Bol. Soc. Paran. Mat. (3s.) **v. 39** 3 (2021): 201–220.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.40607

Certain Results on Lorentzian Para-Kenmotsu Manifolds

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abstract: The object of the present paper is to study Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First, we study Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying the curvature conditions $\bar{R}\cdot\bar{S}=0$ and $\bar{S}\cdot\bar{R}=0$. Next, we study ϕ-conformally flat, ϕ-conharmonically flat, ϕ-concircularly flat, ϕ-projectively flat and conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection and it is shown that in each of these cases the manifold is a generalized η-Einstein manifold.

Key Words: Lorentzian para-Kenmotsu manifold, η-Einstein manifold, Curvature tensor, Quarter-symmetric metric connection.

Contents

2010 *Mathematics Subject Classification:* 53D15, 53C05, 53C25.

1. Introduction

Let (M, g), be an n-dimensional connected semi-Riemannian manifold of class C^∞ and ∇ be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R, the projective curvature tensor P, the concircular curvature tensor V, the conharmonic curvature tensor K and the conformal curvature tensor C of (M, g) are defined by [16]

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \tag{1.2}$$

$$V(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{1.4}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{1.5}$$

respectively, where r is the scalar curvature, S is the Ricci tensor and Q is the Ricci operator such that $S(X,Y) = g(QX,Y)$.

A linear connection $\bar{\nabla}$ defined on (M, g) is said to be a quarter-symmetric connection [8] if its torsion tensor T

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.6}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a $(1,1)$-tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0, \tag{1.7}$$

where $X, Y, Z \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. If we change ϕX by X, then the connection is known as semi-symmetric metric connection [7]. Thus the notion of quarter-symmetric connection generalizes the notion of semi-symmetric

Submitted November 27, 2017. Published May 06, 2018

connection. A quarter-symmetric metric connection have been studied by many geometers in several ways to a different extent such as ([1], [3], [5], [6], [9]- [12], [15]) and many others.

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in a Lorentzian para-Kenmotsu manifold M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{1.8}$$

The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian para-Kenmotsu manifolds. In Section 3, we establish the relation between the curvature tensors of the Riemannian connection and the quarter-symmetric metric connection in a Lorentzian para-Kenmotsu manifold. Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying the curvature conditions $\bar{R}\cdot\bar{S} = 0$ and $\bar{S}\cdot\bar{R} = 0$ have studied in Sections 4 and 5 respectively. Sections 6, 7, 8, 9 and 10 are devoted to study ϕ-conformally flat, ϕ-conharmonically flat, ϕ-concircularly flat, ϕ-projectively flat and conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection, respectively.

2. Preliminaries

Let M be an n-dimensional Lorentzian metric manifold. If it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on M and g is a Lorentz metric, satisfying [2]

$$\phi^2 X = X + \eta(X)\xi, \qquad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

$$\eta(\xi) = -1, \qquad g(X, \xi) = \eta(X) \tag{2.2}$$

for any X, Y on M, then it is called Lorentzian almost paracontact manifold. In the Lorentzian almost paracontact manifold, the following relations hold:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.3}$$

$$\Phi(X, Y) = \Phi(Y, X), \tag{2.4}$$

where $\Phi(X, Y) = g(X, \phi Y)$.
If ξ is a killing vector field, the (para) contact structure is called K-(para) contact. In such a case, we have

$$\nabla_X \xi = \phi X. \tag{2.5}$$

A Lorentzian almost paracontact manifold M is called Lorentzian para-Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.6}$$

for any vector fields X, Y on M.

Now, we define a new manifold called Lorentzian para-Kenmostu manifold:

Definition 2.1. *A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmostu manifold if*

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.7}$$

for any vector fields X, Y on M.

In the Lorentzian para-Kenmostu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Further, on a Lorentzian para-Kenmotsu manifold M, the following relations hold:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.10}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.13}$$

$$S(X, \xi) = (n-1)\eta(X),\ S(\xi, \xi) = -(n-1), \tag{2.14}$$

$$Q\xi = (n-1)\xi, \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{2.16}$$

for any vector fields X, Y and Z on M.

Example 2.2. *We consider the 3-dimensional manifold*

$$M^3 = \left\{(x, y, z) \in R^3 : z > 0\right\},$$

where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M^3 given by

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M^3 and hence form a basis of T_pM^3. Define a Lorentzian metric g on M^3 as

$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1,\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M^3 defined as $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field on M^3 defined as

$$\phi e_1 = -e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi,\xi) = -1,\ \phi^2 X = X + \eta(X)\xi,\ \eta(\phi X) = 0,$$

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$.
Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1,e_2] = 0,\ [e_2,e_1] = 0,\ [e_1,e_3] = -e_1,\ [e_3,e_1] = e_1,\ [e_2,e_3] = -e_2,\ [e_3,e_2] = e_2.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -e_1,\ \nabla_{e_2} e_1 = 0, \tag{2.17}$$

$$\nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_3 = -e_2,\ \nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

$$\textit{Let} \qquad X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M).$$

