule of $M/K$. Let $x + K \in N/K$. Then $x \in N$ and since $N$ is graded, $x = \sum_{g\in G} x_g$ where $x_g \in N$ for all $g \in G$ and then $(x + K)_g = x_g + K \in N/K$ for all $g \in G$. Hence, $N/K$ is a graded $R$-submodule of $M/K$. Conversely, let $x \in N$. Then $x = \sum_{g\in G} x_g$ where $x_g \in M_g$ for all $g \in G$ and then $(x_g + K) \in (M_g + K)/K = (M/K)_g$ for all $g \in G$ such that

$$\sum_{g\in G}(x + K)_g = \sum_{g\in G}(x_g + K) = \left(\sum_{g\in G} x_g\right) + K = x + K \in N/K.$$

Since $N/K$ is graded, $x_g + K \in N/K$ for all $g \in G$ which implies that $x_g \in N$ for all $g \in G$. Hence, $N$ is a graded $R$-submodule of $M$. $\square$

Graded multiplication modules have been introduced and studied by Escoriza and Torrecillas in [8]. A graded $R$-module $M$ is said to be graded multiplication if for every graded $R$-submodule $N$ of $M$, $N = IM$ for some graded ideal $I$ of $R$. In this case, we can take $I = (N :_R M)$. Graded multiplication modules have been studied by several authors, for example, see [1], [5] and [13].

Graded prime submodules have been introduced and studied by S. Ebrahimi Atani in [6]. A proper graded $R$-submodule $N$ of a graded $R$-module $M$ is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $r \in (N :_R M)$ or $m \in N$. Graded prime submodules have been studied by several authors, for example, see [2] and [3]. The set of all graded prime submodules of $M$ is denoted by $GSpec(M)$.

Graded weak multiplication modules have been introduced and studied by F. Farzalipour and P. Ghiasvand in [10]. A graded $R$-module $M$ is said to be graded weak multiplication if $GSpec(M) = \emptyset$ or for every graded prime $R$-submodule $N$ of $M$, $N = IM$ for some graded deal $I$ of $R$. Graded weak multiplication modules have been studied by several authors, for example, see [4].

Graded semiprime submodules have been introduced by S. C. Lee and R. Varmazyar in [14]. A proper graded $R$-submodule $N$ of $M$ is said to be graded semiprime if whenever $I$ is a graded ideal of $R$ and $K$ is a graded $R$-submodule of $M$ such that $I^n K \subseteq N$ for some positive integer $n$, then $IK \subseteq N$. A graded $R$-module $M$ is said to be graded semiprime if $\{0\}$ is a graded semiprime $R$-submodule of $M$. Graded semiprime submodules are also studied in [11]. The set of all graded semiprime $R$-submodules of $M$ is denoted by $GSSpec(M)$.

Motivated from the concepts of graded multiplication modules in [8] and graded weak multiplication modules in [10], we introduce a new class of graded $R$-modules, called graded semiprime multiplication modules. A graded $R$-module $M$ is said to be graded semiprime multiplication if $GSSpec(M) = \emptyset$ or for every graded semiprime $R$-submodule $N$ of $M$, $N = IM$ for some graded deal $I$ of $R$.

In this article, several results have been introduced, for example, we prove that if $K$ and $N$ are $R$-submodules of $M$ such that $K \subseteq N$, then $N$ is a graded semiprime $R$-submodule of $M$ if and only if $N/K$ is a graded semiprime $R$-submodule of $M/K$ (Theorem 2.2). Also, we prove that $N$ is a graded prime $R$-submodule of $M$ if and only if $N$ is a graded semiprime and a graded primary $R$-submodule of $M$ (Theorem 2.4). Moreover, we prove that if $M$ is a graded semiprime multiplication $R$-module, $J$ is an ideal of $R$ and $K$ is an $R$-submodule of $M$ such that $J \subseteq (K :_R M)$, then $M/K$ is a graded semiprime multiplication $R/J$-module (Theorem 2.6). Finally, We define the torsion set of $M$ with respect to the homogeneous elements of $R$ to be $HT(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in h(R)\}$. We prove that if $M$ is a graded semiprime multiplication $R$-module over an integral domain $R$, then either $HT(M) = \{0\}$ or $HT(M) = M$ (Theorem 2.16).

