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Zariski Topology on the Spectrum of Graded Pseudo Prime Submodules

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ABSTRACT: In this article, we introduce the concept of graded pseudo prime submodules of graded modules that is a generalization of the graded prime ideals over commutative rings. We study the Zariski topology on the graded spectrum of graded pseudo prime submodules. We investigate different properties of this topological space and connect them to the algebraic properties of the graded modules under consideration.

Key Words: Graded spectrum, Graded topological modules, Zariski topology, Graded prime submodules, Graded pseudo prime submodules.

Contents

1 Introduction **17**

2 Graded Pseudo Prime Submodules **18**

3 Graded Pseudo Prime Spectrum of Graded Topological Modules **23**

1. Introduction

Inspired by the Zariski topology defined on the graded prime spectrum of a graded module ([12]), we introduce the Zariski topology on the graded pseudo prime spectrum of graded pseudo prime submodules of a certain graded module M over a commutative graded ring R and study different properties of M and connect them to other properties of the new topological space. Similar studies are introduced in [3] and [6].

Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \oplus_{g\in G}R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous elements of degree g. The identity component R_e of R is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \cup_{g\in G}R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$.

Let M be a left R - module. Then M is a G - graded R - module (shortly, M is gr-R- module) if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \oplus_{g\in G}M_g$ and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of M_g are called homogeneous elements of degree g. If $x \in M$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in M_g. Clearly, M_g

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is an R_e - submodule of M for all $g \in G$. Also we write $h(M) = \cup_{g\in G} M_g$ and $supp(M, G) = \{g \in G : M_g \neq 0\}$.

Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal if $I = \oplus_{g\in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded.

Let M be a G-gr-R-module and N be an R-submodule of M. Then N is called a G-graded R-submodule of M if $N = \oplus_{g\in G}(N \cap M_g)$, i.e., if $x \in N$ and $x = \sum_{g\in G} x_g$, then $x_g \in N$ for all $g \in G$. Also, an R-submodule of a G-graded R-module need not be G-graded.

If M is a graded R-module and N is an R-submodule of M, then M/N may be made into a graded R-module by putting $(M/N)_g = (M_g + N)/N$.

For more terminology on graded rings and graded modules, one can look in [11].

Graded prime ideals have been introduced by M. Refai, M. Hailat and S. Obeidat in [13]; a proper graded ideal I of a graded ring R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$.

Graded prime submodules have been introduced by S. E. Atani in [5]; a proper graded R-submodule N of a graded R-module M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$ where $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R, see([8], Lemma 2.1).

Theorem 1.1. *([5], Proposition 2.7) Let N be a graded R-submodule of a graded R-module M. If N is graded prime, then $(N :_R M)$ is a graded prime ideal of R.*

The idea of the concept of graded pseudo prime submodules comes from the following question: Is the converse of Theorem 1.1 true? In fact, the following example shows that the answer is No.

Example 1.2. *Let $G = \mathbb{Z}_2$, $R = \mathbb{Z}$ and $M = \mathbb{Z} \bigoplus \mathbb{Z}$. Then R and M are trivially G-graded by $R_0 = R$, $R_1 = \{0\}$, $M_0 = M$ and $M_1 = \{0\}$. Choose $m = (2, 0) \in h(M)$, then $N = Rm$ is a graded R-submodule of M with $(N :_R M) = \{0\}$ that is a graded prime ideal of R. On the other hand, N is not graded prime R-submodule of M since $2 \in h(R)$ and $(3, 0) \in h(M)$ such that $2(3, 0) \in N$ but $2 \notin (N :_R M)$ and $(3, 0) \notin N$.*

We call a proper graded R-submodule N of a graded R-module M graded pseudo prime if $(N :_R M)$ is a graded prime ideal of R. By Theorem 1.1, every graded prime submodule is graded pseudo prime but the converse is not true in general by Example 1.2.

In this article, we study the Zariski topology on the graded spectrum of graded pseudo prime submodules. We study different properties of the underlying modules and connect them to other properties of this topological space.

2. Graded Pseudo Prime Submodules

In this section, we introduce and study the concept of graded pseudo prime submodules. Several concepts are given and have been connected to the concept of graded multiplication modules that has been introduced in [7]. Throughout this article, R will be a commutative graded ring.