Also, one can easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \textit{and} \quad (\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Now let

$$X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3,$$

$$Y = \sum_{j=1}^{3} Y^j e_j = Y^1 e_1 + Y^2 e_2 + Y^3 e_3,$$

$$Z = \sum_{k=1}^{3} Z^k e_k = Z^1 e_1 + Z^2 e_2 + Z^3 e_3$$

for all $X, Y, Z \in \chi(M)$. It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.18}$$

From the equations (2.17) and (2.18), it can be easily verified that

$$R(e_1,e_2)e_1 = -e_2,\ R(e_1,e_3)e_1 = -e_3,\ R(e_2,e_3)e_1 = 0, \tag{2.19}$$

$$R(e_1,e_2)e_2 = e_1,\ R(e_1,e_3)e_2 = 0,\ R(e_2,e_3)e_2 = -e_3,$$

$$R(e_1,e_2)e_3=0,\ R(e_1,e_3)e_3=-e_1,\ R(e_2,e_3)e_3=-e_2.$$

With the help of above expressions of the curvature tensors, it follows that

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y.$$

Hence, the manifold (M^3,ϕ,ξ,η,g) is a Lorentzian para-Kenmotsu manifold of constant curvature 1 and is locally isometric to the unit sphere $S^3(1)$.

Definition 2.3. *A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form [4]*

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where a and b are scalar functions on M.
A Lorentzian para-Kenmotsu manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)+c\Phi(X,Y),$$

where a, b, c are scalar functions on M and $\Phi(X,Y)=g(\phi X,Y)$. If $c=0$, then the manifold reduces to an η-Einstein manifold.

3. Curvature tensor of Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

The curvature tensor $\bar{R}$ of a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

From the equations (1.8), (2.1), (2.3), (2.7)-(2.9), we get

$$\bar{R}(X,Y)Z=R(X,Y)Z+g(Y,Z)\phi X-g(X,Z)\phi Y+g(\phi Y,Z)X \tag{3.2}$$
$$-g(\phi X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y,$$

where

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of the connection ∇. Taking inner product of (3.2) with W, we have

$$\bar{R}(X,Y,Z,W)=R(X,Y,Z,W)+g(Y,Z)g(\phi X,W)-g(X,Z)g(\phi Y,W) \tag{3.3}$$
$$+g(\phi Y,Z)g(X,W)-g(\phi X,Z)g(Y,W)+g(\phi Y,Z)g(\phi X,W)-g(\phi X,Z)g(\phi Y,W),$$

where $\bar{R}(X,Y,Z,W)=g(\bar{R}(X,Y)Z,W)$ and $R(X,Y,Z,W)=g(R(X,Y)Z,W)$. Contracting (3.3) over X and W, we get

$$\bar{S}(Y,Z)=S(Y,Z)+(n-2+\psi)g(\phi Y,Z)+(\psi-1)g(Y,Z)-\eta(Y)\eta(Z), \tag{3.4}$$

where S and $\bar{S}$ are the Ricci tensors with respect to the connections ∇ and $\bar{\nabla}$, respectively on M and ψ=trace ϕ.
From (3.4), we have

$$\bar{Q}Y = QY + (n-2+\psi)\phi Y + (\psi-1)Y - \eta(Y)\xi, \tag{3.5}$$

where Q and $\bar{Q}$ are the Ricci operators with respect to the connections ∇ and $\bar{\nabla}$, respectively on M. Contracting (3.4) over Y and Z, we get

$$\bar{r} = r + (2\psi-1)(n-1) + \psi^2, \tag{3.6}$$

where r and $\bar{r}$ are the scalar curvatures with respect to the connections ∇ and $\bar{\nabla}$, respectively on M.
Writing two more equations by the cyclic permutations of X, Y and Z, we have

$$\bar{R}(Y,Z)X = R(Y,Z)X + g(Z,X)\phi Y - g(Y,X)\phi Z + g(\phi Z,X)Y \tag{3.7}$$

$$-g(\phi Y,X)Z + g(\phi Z,X)\phi Y - g(\phi Y,X)\phi Z,$$

$$\bar{R}(Z,X)Y = R(Z,X)Y + g(X,Y)\phi Z - g(Z,Y)\phi X + g(\phi X,Y)Z \tag{3.8}$$

$$-g(\phi Z,Y)X + g(\phi X,Y)\phi Z - g(\phi Z,Y)\phi X.$$

By adding (3.2), (3.7) and (3.8) and using the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{3.9}$$

Thus we can state that, if the manifold is a Lorentzian para-Kenmotsu, then the curvature tensor with respect to the quarter-symmetric metric connection satisfies the first Bianchi identity.
From (3.2), clearly

$$\bar{R}(X,Y,Z,W) = -\bar{R}(Y,X,Z,W), \tag{3.10}$$

$$\bar{R}(X,Y,Z,W) = -\bar{R}(X,Y,W,Z). \tag{3.11}$$

$$\bar{R}(X,Y,Z,W) = \bar{R}(Z,W,X,Y). \tag{3.12}$$

Combining equations (3.10)-(3.12), we have

$$\bar{R}(X,Y,Z,W) = \bar{R}(Y,X,W,Z) = \bar{R}(W,Z,Y,X). \tag{3.13}$$

Thus, in view of the equations (3.10)-(3.12), we can state the following theorem:

Theorem 3.1. *The curvature tensor of type $(0,4)$ of a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is*
(i) *Skew-symmetric in first two slots,*
(ii) *Skew-symmetric in last two slots,*
(iii) *Symmetric in pair of slots.*

Lemma 3.2. *Let M be an n-dimensional Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then*

$$\bar{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y + \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.14}$$

$$\bar{R}(\xi,X)Y = -\bar{R}(X,\xi)Y = g(X,Y)\xi - \eta(Y)X - \eta(Y)\phi X + g(\phi X,Y)\xi, \tag{3.15}$$

$$\bar{R}(\xi,X)\xi = \eta(X)\xi + X + \phi X, \tag{3.16}$$

$$\bar{S}(X,\xi) = (n+\psi-1)\eta(X), \qquad \bar{S}(\xi,\xi) = -(n+\psi-1), \tag{3.17}$$

$$\bar{Q}\xi = (n+\psi-1)\xi \tag{3.18}$$

for all X, Y on M.

4. Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying $\bar{R}.\bar{S} = 0$

In this section we consider a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{R}(X,Y).\bar{S} = 0. \tag{4.1}$$

Then we have

$$\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) = 0 \tag{4.2}$$

for any vector fields $X, Y, U, V \in \chi(M)$. Putting $X = \xi$ in (4.2), it follows that

$$\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U,\bar{R}(\xi,Y)V) = 0. \tag{4.3}$$

In view of (3.15) and (3.17), (4.3) yields

$$\begin{aligned}(n+\psi-1)g(Y,U)\eta(V) - \eta(U)\bar{S}(Y,V) - \eta(U)\bar{S}(\phi Y,V) \\ +(n+\psi-1)g(\phi Y,U)\eta(V) + (n+\psi-1)g(Y,V)\eta(U) - \eta(V)\bar{S}(U,Y) \\ -\eta(V)\bar{S}(U,\phi Y) + (n+\psi-1)g(\phi Y,V)\eta(U) = 0.\end{aligned} \tag{4.4}$$

By taking $U = \xi$ in (4.4) and using (2.2), we get

$$\bar{S}(Y,V) + \bar{S}(\phi Y,V) = (n+\psi-1)g(Y,V) + (n+\psi-1)g(\phi Y,V). \tag{4.5}$$

In view of (3.4), (4.5) takes the form

$$S(Y,V)+S(\phi Y,V) = (2-\psi)g(Y,V)+(3-n-\psi)\eta(Y)\eta(V)+(2-\psi)g(\phi Y,V). \tag{4.6}$$

Thus we can state the following theorem:

Theorem 4.1. *For a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{R}.\bar{S} = 0$, the Ricci tensor S is given by (4.6).*

5. Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying $\bar{S}.\bar{R} = 0$

In this section we consider a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$(\bar{S}(X, Y).\bar{R})(U, V)Z = 0 \tag{5.1}$$

for any vector fields $X, Y, Z, U, V \in \chi(M)$.
This implies that

$$(X \wedge_{\bar{S}} Y)\bar{R}(U, V)Z + \bar{R}((X \wedge_{\bar{S}} Y)U, V)Z + \bar{R}(U, (X \wedge_{\bar{S}} Y)V)Z \tag{5.2}$$

$$+\bar{R}(U, V)(X \wedge_{\bar{S}} Y)Z = 0,$$

where the endomorphism $X \wedge_{\bar{S}} Y$ is defined by

$$(X \wedge_{\bar{S}} Y)Z = \bar{S}(Y, Z)X - \bar{S}(X, Z)Y. \tag{5.3}$$

Taking $Y = \xi$ in (5.2), we have

$$(X \wedge_{\bar{S}} \xi)\bar{R}(U, V)Z + \bar{R}((X \wedge_{\bar{S}} \xi)U, V)Z + \bar{R}(U, (X \wedge_{\bar{S}} \xi)V)Z \tag{5.4}$$

$$+\bar{R}(U, V)(X \wedge_{\bar{S}} \xi)Z = 0.$$

From (3.17), (5.3) and (5.4), we have

$$(n + \psi - 1)[\eta(\bar{R}(U, V)Z)X + \eta(U)\bar{R}(X, V)Z + \eta(V)\bar{R}(U, X)Z \tag{5.5}$$

$$+\eta(Z)\bar{R}(U, V)X] - \bar{S}(X, \bar{R}(U, V)Z)\xi - \bar{S}(X, U)\bar{R}(\xi, V)Z$$

$$-\bar{S}(X, V)\bar{R}(U, \xi)Z - \bar{S}(X, Z)\bar{R}(U, V)\xi = 0.$$

Taking inner product of (5.5) with ξ, we get

$$(n + \psi - 1)[\eta(\bar{R}(U, V)Z)\eta(X) + \eta(U)\eta(\bar{R}(X, V)Z) \tag{5.6}$$

$$+\eta(V)\eta(\bar{R}(U, X)Z) + \eta(Z)\eta(\bar{R}(U, V)X)] + \bar{S}(X, \bar{R}(U, V)Z)$$

$$-\bar{S}(X, U)\eta(\bar{R}(\xi, V)Z) - \bar{S}(X, V)\eta(\bar{R}(U, \xi)Z) - \bar{S}(X, Z)\eta(\bar{R}(U, V)\xi) = 0.$$