## 2. Graded Semiprime Submodules and Graded Semiprime Multiplication Modules

In this section, we introduce the concept of graded semiprime multiplication modules. Also, we introduce several results concerning graded semiprime submodules. We begin our results with the following theorem.

**Theorem 2.1.** *Let $M$ be a graded $R$-module and $N$ be a graded $R$-submodule of $M$. Then $N$ is graded semiprime if and only if whenever $r \in h(R)$ and $m \in h(M)$ such that $r^n m \in N$ for some positive integer $n$, then $rm \in N$.*

**Proof:** Suppose that $N$ is a graded semiprime $R$-submodule of $M$. Let $r \in h(R)$ and $m \in h(M)$ such that $r^n m \in N$ for some positive integer $n$. Then $I = Rr$ is a graded ideal of $R$ and $K = Rm$ is a graded $R$-submodule of $M$ such that $I^n K \subseteq N$. Since $N$ is graded semiprime, $IK \subseteq N$ and then $rm \in N$. Conversely, let $I$ be a graded ideal of $R$ and $K$ be a graded $R$-submodule of $M$ such that $I^n K \subseteq N$ for some positive integer $n$. Assume that $a \in I$ and $x \in K$. Since $I$ and $K$ are graded, $a = \sum\limits_{g \in G} a_g$ where $a_g \in I$ for all $g \in G$ and $x = \sum\limits_{g \in G} x_g$ where $x_g \in K$ for all $g \in G$. For every $g, h \in G$, $a_g^n x_h \in I^n K \subseteq N$, so by assumption, $a_g x_h \in N$ for all $g, h \in G$ and then $ax \in N$ which implies that $IK \subseteq N$. Hence, $N$ is a graded semiprime $R$-submodule of $M$. $\square$

**Theorem 2.2.** *Let $M$ be a graded $R$-module, $K$ and $N$ be $R$-submodules of $M$ such that $K \subseteq N$. Then $N$ is a graded semiprime $R$-submodule of $M$ if and only if $N/K$ is a graded semiprime $R$-submodule of $M/K$.*

**Proof:** Suppose that $N$ is a graded semiprime $R$-submodule of $M$. By Lemma 1.2, $N/K$ is a graded $R$-submodule of $M/K$. Let $r \in h(R)$ and $m + K \in h(M/K)$ such that $r^n(m + K) \in N/K$ for some positive integer $n$. Then $m \in h(M)$ such that $r^n m \in N$ and since $N$ is graded semiprime, $rm \in N$ which implies that $r(m + K) \in N/K$. Hence, $N/K$ is a graded semiprime $R$-submodule of $M/K$. Conversely, by Lemma 1.2, $N$ is a graded $R$-submodule of $M$. Let $r \in h(R)$ and $m \in h(M)$ such that $r^n m \in N$ for some positive integer $n$. Then $m+K \in h(M/K)$ such that $r^n(m + K) = r^n m + K \in N/K$ and since $N/K$ is graded semiprime, $r(m + K) \in N/K$ which implies that $rm \in N$. Hence, $N$ is a graded semiprime $R$-submodule of $M$. $\square$

Graded primary submodules have been introduced and studied by S. Ebrahimi Atani and F. Farzalipour in [7]. A proper graded $R$-submodule $N$ of $M$ is said to be graded primary if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r^n \in (N :_R M)$ for some positive integer $n$. Graded primary submodules are deeply studied in [16] and [12].

The following example shows that a graded primary submodule need not be graded semiprime, and a graded semiprime submodule need not be graded primary.

