Multiplicative Sum Zagreb Index of Splice, Bridge, and Bridge-Cycle Graphs

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ABSTRACT: The multiplicative sum Zagreb index is a graph invariant defined as the product of the sums of the degrees of pairs of adjacent vertices in a graph. In this paper, we compute the multiplicative sum Zagreb index of some composite graphs such as splice graphs, bridge graphs, and bridge-cycle graphs in terms of the multiplicative sum Zagreb indices of their components. Then, we apply our results to compute the multiplicative sum Zagreb index for several classes of chemical graphs and nanostructures.

Key Words: Chemical graph, Multiplicative sum Zagreb index, Splice, Link, Bridge graph, Bridge-cycle graph.

Contents

1. Introduction

A *topological index*, also known as *graph invariant*, is a numeric quantity that is mathematically derived in a direct and unambiguous manner from the structural graph of a molecule. It is used in theoretical chemistry for the design of chemical compounds with given physico-chemical properties or given pharmacologic and biological activities [8,22].

The *Zagreb indices* are among the oldest topological indices, and were introduced by Gutman and Trinajstić [14] in 1972. These indices have since been used to study molecular complexity, chirality, ZE-isomerism, and hetero-systems. The first and second Zagreb indices of a simple graph G are denoted by $M_1(G)$ and $M_2(G)$, respectively and defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v),$$

where $V(G)$, $E(G)$, and $d_G(u)$ denote the vertex set of G, edge set of G, and degree of the vertex u in G, respectively. The first Zagreb index can also be expressed as a sum over edges of G,

$$M_1(G) = \sum_{uv \in E(G)} \big(d_G(u) + d_G(v)\big).$$

The multiplicative versions of Zagreb indices were introduced by Todeschini and Consonni [20] in 2010. The first and second *multiplicative Zagreb indices* of G are denoted by $\Pi_1(G)$ and $\Pi_2(G)$, respectively and defined as

$$\Pi_1(G) = \prod_{u \in V(G)} d_G(u)^2 \ and \ \Pi_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).$$

The second multiplicative Zagreb index can also be expressed as a product over vertices of G [13],

$$\Pi_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)}.$$

For more information on multiplicative Zagreb indices, see [7,10,12,15,17,21].

In 2012, Eliasi *et al.* [11] introduced another multiplicative version of the first Zagreb index called *multiplicative sum Zagreb index*. The multiplicative sum Zagreb index of G is denoted by $\Pi_1^*(G)$ and defined as

$$\Pi_1^*(G) = \prod_{uv \in E(G)} \big(d_G(u) + d_G(v)\big).$$

In [6,16], some lower and upper bounds on the multiplicative sum Zagreb index of several composite graphs such as union, join, corona product, rooted product, hierarchical product, composition, direct product, Cartesian product, and strong product were presented. In this paper, we present exact formulae for computing the multiplicative sum Zagreb index of some other composite graphs such as splice graphs, bridge graphs, and bridge-cycle graphs. In addition, the multiplicative sum Zagreb index of several classes of chemical graphs and nanostructures are computed.

2. Definitions and Preliminaries

In this section, we recall the definitions of splice, bridge, and bridge-cycle graphs and state some preliminary results about these graphs.

Definition 2.1. *Let G_1 and G_2 be two finite disjoint graphs. For given vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, a splice or coalescence of G_1 and G_2 by vertices v_1 and v_2 is denoted by $(G_1.G_2)(v_1, v_2)$ and defined by identifying the vertices v_1 and v_2 in the union of G_1 and G_2.*

Definition 2.2. *Let $\{G_i\}_{i=1}^d$, $d \geq 2$, be a set of finite pairwise disjoint graphs with $v_i \in V(G_i)$. The bridge graph $B = B(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is the graph obtained from the graphs $G_1, G_2, ..., G_d$ by connecting the vertices v_i and v_{i+1} by an edge for all $1 \leq i \leq d-1$.*

Remark 2.3. *In the case $d = 2$, the bridge graph $B(G_1, G_2; v_1, v_2)$ is also called the link of G_1 and G_2 and denoted by $(G_1 \sim G_2)(v_1, v_2)$.*

Definition 2.4. *The bridge-cycle graph $BC = BC(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$, $d \geq 3$, of $\{G_i\}_{i=1}^{d}$ with respect to the vertices $\{v_i\}_{i=1}^{d}$ is obtained by connecting the vertices v_1 and v_d by an edge in the bridge graph $B = B(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$.*

In the following lemma, the degree of an arbitrary vertex in splice, bridge, and bridge-cycle graphs are computed. The results follow easily from the above definitions and the proof of the lemma is therefore omitted.

Lemma 2.5. *(i) The degree of a vertex u in the splice graph $S = (G_1.G_2)(v_1, v_2)$ is given by*

$$d_S(u) = \begin{cases} d_{G_i}(u) & u \in V(G_i) - \{v_i\}, \ 1 \leq i \leq 2, \\ \delta_1 + \delta_2 & u = v_1 \ or \ u = v_2, \end{cases}$$

(ii) The degree of a vertex u in the bridge graph $B = B(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ is given by

$$d_B(u) = \begin{cases} d_{G_i}(u) & u \in V(G_i) - \{v_i\}, \ 1 \leq i \leq d, \\ \delta_i + 1 & u = v_i, \ i \in \{1, d\}, \\ \delta_i + 2 & u = v_i, \ 2 \leq i \leq d - 1, \end{cases}$$

(iii) The degree of a vertex u in the bridge-cycle graph $BC = BC(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ is given by

$$d_{BC}(u) = \begin{cases} d_{G_i}(u) & u \in V(G_i) - \{v_i\}, \ 1 \leq i \leq d, \\ \delta_i + 2 & u = v_i, \ 1 \leq i \leq d, \end{cases}$$

where for $1 \leq i \leq d$, $\delta_i = d_{G_i}(v_i)$.