Definition 2.1. *Let N be a proper graded R-submodule of a graded R-module M. Then N is said to be graded pseudo prime if $(N :_R M)$ is a graded prime ideal of R.*

Clearly, every graded prime submodule is graded pseudo prime by Theorem 1.1 but the converse is not true in general by Example 1.2.

Definition 2.2. *The graded pseudo prime spectrum of a graded R-module M is defined to be the set of all graded pseudo prime submodules of M, and it will be denoted by Y_M. For any graded prime ideal J of R, the collection of all graded pseudo prime submodules N of M with $(N :_R M) = J$ is denoted by $Y_{M,J}$.*

Definition 2.3. *For a graded R-submodule N of a graded R-module M, we define $X(N) = \{K \in Y_M : N \subseteq K\}$.*

We denote the set of all graded prime ideals of R by $GSpec(R)$.

Definition 2.4. *Suppose that $Y_M \neq \emptyset$. Then the map*

$$\varphi : Y_M \to GSpec(R/Ann(M))$$

defined by $\varphi(K) = (K :_R M)/Ann(M)$ will be called the natural map of Y_M. Note that, $Ann(M) = (0 :_R M)$.

Definition 2.5. *A graded R-module M is said to be graded pseudo primeful if either $M = \{0\}$ or $M \neq \{0\}$ and φ is surjective. If φ is injective, then M is called graded pseudo injective.*

Graded multiplication modules have been introduced by Escoriza and Torrecillas in [7]; a graded R-module M is said to be graded multiplication if for every graded R-submodule N of M, $N = IM$ for some graded deal I of R. Graded multiplication modules have been studied by several authors, for example, see [1], [2] and [9].

Definition 2.6. *Let M be a graded R-module and P be a graded maximal ideal of R. We say that M is graded P-cyclic if there exist $p \in P \bigcap h(R)$ and $m \in h(M)$ such that $(1 - p)M \subseteq Rm$.*

Theorem 2.7. *If M is a graded P-cyclic R-module for every graded maximal ideal of R, then M is a graded multiplication R-module.*

Proof: Let N be a graded R-submodule of M. Assume that $J = (N :_R M)$. Then J is a graded ideal of R and then JM is a graded R-submodule of M such that $JM \subseteq N$. Let $h \in G$ and $x_h \in N_h = M_h \bigcap N$. Suppose that $I = \{r \in R : rx_h \in JM\}$. Clearly, I is an ideal of R. To prove that I is a graded ideal, let $r \in I$. Then $r = \sum_{g \in G} r_g$ where $r_g \in R_g$ for all $g \in G$. Note that,

$r_g x_h \in R_g M_h \subseteq M_{gh} \subseteq h(M)$ such that $\sum_{g\in G} r_g x_h = \left(\sum_{g\in G} r_g\right) x_h = r x_h \in JM$ and since JM is graded, $r_g x_h \in JM$ for all $g \in G$, which implies by the definition of I that $r_g \in I$, for all $g \in G$. Hence, I is a graded ideal of R. Suppose that $I \neq R$. Then there exists a graded maximal ideal P of R such that $I \subseteq P$. By assumption, M is graded P-cyclic, i.e., there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$ and then $(1-p)N$ is a graded R-submodule of Rm and hence $(1-p)N = Km$ where $K = \{r \in R : rm \in (1-p)N\}$ is a graded ideal of R. Note that, $(1-p)KM = K(1-p)M \subseteq KRm \subseteq Km \subseteq N$ and then $(1-p)K \subseteq J$. So, $(1-p)^2 x_h \in (1-p)^2 N = (1-p)Km \subseteq JM$ and this gives the contradiction $(1-p)^2 \in I \subseteq P$ (if $(1-p)^2 \in P$, then $1-2p+p^2 \in P$ but since $p \in P$, $2p-p^2 \in P$ and then $1 \in P$ which is impossible as P is maximal ideal of R). Thus, $I = R$ and then $x_h \in JM$ for all $h \in G$. Now, let $x \in N$. Then $x = \sum_{h\in G} x_h$ where $x_h \in N$ for all $h \in G$, $x \in JM$ and hence $N = JM$. Thus, M is a graded multiplication R-module. □