**Example 2.3.** *Let $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $R$ is trivially graded by $R_0 = R$ and $R_1 = \{0\}$. Also, $M$ is graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Clearly, $N = 4\mathbb{Z}\bigoplus\langle 0\rangle$ is a graded primary $R$-submodule of $M$, but $N$ is not graded semiprime since $2 \in h(R)$ and $(3,0) \in h(M)$ such that $2^2(3,0) \in N$ and $2(3,0) \notin N$. On the other hand, $K = 6\mathbb{Z}\bigoplus\langle 0\rangle$ is a graded semiprime $R$-submodule of $M$ that is not graded primary.*

However, we introduce the following theorem.

**Theorem 2.4.** *Let $M$ be a graded $R$-module and $N$ be a graded $R$-submodule of $M$. Then $N$ is graded prime if and only if $N$ is graded semiprime and graded primary.*

**Proof:** Suppose that $N$ is a graded prime $R$-submodule of $M$. Clearly, $N$ is graded primary. Let $I$ be a graded ideal of $R$ and $K$ be a graded $R$-submodule of $M$ such that $I^nK \subseteq N$ for some positive integer $n$. Then either $I \subseteq (N :_R M)$ or $K \subseteq N$ and hence $IK \subseteq N$. Thus, $N$ is a graded semiprime $R$-submodule of $M$. Conversely, let $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$. Suppose that $m \notin N$. Since $N$ is graded primary, $r^n \in (N :_R M)$ for some positive integer $n$. Let $x \in M$. Then $x = \sum_{g\in G} x_g$ where $x_g \in M_g$ for all $g \in G$ and then $r^n x_g \in r^n M \subseteq N$ for all $g \in G$. Since $N$ is graded semiprime, $rx_g \in N$ for all $g \in G$ and then $rx \in N$. So, $rM \subseteq N$, i.e., $r \in (N :_R M)$. Hence, $N$ is a graded prime $R$-submodule of $M$. □

**Definition 2.5.** *Let $M$ be a graded $R$-module. Then $M$ is said to be graded semiprime multiplication if $GSSpec(M) = \emptyset$ or for every graded semiprime $R$-submodule $N$ of $M$, $N = IM$ for some graded deal $I$ of $R$.*

It is easy to prove that if $M$ is a graded semiprime multiplication $R$-module, then $N = (N :_R M)M$ for every graded semiprime $R$-submodule of $M$. Also, by Theorem 2.4, one can see that the class of graded semiprime multiplication modules contains the class of graded weak multiplication modules.

**Theorem 2.6.** *Let $M$ be a graded semiprime multiplication $R$-module. If $J$ is an ideal of $R$ and $K$ is an $R$-submodule of $M$ such that $J \subseteq (K :_R M)$, then $M/K$ is a graded semiprime multiplication $R/J$-module.*

**Proof:** Let $N/K$ be a graded semiprime submodule of $M/K$. Then by Theorem 2.2, $N$ is a graded semiprime submodule of $M$ and then $N = (N :_R M)M$ and hence $N/K = (N/K :_{R/J} M/K)(M/K)$. Thus, $M/K$ is a graded semiprime multiplication $R/J$-module. □

**Corollary 2.7.** *Let $M$ be a graded semiprime multiplication $R$-module. Then $M/K$ is a graded semiprime multiplication $R$-module for every $R$-submodule $K$ of $M$.*

**Proof:** Apply Theorem 2.6 with $J = \{0\}$. □

**Lemma 2.8.** *Let $M$ be a graded $R$-module and $K$ be a graded $R$-module of $M$. Then $K$ is a graded semiprime $R$-submodule of $M$ if and only if $M/K$ is a graded semiprime $R$-module.*