By taking $U = Z = \xi$ in (5.6) and using (3.14)-(3.17) and (10), we get

$$\bar{S}(X, V) + \bar{S}(X, \phi V) = -(n + \psi - 1)g(X, V) - 2(n + \psi - 1)\eta(X)\eta(V) \tag{5.7}$$

$$-(n + \psi - 1)g(\phi V, X).$$

In view of (3.4), (5.7) takes the form

$$S(X, V) + S(X, \phi V) = -(2n + 3\psi - 4)g(X, V) - (3n + 3\psi - 5)\eta(X)\eta(V) \tag{5.8}$$

$$-(2n + \psi)g(\phi X, V).$$

Thus we can state the following theorem:

Theorem 5.1. *For a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{S}.\bar{R} = 0$, the Ricci tensor S is given by (5.8).*

6. ϕ-conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.5), the conformal curvature tensor $\bar{C}$ with respect to the quarter-symmetric metric connection is defined by

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \tag{6.1}$$

$$-g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],$$

where $\bar{R}$, $\bar{S}$ and $\bar{r}$ are the Riemannian curvature tensor, the Ricci tensor and the scalar curvature with respect to the connection $\bar{\nabla}$, respectively on M.

Definition 6.1. *A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ϕ-conformally flat if [13,14]*

$$\phi^2\bar{C}(\phi X,\phi Y)\phi Z = 0 \tag{6.2}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-conformally flat Lorentzian para-Kenmotsu manifold with the quarter-symmetric metric connection. Then from (6.2), it follows that

$$g(\bar{C}(\phi X,\phi Y)\phi Z,\phi W) = 0. \tag{6.3}$$

From the equations (6.1) and (6.3), we have

$$g[\bar{R}(\phi X,\phi Y)\phi Z,\phi W] = \frac{1}{(n-2)}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi W) \tag{6.4}$$

$$-\bar{S}(\phi X,\phi Z)g(\phi Y,\phi W) + \bar{S}(\phi X,\phi W)g(\phi Y,\phi Z) - \bar{S}(\phi Y,\phi W)g(\phi X,\phi Z)]$$

$$-\frac{\bar{r}}{(n-1)(n-2)}[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)].$$

In view of (2.1), (2.3), (3.2) and (3.4), (6.4) takes the form

$$g[R(\phi X,\phi Y)\phi Z,\phi W] - g(\phi X,\phi Z)g(Y,\phi W) + g(\phi Y,\phi Z)g(X,\phi W) \tag{6.5}$$

$$+g(\phi X,\phi W)g(Y,\phi Z) - g(\phi Y,\phi W)g(X,\phi Z)$$

$$+g(Y,\phi Z)g(X,\phi W) - g(X,\phi Z)g(Y,\phi W)$$

$$= \frac{1}{(n-2)}[\{S(\phi Y,\phi Z) + (n-2+\psi)g(Y,\phi Z) + (\psi-1)g(\phi Y,\phi Z)\}g(\phi X,\phi W)$$

$$-\{S(\phi X,\phi Z) + (n-2+\psi)g(X,\phi Z) + (\psi-1)g(\phi X,\phi Z)\}g(\phi Y,\phi W)$$

$$+\{S(\phi X,\phi W) + (n-2+\psi)g(X,\phi W) + (\psi-1)g(\phi X,\phi W)\}g(\phi Y,\phi Z)$$

$$-\{S(\phi Y,\phi W) + (n-2+\psi)g(Y,\phi W) + (\psi-1)g(\phi Y,\phi W)\}g(\phi X,\phi Z)]$$

$$-\frac{\bar{r}}{(n-1)(n-2)}[g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)].$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = W = e_i$ in (6.5) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i]-\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(Y,\phi e_i)+g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(e_i,\phi e_i) \tag{6.6}$$
$$+g(Y,\phi Z)\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i)-\sum_{i=1}^{n-1} g(\phi Y,\phi e_i)g(e_i,\phi Z)+\sum_{i=1}^{n-1} g(Y,\phi Z)g(e_i,\phi e_i)$$
$$-\sum_{i=1}^{n-1} g(e_i,\phi Z)g(Y,\phi e_i)=\frac{1}{(n-2)}[\{S(\phi Y,\phi Z)+(n-2+\psi)g(Y,\phi Z)$$
$$+(\psi-1)g(\phi Y,\phi Z)\}\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i)-\sum_{i=1}^{n-1} S(\phi e_i,\phi Z)g(\phi Y,\phi e_i)$$
$$-(n-2+\psi)\sum_{i=1}^{n-1} g(e_i,\phi Z)g(\phi Y,\phi e_i)-(\psi-1)\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)$$
$$+g(\phi Y,\phi Z)\sum_{i=1}^{n-1} S(\phi e_i,\phi e_i)+(n-2+\psi)g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(e_i,\phi e_i)$$
$$+(\psi-1)g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i)-\sum_{i=1}^{n-1} S(\phi Y,\phi e_i)g(\phi e_i,\phi Z)$$
$$-(n-2+\psi)\sum_{i=1}^{n-1} g(Y,\phi e_i)g(\phi e_i,\phi Z)-(\psi-1)\sum_{i=1}^{n-1} g(\phi Y,\phi e_i)g(\phi e_i,\phi Z)]$$
$$-\frac{\bar{r}}{(n-1)(n-2)}[g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i)-\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)].$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i]=S(\phi Y,\phi Z)-g(\phi Y,\phi Z), \tag{6.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)=g(\phi Y,\phi Z), \tag{6.8}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(Y,\phi e_i)=g(Y,\phi Z), \tag{6.9}$$