Corollary 2.8. *Let M be a graded finitely generated R-module, and J a graded maximal ideal of R. If M/JM is graded cyclic, then M is a graded multiplication R-module.*

Proof: Let P be a graded maximal ideal of R. Then there exists $m \in M$ such that $M = Rm + PM$. Thus, $M/Rm = P(M/Rm)$. Since M/Rm is graded finitely generated, there exists $p \in P$ such that $(1-p)(M/Rm) = \{0\}$ by the usual determinant argument(see [10]). So, $(1-p)M \subseteq Rm$ which implies that M is graded P-cyclic and then by Theorem 2.7, M is a graded multiplication R-module. □

Lemma 2.9. *Let M be a graded finitely generated R-module, and J a graded maximal ideal of R. If $|Y_{M,J}| \leq 1$, then M/JM is a graded simple R-module.*

Proof: Let J be a graded maximal ideal of R. If $JM = M$, then we are done. Suppose that $JM \neq M$. Let N/JM be a proper graded R-submodule of M/JM. Then $J = (JM :_R M) = (N :_R M)$. So, $N, JM \in Y_{M,J}$ and then by assumption, $N = JM$. So, JM is a maximal submodule of M, and hence M/JM is graded simple. □

Theorem 2.10. *Let M be a graded finitely generated R-module. If M is graded pseudo injective, then M is graded multiplication.*

Proof: Since M is graded pseudo injective, $|Y_{M,J}| \leq 1$ for every graded maximal ideal J of R and then by Lemma 2.9, M/JM is graded simple and so graded cyclic and then by Corollary 2.8, M is a graded multiplication R-module. □

We state the following concepts.

Definition 2.11. *A graded R-submodule N of a graded R-module M is said to be graded pseudo semiprime if N is an intersection of graded pseudo prime R-submodules of M.*

Definition 2.12. *Let N be a graded pseudo prime R-submodule of a graded R-module M. Then N is said to be graded extraordinary if whenever K and L are graded pseudo semiprime R-submodules of M such that $K \bigcap L \subseteq N$, then either $K \subseteq N$ or $L \subseteq N$.*

Definition 2.13. *A graded R-module M is said to be graded topological module if $Y_M = \emptyset$ or every graded pseudo prime R-submodule of M is graded extraordinary.*

Definition 2.14. *Let $W \subseteq Y_M$. We denote the intersection of all elements in W by $\mu(W)$.*

Definition 2.15. *For a graded R-submodule N of a graded R-module M, the graded pseudo prime radical of N will be denoted by $GPrad(N)$ and is defined by the intersection of all graded pseudo prime R-submodules of M containing N, i.e., $GPrad(N) = \mu(X(N)) = \bigcap\limits_{P \in X(N)} P$.*

If $X(N) = \emptyset$, then we say that $GPrad(N) = M$. Also, N is said to be graded pseudo prime radical if $GPrad(N) = N$.

Lemma 2.16. *Let M be a graded R-module. Then M is graded topological if and only if $X(N) \bigcup X(L) = X(N \bigcap L)$ for every graded pseudo semiprime R-submodules N and L of M.*

Proof: Suppose that M is a graded topological R-module. Let N and L be graded pseudo semiprime R-submodules of M. Clearly, $X(N) \bigcup X(L) \subseteq X(N \bigcap L)$. Let $K \in X(N \bigcap L)$. Then $N \bigcap L \subseteq K$ and then by assumption, either $N \subseteq K$ or $L \subseteq K$, which yields, either $K \in X(N)$ or $K \in X(L)$ which proves that $X(N \bigcap L) \subseteq X(N) \bigcup X(L)$. Hence, $X(N) \bigcup X(L) = X(N \bigcap L)$. Conversely, let K be a graded pseudo prime R-submodule of M. Assume that N and L are graded pseudo semiprime R-submodules of M such that $N \bigcap L \subseteq K$. Then $K \in X(N \bigcap L) = X(N) \bigcup X(L)$ by assumption. So, $K \in X(N)$ or $K \in X(L)$ that is either $N \subseteq K$ or $L \subseteq K$. Hence, M is a graded topological R-module. □