**Proof:** Suppose that $K$ is a graded semiprime $R$-submodule of $M$. Let $r \in h(R)$ and $m + K \in h(M/K)$ such that $r^n(m + K) = 0 + K$ for some positive integer $n$. Then $m \in h(M)$ such that $r^n m + K = 0 + K$, i.e., $r^n m \in K$. Since $K$ is graded semiprime, $rm \in K$ and then $rm + K = 0 + K$, i.e., $r(m + K) = 0 + K$. Hence, $M/K$ is a graded semiprime $R$-module. Conversely, let $r \in h(R)$ and $m \in h(M)$ such that $r^n m \in K$ for some positive integer $n$. Then $m + K \in h(M/K)$ such that $r^n(m+K) = r^n m+K = 0+K$. Since, $M/K$ is graded semiprime, $r(m+K) = 0+K$ which implies that $rm \in K$. Hence, $K$ is a graded semiprime $R$-submodule of $M$. □

Let $R$ and $S$ be two $G$-graded rings. A homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$. One can prove that if $I$ is a graded ideal of $R$ and $J$ is a graded ideal of $S$, then $f(I)$ is a graded ideal of $S$ and $f^{-1}(J)$ is a graded ideal of $R$ (see [15]).

**Lemma 2.9.** *Let $R$ and $S$ be two $G$-graded rings. Suppose that $f : R \to S$ is a graded homomorphism. If $f$ is surjective, then $f(R_g) = S_g$ for all $g \in G$.*

**Proof:** Let $g \in G$. Since $f$ is graded homomorphism, $f(R_g) \subseteq S_g$. Let $s_g \in S_g$. If $s_g = 0$, then $s_g = f(0_R) \in f(R_g)$. Suppose that $s_g \neq 0$. Since $f$ is surjective, there exists $r \in R - \{0\}$ such that $f(r) = s_g$. Assume that $r = \sum_{i=1}^{n} r_{g_i}$ where $r_{g_i} \in R_{g_i} - \{0\}$, $g_i \neq g_j$ for $i \neq j$. Then $s_g = f(r) = \sum_{i=1}^{n} f(r_{g_i}) = \sum_{i=1}^{k} f(r_{g_{t_i}})$ where $1 \leq t_i \leq n$ and $f(r_{g_{t_i}}) \neq 0$ for all $1 \leq i \leq k$. Since $f(r_{g_{t_i}}) \in S_{g_{t_i}}$, $s_g \in S_g \bigcap \sum_{i=1}^{k} S_{g_{t_i}}$. Hence, $g = g_{t_1} = ........ = g_{t_n}$ and hence $k = 1$ and $f(r_{g_{t_i}}) = f(r_g) = s_g$. So, $S_g \subseteq f(R_g)$ and hence $f(R_g) = S_g$. □

**Lemma 2.10.** *Let $R$ and $S$ be two $G$-graded rings. Suppose that $f : R \to S$ is a surjective graded homomorphism and $M$ is a graded $S$-module. If $M$ is graded semiprime as an $R$-module, then $M$ is graded semiprime as an $S$-module.*

**Proof:** Let $s \in h(S)$ and $m \in h(M)$ such that $s^n m = 0$ for some positive integer $n$. Since $s \in h(S)$, there exists $g \in G$ such that $s \in S_g = f(R_g)$ by Lemma 2.9 and then there exists $r \in R_g$ such that $f(r) = s$. So, $r \in h(R)$ such that $r^n m = f(r^n)m = (f(r))^n m = s^n m = 0$. Since $M$ is graded semiprime as an $R$-module, $rm = 0$ and then $sm = f(r)m = rm = 0$. Hence, $M$ is graded semiprime as an $S$-module. □

**Lemma 2.11.** *Let $R$ and $S$ be two $G$-graded rings. Suppose that there exists a surjective graded homomorphism from $R$ to $S$ and $M$ is a graded $S$-module. If $K$ is a graded semiprime $R$-submodule of $M$, then $K$ is a graded semiprime $S$-submodule of $M$.*

**Proof:** By Lemma 2.8, $M/K$ is a graded semiprime $R$-module and then by Lemma 2.10, $M/K$ is a graded semiprime $S$-module and hence by Lemma 2.8, $K$ is a graded semiprime $S$-submodule of $M$. □