$$\sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = g(Y, Z) + \eta(Y)\eta(Z), \tag{6.10}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n-1), \tag{6.11}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{6.12}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{6.13}$$

$$\sum_{i=1}^{n-1} g(e_i, \phi e_i) = \psi. \tag{6.14}$$

By virtue of (6.7)-(6.14), the equation (6.6) can be written as

$$S(\phi Y, \phi Z) + (\psi - 2)g(\phi Y, \phi Z) + (n + \psi - 3)g(Y, \phi Z) \tag{6.15}$$
$$= \frac{1}{(n-2)}[(n-3)S(\phi Y, \phi Z) + (n-3)(n+\psi-2)g(Y, \phi Z)$$
$$+\{2(n-2)(\psi-1) + r - n + 1 + (n-2+\psi)\psi\}g(\phi Y, \phi Z)] - \frac{\bar{r}}{(n-1)}g(\phi Y, \phi Z)$$

from which it follows that

$$S(\phi Y, \phi Z) = [r + 2n\psi + \psi^2 - 4\psi - n + 1 - \frac{(n-2)\bar{r}}{n-1}]g(\phi Y, \phi Z) - \psi g(Y, \phi Z). \tag{6.16}$$

In view of (2.1), (3.6) and (3.7), (6.16) yields

$$S(Y, Z) = [\frac{r + \psi^2}{n-1} - 1]g(Y, Z) + [\frac{r + \psi^2}{n-1} - n]\eta(Y)\eta(Z) - \psi g(Y, \phi Z).$$

Thus we can state the following theorem:

Theorem 6.2. *An n-dimensional ϕ-conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with respect to the connection ∇.*

7. ϕ-conharmonically flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.4), the conharmonic curvature tensor $\bar{K}$ with respect to the quarter-symmetric metric connection is defined by

$$\bar{K}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y], \tag{7.1}$$

where $\bar{R}$, $\bar{S}$ and $\bar{Q}$ are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator with respect to the connection $\bar{\nabla}$, respectively on M.

Definition 7.1. *A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ϕ-conharmonically flat if*

$$\phi^2 \bar{K}(\phi X, \phi Y)\phi Z = 0 \tag{7.2}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-conharmonically flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (7.2), it follows that

$$g(\bar{K}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{7.3}$$

From the equations (7.1) and (7.3), we have

$$g[\bar{R}(\phi X, \phi Y)\phi Z, \phi W] = \frac{1}{(n-2)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) \tag{7.4}$$

$$-\bar{S}(\phi X, \phi Z)g(\phi Y, \phi W) + \bar{S}(\phi X, \phi W)g(\phi Y, \phi Z) - \bar{S}(\phi Y, \phi W)g(\phi X, \phi Z)].$$

In view of (2.1), (2.3), (3.2) and (3.4), (7.4) takes the form

$$g[R(\phi X, \phi Y)\phi Z, \phi W] - g(\phi X, \phi Z)g(Y, \phi W) + g(\phi Y, \phi Z)g(X, \phi W) \tag{7.5}$$

$$+g(\phi X, \phi W)g(Y, \phi Z) - g(\phi Y, \phi W)g(X, \phi Z)$$

$$+g(Y, \phi Z)g(X, \phi W) - g(X, \phi Z)g(Y, \phi W)$$

$$= \frac{1}{(n-2)}[\{S(\phi Y, \phi Z) + (n-2+\psi)g(Y, \phi Z) + (\psi-1)g(\phi Y, \phi Z)\}g(\phi X, \phi W)$$

$$-\{S(\phi X, \phi Z) + (n-2+\psi)g(X, \phi Z) + (\psi-1)g(\phi X, \phi Z)\}g(\phi Y, \phi W)$$

$$+\{S(\phi X, \phi W) + (n-2+\psi)g(X, \phi W) + (\psi-1)g(\phi X, \phi W)\}g(\phi Y, \phi Z)$$

$$-\{S(\phi Y, \phi W) + (n-2+\psi)g(Y, \phi W) + (\psi-1)g(\phi Y, \phi W)\}g(\phi X, \phi Z)].$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = W = e_i$ in (7.5) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) + g(\phi Y, \phi Z)\sum_{i=1}^{n-1} g(e_i, \phi e_i) \tag{7.6}$$