Theorem 2.17. *Every graded multiplication module is graded topological module.*

Proof: Suppose that M is a graded multiplication R-module. Let N be a graded pseudo prime R-submodule of M. Assume that K and L are two graded pseudo semiprime R-submodules of M such that $K \bigcap L \subseteq N$. Then there exist graded ideals I and J of R such that $X(K) = X(IM)$ and $X(L) = X(JM)$. Suppose that $K = \bigcap\limits_{i \in \Lambda} P_i$ for some graded pseudo prime R-submodules $\{P_i\}_{i \in \Lambda}$. Now, for every $i \in \Lambda$, $P_i \in X(K) \subseteq X(K) \bigcup X(L) = X(IM) \bigcup X(JM) = X((I \bigcap J)M)$ by

Lemma 2.16. This implies that $(I \bigcap J)M \subseteq \bigcap_{i \in \Lambda} P_i = K$. Similarly, $(I \bigcap J)M \subseteq L$. So, $(I \bigcap J)M \subseteq K \bigcap L \subseteq N$, i.e., $I \bigcap J \subseteq (N :_R M)$ which implies $K \subseteq N$ or $L \subseteq N$. Hence, M is a graded topological R-module. $\square$

Lemma 2.18. *If M is a graded topological R-module and N is an R-submodule of M, then M/N is a graded topological R-module.*

Proof: Any graded pseudo semiprime R-submodule of M/N has the form S/N where S is a graded pseudo semiprime R-submodule of M containing N. Apply Lemma 2.16. $\square$

Lemma 2.19. *Let M be a G-graded R-module where R is a field such that $R_g + R_h \subseteq h(R)$ for all $g, h \in G$. Then M contains a graded extraordinary R-submodule if and only if M is of dimension 1 over R.*

Proof: Since R is a field, M is a vector space over R and then every proper subspace of M is prime. Suppose that M contains a graded extraordinary R-submodule N and M is not of dimension 1 over R. Clearly, $N \neq M$ and $N \neq \{0\}$. So, there exists $m \in M$ such that $m \notin N$ and there exists $0 \neq n \in N$ and then there exist $g, h \in G$ such that $m_g \notin N$ and $n_h \neq 0$. Now, Rm_g and $R(m_g + n_h)$ are graded R-submodules of M such that $Rm_g \bigcap R(m_g + n_h) = \{0\} \subseteq N$ but $Rm_g \nsubseteq N$ and $R(m_g + n_h) \nsubseteq N$ which is a contradiction. The converse is clear. $\square$

Lemma 2.20. *If M is a G-graded topological R-module such that $R_g + R_h \subseteq h(R)$ for all $g, h \in G$, then M/JM is a graded simple R-module for every graded maximal ideal J of R.*

Proof: Let J be a graded maximal ideal of R. If $M = JM$, then we are done. Suppose that $M \neq JM$. Since M is graded topological, by Lemma 2.18, the R/J-module M/JM is graded topological. In particular, M/JM contains a graded extraordinary R/J-submodule. Apply Lemma 2.19 on R/J and M/JM to obtain that M/JM is of dimension 1 over R/J, i.e., M/JM is a graded simple R-module. $\square$

Theorem 2.21. *Let M be a G-graded finitely generated R-module such that $R_g + R_h \subseteq h(R)$ for all $g, h \in G$. Then M is graded multiplication if and only if M is graded topological.*

Proof: Suppose that M is a graded topological R-module. Then by Lemma 2.20, M/JM is a graded simple R-module for every graded maximal ideal J of R and so graded cyclic and then by Corollary 2.8, M is a graded multiplication R-module. The converse holds by Theorem 2.17. $\square$

3. Graded Pseudo Prime Spectrum of Graded Topological Modules

In this section, we assume that M is always a graded topological R-module. Then $\emptyset = X(M)$, $Y_M = X(\{0\})$ and for any family of graded R-submodules $\{N_i\}_{i\in\Lambda}$ of M, $\bigcap_{i\in\Lambda} X(N_i) = X\left(\sum_{i\in\Lambda} N_i\right)$. Also, for any graded R-submodules N and L of M, $X(N)\bigcup X(L) = X(N\bigcap L)$. Thus, if $\sigma(M)$ denotes the collection of all subsets $X(N)$ of Y_M, then $\sigma(M)$ satisfies the axioms of a topological space for the closed subsets. This topology is called the Zariski topology.