**Theorem 2.12.** *Let $R$ and $S$ be two $G$-graded rings. Suppose that $f : R \to S$ is a surjective graded homomorphism and $M$ is a graded $S$-module. If $M$ is a graded semiprime multiplication $S$-module, then $M$ is a graded semiprime multiplication $R$-module.*

**Proof:** Let $K$ be a graded semiprime $R$-submodule of $M$. Then by Lemma 2.11, $K$ is a graded semiprime $S$-submodule of $M$. Since $M$ is graded semiprime multiplication as an $S$-module, $K = JM$ for some graded ideal $J$ of $S$. By Lemma 2.9, $I = f^{-1}(J)$ is a graded ideal of $R$ such that $f(I) = f(f^{-1}(J)) \bigcap f(R) = J$ and hence $IM = f(I)M = JM = K$. Thus, $M$ is a graded semiprime multiplication $R$-module. □

**Definition 2.13.** *Let $M$ be a graded $R$-module. We define the torsion set of $M$ with respect to the homogeneous elements of $R$ to be*
$HT(M) = \{m \in M : rm = 0$ *for some nonzero* $r \in h(R)\}$.

**Lemma 2.14.** *If $M$ is a graded $R$-module over an integral domain $R$, then $HT(M)$ is a graded $R$-submodule of $M$.*

**Proof:** Let $m, n \in HT(M)$. Then there exist $r, s \in h(R) - \{0\}$ such that $rm = sn = 0$. Since $r, s \in h(R)$, there exist $g, h \in G$ such that $r \in R_g$ and $r \in R_h$ and then $rs \in R_g R_h \subseteq R_{gh} \subseteq h(R)$. Since $R$ is an integral domain, $rs \in h(R) - \{0\}$ such that $rs(m - n) = rsm - rsn = s(rm) - r(sn) = 0$ which implies that $m - n \in HT(M)$. Let $t \in R$. Then $r(tm) = t(rm) = 0$ which implies that $tm \in HT(M)$. Hence, $HT(M)$ is an $R$-submodule of $M$. We show that $HT(M)$ is graded. Let $m \in HT(M)$. Then there exists a nonzero $r \in h(R)$ such that $rm = 0$. Assume that $m = \sum_{g \in G} m_g$ where $m_g \in M_g$ for all $g \in G$. Since $r \in h(R)$, $r \in R_h$ for some $h \in G$ and then $rm_g \in R_h M_g \subseteq M_{gh}$ for all $g \in G$. So, $rm_g \in h(M)$ for all $g \in G$ such that

$$\sum_{g \in G} rm_g = r\left(\sum_{g \in G} m_g\right) = rm = 0 \in \{0\}$$

and since $\{0\}$ is a graded $R$-submodule, $rm_g \in \{0\}$ for all $g \in G$, i.e., $rm_g = 0$ for all $g \in G$ which implies that $m_g \in HT(M)$ for all $g \in G$. Hence, $HT(M)$ is a graded $R$-submodule of $M$. □

**Lemma 2.15.** *Let $M$ be a graded $R$-module over an integral domain $R$. If $HT(M) \neq M$, then $HT(M)$ is a graded prime $R$-submodule of $M$ with $(HT(M) :_R M) = \{0\}$.*

**Proof:** By Lemma 2.14, $HT(M)$ is a graded $R$-submodule of $M$. Let $r \in h(R)$ and $m \in h(M)$ such that $rm \in HT(M)$. Then there exists a nonzero $s \in h(R)$ such that $s(rm) = 0$. If $r = 0$, then $r \in (HT(M) :_R M)$. Suppose that $r \neq 0$, then $sr \in h(R) - \{0\}$ such that $sr(m) = s(rm) = 0$ which implies that $m \in HT(M)$. Hence, $HT(M)$ is a graded prime $R$-submodule of $M$. We show that $(HT(M) :_R M) = \{0\}$. Let $r \in (HT(M) :_R M)$. Then $rM \subseteq HT(M)$. Since $HT(M) \neq M$, there exists $m \in M$ such that $m \notin HT(M)$ and then $rm \in rM \subseteq HT(M)$ which implies that there exists a nonzero $s \in h(R)$ such that $s(rm) = 0$. Since $m \notin HT(M)$, $sr = 0$ and since $s \neq 0$, $r = 0$. Hence, $(HT(M) :_R M) = \{0\}$. □