$$+g(Y, \phi Z)\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) + \sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i)$$

$$-\sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = \frac{1}{(n-2)}[\{S(\phi Y, \phi Z) + (n-2+\psi)g(Y, \phi Z)$$

$$+(\psi-1)g(\phi Y, \phi Z)\}\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)$$

$$-(n-2+\psi)\sum_{i=1}^{n-1} g(e_i,\phi Z)g(\phi Y,\phi e_i) - (\psi-1)\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)$$

$$+g(\phi Y,\phi Z)\sum_{i=1}^{n-1} S(\phi e_i,\phi e_i) + (n-2+\psi)g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(e_i,\phi e_i)$$

$$+(\psi-1)g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) - \sum_{i=1}^{n-1} S(\phi Y,\phi e_i)g(\phi e_i,\phi Z)$$

$$-(n-2+\psi)\sum_{i=1}^{n-1} g(Y,\phi e_i)g(\phi e_i,\phi Z) - (\psi-1)\sum_{i=1}^{n-1} g(\phi Y,\phi e_i)g(\phi e_i,\phi Z)].$$

By virtue of (6.7)-(6.14), the equation (7.6) can be written as

$$S(\phi Y,\phi Z) + (\psi-2)g(\phi Y,\phi Z) + (n+\psi-3)g(Y,\phi Z) \tag{7.7}$$

$$= \frac{1}{(n-2)}[(n-3)S(\phi Y,\phi Z) + (n-3)(n+\psi-2)g(Y,\phi Z)$$

$$+\{2(n-2)(\psi-1) + r - n + 1 + (n-2+\psi)\psi\}g(\phi Y,\phi Z)]$$

from which it follows that

$$S(\phi Y,\phi Z) = [r + 2n\psi + \psi^2 - 4\psi - n + 1]g(\phi Y,\phi Z) - \psi g(Y,\phi Z). \tag{7.8}$$

In view of (2.1) and (2.16), (7.8) yields

$$S(Y,Z) = [r + 2n\psi + \psi^2 - 4\psi - n + 1]g(Y,Z) \tag{7.9}$$

$$+[r + 2n\psi + \psi^2 - 4\psi - 2n + 2]\eta(Y)\eta(Z) - \psi g(Y,\phi Z).$$

Contracting (7.9) over Y and Z gives

$$r = \frac{(4\psi - \psi^2 - 2n\psi)(n-1) + (n-1)(n-2) + \psi^2}{n-2}. \tag{7.10}$$

By using this value of r in (7.9), we get

$$S(Y,Z) = -2\psi g(Y,Z) - (n+2\psi-1)\eta(Y)\eta(Z) - \psi g(Y,\phi Z).$$

Thus we can state the following theorem:

Theorem 7.2. *An n-dimensional ϕ-conhamonically flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with the scalar curvature r given by (7.10).*

8. ϕ-concircularly flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.3), the concircular curvature tensor $\bar{V}$ with respect to the quarter-symmetric metric connection is defined by

$$\bar{V}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{8.1}$$

where $\bar{R}$ and $\bar{r}$ are the Riemannian curvature tensor and the scalar curvature with respect to the connection $\bar{\nabla}$, respectively on M.

Definition 8.1. *A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ϕ-concircularly flat if*

$$\phi^2\bar{V}(\phi X,\phi Y)\phi Z = 0 \tag{8.2}$$

for all X, Y, Z on M.

Let M be an n-dimensional ϕ-concircularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (8.2), it follows that

$$g(\bar{V}(\phi X,\phi Y)\phi Z,\phi W) = 0. \tag{8.3}$$

From the equations (8.1) and (8.3), we have

$$g[\bar{R}(\phi X,\phi Y)\phi Z,\phi W] = \frac{\bar{r}}{n(n-1)}[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{8.4}$$

In view of (3.2), (8.4) takes the form

$$\begin{aligned} & g[R(\phi X,\phi Y)\phi Z,\phi W] - g(\phi X,\phi Z)g(Y,\phi W) + g(\phi Y,\phi Z)g(X,\phi W) \\ & +g(\phi X,\phi W)g(Y,\phi Z) - g(\phi Y,\phi W)g(X,\phi Z) + g(Y,\phi Z)g(X,\phi W) \\ & -g(X,\phi Z)g(Y,\phi W) = \frac{\bar{r}}{n(n-1)}[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{8.5}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = W = e_i$ in (8.5) and sum up with respect to i, then we have

$$\begin{aligned} & \sum_{i=1}^{n-1} g[R(\phi e_i,\phi Y)\phi Z,\phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(Y,\phi e_i) + g(\phi Y,\phi Z)\sum_{i=1}^{n-1} g(e_i,\phi e_i) \\ & +g(Y,\phi Z)\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) - \sum_{i=1}^{n-1} g(\phi Y,\phi e_i)g(e_i,\phi Z) + \sum_{i=1}^{n-1} g(Y,\phi Z)g(e_i,\phi e_i) \\ & -\sum_{i=1}^{n-1} g(e_i,\phi Z)g(Y,\phi e_i) = \frac{\bar{r}}{n(n-1)}\sum_{i=1}^{n-1}[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \end{aligned} \tag{8.6}$$