We investigate the topological properties of this topology and we introduce some results about the relationship between algebraic properties of graded topological modules and topological properties of the Zariski topology on the graded pseudo prime spectrum of them.

We define $X^R(N) = \{K \in Y_R : N \subseteq K\}$. In this section, for every graded ideal $J \in X^R(Ann(M))$, R^* and J^* denote $R/Ann(M)$ and $J/Ann(M)$, respectively. We begin our applications of Zariski topology with the following.

Theorem 3.1. *Let M be a graded pseudo primeful R-module such that Y_M is connected. Then Y_{R^*} is connected and the only idempotent elements of R^* are 0^* and 1^*, where 0^* is the zero of $R^* = R/Ann(M)$ and 1^* is the unity of R^*.*

Proof: Since the natural map $\varphi : Y_M \to GSpec(R/Ann(M))$ is surjective, it is enough to prove that φ is continuous with respect to the Zariski topology. Let J be a graded ideal of R containing $Ann(M)$ and let $K \in \varphi^{-1}(X^{R^*}(J^*))$. Then there exists $I^* \in X^{R^*}(J^*)$ such that $\varphi(K) = I^*$ and then $I = (K :_R M) \supseteq J$ which implies that $JM \subseteq K$ and hence $K \in X(JM)$. Let $L \in X(JM)$. Then $(L :_R M) \supseteq (JM : M) \supseteq J$ and then $L \in \varphi^{-1}(X^{R^*}(J^*))$. So, $\varphi^{-1}(X^{R^*}(J^*)) = X(JM)$ which means that φ is continuous. □

Lemma 3.2. *Let M be a graded R-module, $W \subseteq Y_M$, $J \in GSpec(R)$ and $K \in Y_{M,J}$. Then $Cl(W) = X(\mu(W))$ and hence W is closed if and only if $W = X(\mu(W))$. In particular, $Cl(\{K\}) = X(K)$.*

Proof: Clearly, $W \subseteq X(\mu(W))$. Let $X(N)$ be any closed subset of Y_M containing W. Since $\mu(W) \supseteq \mu(X(N))$, $X(\mu(W)) \subseteq X(\mu(X(N))) = X(GPrad(N)) = X(N)$ which means that $X(\mu(W))$ is the smallest closed subset of Y_M containing W. Hence, $Cl(W) = X(\mu(W))$. □

Theorem 3.3. *Let M be a graded R-module and $W \subseteq Y_M$. If $\{0\} \in W$, then W is dense in Y_M.*

Proof: It is clear by Lemma 3.2. □

Theorem 3.4. *Let M be a graded R-module. Then Y_M is a T_0-space.*

Proof: Let N and K be two distinct points of Y_M. Then by Lemma 3.2, $Cl(\{N\}) = X(N) \neq X(K) = Cl(\{K\})$. Hence, Y_M is a T_0-space. $\square$

Theorem 3.5. *Let M be a graded R-module. Then Y_M is a T_1-space if and only if every element of Y_M is maximal element.*

Proof: Suppose that every element of Y_M is maximal element. Let $Q \in Y_M$. Then Q is maximal in Y_M and then by Lemma 3.2, $Cl(\{Q\}) = X(Q) = \{Q\}$ which means that $\{Q\}$ is closed and hence Y_M is a T_1-space. Conversely, let $Q \in Y_M$. Since Y_M is a T_1-space, $\{Q\}$ is closed and then $\{Q\} = Cl(\{Q\}) = X(\mu(\{Q\})) = X(Q)$ and hence Q is maximal in Y_M. $\square$

In [4], a topological space T is said to be irreducible if for any decomposition $T = B_1 \bigcup B_2$ with closed subsets B_1 and B_2 of T, we have either $B_1 = T$ or $B_2 = T$. A nonempty subset A of a topological space T is said to be irreducible if it cannot be expressed as the union $A = A_1 \bigcup A_2$ of two proper subsets, each one of which is closed in A. The empty set is not considered to be irreducible. Also, an irreducible component of a topological space T is a maximal irreducible subset of T. Since every singleton of Y_M is irreducible, its closure is also irreducible. So, using Lemma 3.2, we can state the following theorem.