**Theorem 2.16.** *Let $M$ be a graded $R$-module over an integral domain $R$. If $M$ is a graded semiprime multiplication $R$-module, then either $HT(M) = \{0\}$ or $HT(M) = M$.*

**Proof:** Suppose that $HT(M) \neq M$. Then by Lemma 2.15, $HT(M)$ is a graded prime $R$-submodule of $M$ with $(HT(M) :_R M) = \{0\}$. By Theorem 2.4, $HT(M)$ is a graded semiprime $R$-submodule of $M$ and since $M$ is graded semiprime multiplication, $HT(M) = (HT(M) :_R M)M = \{0\}$. □

## Acknowledgments

The author is very grateful and thankful to the referee for the comments and suggestions that surely improved the article.

## References

1. R. Abu-Dawwas, *Multiplication Components of graded modules, Italian Journal of Pure and Applied Mathematics, 35 (2015), 389-392.*
2. *R. Abu-Dawwas, K. Al-Zoubi and M. Bataineh, Prime submodules of graded modules, Proyecciones Journal of Mathematics, 31 (4) (2012), 355-361.*
3. *R. Abu-Dawwas and M. Refai, Further results on graded prime submodules, International Journal of Algebra, 4 (28) (2010), 1413-1419.*
4. *R. Abu-Dawwas and M. Refai, Some remarks on graded weak multiplication modules, Journal of Contemporary Mathematical Sciences, 6 (14) (2011), 681-686.*
5. *R. Abu-Dawwas and M. Refai, Some remarks on $R_e$-multiplication modules over first strongly graded rings, East-West Journal of Mathematics, 13 (1) (2011), 57-61.*
6. *S. Ebrahimi Atani, On graded prime submodules, Chiang Mai Journal of Science, 33 (1) (2006), 3-7.*
7. *S. Ebrahimi Atani and F. Farzalipour, On graded secondary modules, Turkish Journal of Mathematics, 31 (2007), 371-378.*
8. *J. Escoriza and B. Torrecillas, Multiplication objects in commutative Grothendieck category, Communications in Algebra, 26 (1998), 1867-1883.*

9. F. Farzalipour and P. Ghiasvand, *On the union of graded prime submodules, Thai Journal of Mathematics, 9 (1) (2011), 49-55.*

10. F. Farzalipour and P. Ghiasvand, *On graded weak multiplication modules, Tamkang Journal of Mathematics, 43 (2) (2012), 171-177.*

11. F. Farzalipour and P. Ghiasvand, *On graded semiprime submodules, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 6 (8) (2012), 1169-1172.*

12. P. Ghiasvand and F. Farzalipour, *On graded primary submodules of graded multiplication modules, International Journal of Algebra, 4 (9) (2010), 429-432.*

13. K. Khaksari and F. R. Jahromi, *Multiplication graded modules, International Journal of Algebra, 7 (1) (2013), 17-24.*

14. S. C. Lee and R. Varmazyar, *Semiprime submodules of graded multiplication modules, Journal of Korean Mathematical Society, 49 (2) (2012), 435-447.*

15. *C. Nastasescu and V.F. Oystaeyen,* Methods of Graded Rings, LNM 1836, Berlin-Heidelberg: Springer-Verlag, 2004.

16. K. H. Oral, U. Tekir and A. G. Agargun, *On graded prime and primary submodules, Turkish Journal of Mathematics, 35 (2011), 159-167.*



*Rashid Abu-Dawwas,*
*Department of Mathematics,*
*Yarmouk University,*
*Jordan.*
*E-mail address:* rrashid@yu.edu.jo