By virtue of (6.7)-(6.10), (6.13) and (6.14), the equation (8.6) becomes

$$S(\phi Y,\phi Z)=[\frac{\bar{r}(n-2)}{n(n-1)}-\psi+2]g(\phi Y,\phi Z)-(\psi+n-3)g(Y,\phi Z). \tag{8.7}$$

In view of (2.1) and (2.16), (8.7) takes the form

$$S(Y,Z)=\frac{(n-2)[r+(2\psi-1)(n-1)+\psi^2]-n(n-1)(\psi-2)}{n(n-1)}g(Y,Z) \tag{8.8}$$

$$+\frac{(n-2)[r+(2\psi-1)(n-1)+\psi^2]-n(n-1)(n+\psi-3)}{n(n-1)}\eta(Y)\eta(Z)$$

$$-(n+\psi-3)g(Y,\phi Z).$$

Contracting (8.8) over Y and Z gives

$$r=n(n-\psi)-(\psi-1)^2. \tag{8.9}$$

By using this value of r in (8.8), we get

$$S(Y,Z)=(n-\frac{\psi}{n-1})g(Y,Z)+(1-\frac{\psi}{n-1})\eta(Y)\eta(Z)-(n+\psi-3)g(Y,\phi Z).$$

Thus we can state the following theorem:

Theorem 8.2. *An n-dimensional ϕ-concircularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with the scalar curvature r given by (8.9).*

9. ϕ-projectively flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.2), the projective curvature tensor $\bar{P}$ with respect to the quarter-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{(n-1)}[g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y], \tag{9.1}$$

where $\bar{R}$ and $\bar{Q}$ are the Riemannian curvature tensor and the Ricci operator with respect to the connection $\bar{\nabla}$, respectively on M.

Definition 9.1. *A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ϕ-projectively flat if*

$$\phi^2\bar{P}(\phi X,\phi Y)\phi Z=0 \tag{9.2}$$

for all X,Y,Z on M.

Let M be an n-dimensional ϕ-projectively flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (9.2), it follows that

$$g(\bar{P}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{9.3}$$

From the equations (9.1) and (9.3), we have

$$g[\bar{R}(\phi X, \phi Y)\phi Z, \phi W] = \frac{1}{(n-1)}[\bar{S}(\phi X, \phi W)g(\phi Y, \phi Z) - \bar{S}(\phi Y, \phi W)g(\phi X, \phi Z)]. \tag{9.4}$$

In view of (3.2) and (3.4), (9.4) takes the form

$$\begin{aligned} g[R(\phi X, \phi Y)\phi Z, \phi W] - g(\phi X, \phi Z)g(Y, \phi W) + g(\phi Y, \phi Z)g(X, \phi W) \\ +g(\phi X, \phi W)g(Y, \phi Z) - g(\phi Y, \phi W)g(X, \phi Z) + g(Y, \phi Z)g(X, \phi W) \\ -g(X, \phi Z)g(Y, \phi W) = \frac{1}{(n-1)}[\{S(\phi X, \phi W) + (n-2+\psi)g(X, \phi W) \\ +(\psi-1)g(\phi X, \phi W)\}g(\phi Y, \phi Z) - \{S(\phi Y, \phi W) + (n-2+\psi)g(Y, \phi W) \\ +(\psi-1)g(\phi Y, \phi W)\}g(\phi X, \phi Z)]. \end{aligned} \tag{9.5}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in M. If we put $X = W = e_i$ in (9.5) and sum up with respect to i, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) + g(\phi Y, \phi Z)\sum_{i=1}^{n-1} g(e_i, \phi e_i) \\ +g(Y, \phi Z)\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) \\ +\sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i) - \sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) \\ = \frac{1}{(n-1)}[g(\phi Y, \phi Z)\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi Y, \phi e_i)g(\phi e_i, \phi Z) \\ +(n-2+\psi)g(\phi Y, \phi Z)\sum_{i=1}^{n-1} g(e_i, \phi e_i) + (\psi-1)g(\phi Y, \phi Z)\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) \\ -(n-2+\psi)\sum_{i=1}^{n-1} g(Y, \phi e_i)g(\phi e_i, \phi Z) - (\psi-1)\sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(\phi e_i, \phi Z)]. \end{aligned} \tag{9.6}$$

By virtue of (6.7)-(6.14), the equation (9.6) turns to

$$nS(\phi Y, \phi Z) = (r + n\psi - 3\psi + \psi^2 + 1)g(\phi Y, \phi Z) - (n^2 - 3n + n\psi + 1)g(Y, \phi Z). \tag{9.7}$$

In view of (2.1) and (2.16), (9.7) becomes

$$S(Y,Z) = \frac{r + (n-3+\psi)\psi + 1}{n} g(Y,Z) - \frac{(n^2 - 3n + n\psi + 1)}{n} g(Y,\phi Z) \tag{9.8}$$
$$+ \frac{r - n^2 + n + (n-3+\psi)\psi + 1}{n} \eta(Y)\eta(Z).$$

Contracting (9.8) over Y and Z gives

$$r = n^2 - n\psi + 2\psi - \psi^2 - 1. \tag{9.9}$$

By using this value of r in (9.8), we get

$$S(Y,Z) = (n - \frac{\psi}{n}) g(Y,Z) + (1 - \frac{\psi}{n}) \eta(Y)\eta(Z) - \frac{(n^2 - 3n + n\psi + 1)}{n} g(Y,\phi Z).$$

Thus we can state the following theorem:

Theorem 9.2. *An n-dimensional ϕ-projectively flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with the scalar curvature r given by* (9.9).

10. Conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let M be an n-dimensional conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection, that is, $\bar{C} = 0$. Then from (6.1), it follows that

$$\bar{R}(X,Y)Z = \frac{1}{(n-2)} [\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \tag{10.1}$$
$$- \frac{\bar{r}}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y].$$

By taking $Y = \xi$ in (10.1) and using (2.2), (3.15) and (3.17), we have

$$\eta(Y)X - \eta(X)Y + \eta(Y)\phi X - \eta(X)\phi Y = \frac{1}{(n-2)} [(n+\psi-1)\eta(Y)X \tag{10.2}$$
$$-(n+\psi-1)\eta(X)Y + \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y] - \frac{\bar{r}}{(n-1)(n-2)} [\eta(Y)X - \eta(X)Y].$$

Now taking $Z = \xi$ in (10.2) and then using (2.2), (2.3) and (3.18), we get

$$\bar{Q}X = [n - 2 + \frac{\bar{r}}{n-1}](X + \eta(X)\xi) + (n-2)\phi X - (n+\psi-1)X - 2(n+\psi-1)\eta(X)\xi$$

which by taking inner product with W and using the fact that $g(\bar{Q}X, W) = \bar{S}(X,W)$ gives

$$\bar{S}(X,W) = [\frac{\bar{r}}{n-1} - \psi - 1] g(X,W) + [\frac{\bar{r}}{n-1} - 2\psi - n] \eta(X)\eta(W) + (n-2) g(\phi X, W). \tag{10.3}$$

In view of (3.4) and (3.6), (10.3) takes the form

$$S(X,W) = [\frac{r+\psi^2}{n-1} - 1]g(Y,Z) + [\frac{r+\psi^2}{n-1} - n]\eta(Y)\eta(Z) - \psi g(Y,\phi Z).$$

Thus we can state the following theorem:

Theorem 10.1. *An n-dimensional conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized η-Einstein manifold with respect to the connection ∇.*

Acknowledgments

The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

References

1. Ahmad, M., Jun, J. B. and Haseeb, A., *Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric metric connection*, Bull. Korean Math. Soc. 46, 477-487, (2009).
2. Alegre, P., *Slant submanifolds of Lorentzian Sasakian and Para Sasakian manifolds*, Taiwanese J. Math. 17, 897-910, (2013).
3. Barman, A., *Weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection*, Novi Sad J. Math. 45, 143-153, (2015).
4. Blair, D. E., *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag, 509, (1976).
5. De, U. C., Mandal, D. and Mandal, K., *Some characterizations of Kenmotsu manifolds admitting a quarter-symmetric metric connection*, Bull. Trans. Univ. Brasov 9 (58) no.1, Series III: Math., Inf., Phy., 39-52, (2016).
6. De, U. C. and Sengupta, J., *Quarter-symmetric metric connection on a Sasakian manifold*, Commun. Fac. Sci. Univ. Ank. Series A1 49, 7-13, (2000).
7. Friedmann, A. and Schouten, J. A., *Uber die Geometrie der halbsymmetrischen Ubertragung*, Math. Z. 21, 211-223, (1924).
8. Golab, S., *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N. S.) 29, 249-254, (1975).
9. Haseeb, A., *Some new results on para-Sasakian manifolds with a quarter-symmetric metric connection*, Facta Universitatis (NIŠ), Ser. Math. Inform. 30, 765-776, (2015).
10. Hui, S.K., *On ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection*, Applied Sciences 15, 71-84, (2013).
11. Mandal, K. and De, U. C., *Quarter-symmetric metric connection in a P-Sasakian manifold*, Analele Univ. de Vest, Timisoara Seria Matematica-Informatica 53, 137-150, (2015).
12. Mondal, A. K. and De, U. C., *Some properties of a quarter-symmetric metric connection on a Sasakian manifold*, Bull. Math. Analysis Appl. 3, 99-108. (2009).
13. Özgür, C., *ϕ-conformally flat Lorentzian para-Sasakian manifolds*, Radovi Matematički 12, 99-106, (2003).
14. Prakash, A., *ϕ-conformally flat $(LCS)_n$-manifolds*, Global Journal of Advanced Research on Classical and Modern Geometries 4, 136-143, (2015).

15. Prasad, R. and Haseeb, A., *On a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection*, Novi Sad J. Math. 46, 103-116, (2016).

16. Yano K. and Kon, M., *Structures on Manifolds*, Series in Pure Math., World Scientific, Vol. 3, (1984).

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