Theorem 3.6. *Let M be a graded R-module. Then $X(K)$ is an irreducible closed subset of Y_M for every graded pseudo prime R-submodule K of M.*

Theorem 3.7. *Let M be a graded R-module and $W \subseteq Y_M$. Then $\mu(W)$ is a graded pseudo prime R-submodule of M if and only if W is an irreducible space.*

Proof: Suppose that W is irreducible. Clearly, $\mu(W)$ is a graded R-submodule of M. Let I and J be graded ideals of R such that $IJ \subseteq (\mu(W) :_R M)$. Then $IJM \subseteq \mu(W)$. So, $W \subseteq X(\mu(W)) \subseteq X(IJM) = X(IM) \bigcup X(JM)$ (see [4]). Since W is irreducible, either $W \subseteq X(IM)$ or $W \subseteq X(JM)$ and hence either $\mu(W) \supseteq \mu(X(IM)) = GPrad(IM) \supseteq IM$ or $\mu(W) \supseteq JM$ which implies that either $I \subseteq (\mu(W) :_R M)$ or $J \subseteq (\mu(W) :_R M)$. Hence, $\mu(W)$ is a graded pseudo prime R-submodule of M. Conversely, assume that $W \subseteq W_1 \bigcup W_2$ for closed subsets W_1 and W_2 of Y_M. Then there exist graded R-submodules N and K of M such that $W_1 = X(N)$ and $W_2 = X(K)$ and hence $\mu(W) \supseteq \mu(X(N) \bigcup X(K)) = \mu(X(N) \bigcap X(K)) = \mu(X(N)) \bigcap \mu(X(K)) = GPrad(N) \bigcap GPrad(K)$.

Since M is a graded topological R-module, $\mu(W)$ is a graded extraordinary R-submodule of M and hence either $GPrad(N) \subseteq \mu(W)$ or $GPrad(K) \subseteq \mu(W)$. So, $W \subseteq X(\mu(W)) \subseteq X(GPrad(N)) = X(N) = W_1$ or $W \subseteq W_2$. Hence, W is an irreducible space. $\square$

Corollary 3.8. *Let M be a graded R-module and N be a graded R-submodule of M. Then $X(N)$ is an irreducible subset if and only if $GPrad(N)$ is a graded pseudo prime R-submodule of M.*

Proof: Since $GPrad(N) = \mu(X(N))$, the result is clear by Theorem 3.7. $\square$

If we apply Corollary 3.8 on $N = \{0\}$, we can state the following.

Corollary 3.9. *Let M be a graded R-module. Then Y_M is an irreducible space if and only if $GPrad(\{0\})$ is a graded pseudo prime R-submodule of M.*

Corollary 3.10. *Let M be a graded R-module and $J \in GSpec(R)$. If $Y_{M,J} \neq \emptyset$, then $Y_{M,J}$ is an irreducible space.*

Proof: Since $(\mu(Y_{M,J}) :_R M) = \bigcap_{P \in Y_{M,J}} (P :_R M) = J \in GSpec(R)$, so $\mu(Y_{M,J})$ is pseudo prime which leads by Theorem 3.7 that $Y_{M,J}$ is irreducible. $\square$

Corollary 3.11. *Let M be a graded R-module. If $\{0\} \in Y_M$, then Y_M is an irreducible space.*

Proof: Since $\mu(Y_M) = \{0\} \in Y_M$, by Theorem 3.7, Y_M is an irreducible space. $\square$

Corollary 3.12. *Let M be a graded torsion free R-module. If R is an integral domain, then Y_M is an irreducible space.*

Proof: Since $(0 :_R M) = \{0\} \in GSpec(R)$, by Theorem 3.7, Y_M is an irreducible space. $\square$

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