We refer the reader to [1,2,3,4,9] for more information on computing topological indices of splice and link graphs.

3. Results and Discussion

In this section, we compute the multiplicative sum Zagreb index of the splice, bridge and bridge-cycle graphs. Throughout this section, let G_i be a nontrivial simple connected graph with $v_i \in V(G_i)$ and let $\delta_i = d_{G_i}(v_i)$, where $1 \leq i \leq d$. We denote by $N_G(u)$ the set of all first neighbors of a vertex u in the underlying graph G.

3.1. Splice graphs

In this subsection, we determine the multiplicative sum Zagreb index of the splice graph $S = (G_1.G_2)(v_1, v_2)$ in terms of the multiplicative sum Zagreb indices of the graphs G_1 and G_2.

Theorem 3.1. *The multiplicative sum Zagreb index of the splice of G_1 and G_2 with respect to the vertices v_1 and v_2 is given by*

$$\Pi_1^*\big((G_1.G_2)(v_1,v_2)\big) = \Pi_1^*(G_1)\,\Pi_1^*(G_2) \prod_{a\in N_{G_1}(v_1)} \big(1 + \frac{\delta_2}{d_{G_1}(a)+\delta_1}\big)$$
$$\prod_{a\in N_{G_2}(v_2)} \big(1 + \frac{\delta_1}{d_{G_2}(a)+\delta_2}\big). \tag{3.1}$$

Proof: Let $S = (G_1.G_2)(v_1,v_2)$. By definition of the multiplicative sum Zagreb index,

$$\Pi_1^*(S) = \prod_{ab\in E(S)} \big(d_S(a) + d_S(b)\big).$$

By definition 2.1, $E(S) = E(G_1) \cup E(G_2)$. So, we can consider $\Pi_1^*(S)$ as the product of two expressions as follows:

The first expression P_1 consists of contributions to $\Pi_1^*(S)$ of edges from G_1,

$$P_1 = \prod_{ab\in E(G_1);a,b\neq v_1} \big(d_{G_1}(a) + d_{G_1}(b)\big) \prod_{av_1\in E(G_1)} \big(d_{G_1}(a) + (\delta_1+\delta_2)\big)$$
$$= \frac{\Pi_1^*(G_1)}{\prod_{av_1\in E(G_1)}\big(d_{G_1}(a)+\delta_1\big)} \prod_{av_1\in E(G_1)} \big(d_{G_1}(a) + (\delta_1+\delta_2)\big)$$
$$= \Pi_1^*(G_1) \prod_{a\in N_{G_1}(v_1)} \frac{d_{G_1}(a)+\delta_1+\delta_2}{d_{G_1}(a)+\delta_1}$$
$$= \Pi_1^*(G_1) \prod_{a\in N_{G_1}(v_1)} \big(1 + \frac{\delta_2}{d_{G_1}(a)+\delta_1}\big).$$

The second expression P_2 consists of contributions to $\Pi_1^*(S)$ of edges from G_2. Similar to the previous case, we obtain

$$P_2 = \Pi_1^*(G_2) \prod_{a\in N_{G_2}(v_2)} \big(1 + \frac{\delta_1}{d_{G_2}(a)+\delta_2}\big).$$

Now Eq. (3.1) is obtained by multiplying P_1 and P_2. $\qquad\square$

Suppose that v is a vertex of a graph G and let $G_1 = G_2 = G$ and $v_1 = v_2 = v$. Using Theorem 3.1, we get the following result.

Corollary 3.2. *The multiplicative sum Zagreb index of the splice graph $(G.G)(v,v)$ is given by*

$$\Pi_1^*\big((G.G)(v,v)\big) = \Pi_1^*(G)^2 \prod_{a\in N_G(v)} \big(1 + \frac{\delta}{d_G(a)+\delta}\big)^2,$$

where $\delta = d_G(v)$.

3.2. Bridge graphs

In this subsection, we determine the multiplicative sum Zagreb index of the bridge graph $B = B(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ in terms of the multiplicative sum Zagreb indices of the graphs $G_1, G_2, ..., G_d$.

In the following theorem, the multiplicative sum Zagreb index of the link of two graphs is computed.

Theorem 3.3. *The multiplicative sum Zagreb index of the link of G_1 and G_2 with respect to the vertices v_1 and v_2 is given by*

$$\Pi_1^*\big((G_1 \sim G_2)(v_1, v_2)\big) = (\delta_1 + \delta_2 + 2)\Pi_1^*(G_1) \, \Pi_1^*(G_2) \prod_{a \in N_{G_1}(v_1)} \big(1 + \frac{1}{d_{G_1}(a) + \delta_1}\big)$$

$$\prod_{a \in N_{G_2}(v_2)} \big(1 + \frac{1}{d_{G_2}(a) + \delta_2}\big).$$

(3.2)

Proof: Let $G = (G_1 \sim G_2)(v_1, v_2)$. By definition of the multiplicative sum Zagreb index,

$$\Pi_1^*(G) = \prod_{ab \in E(G)} \big(d_G(a) + d_G(b)\big).$$

By definition of the link of graphs, $E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2\}$. So, we can consider $\Pi_1^*(G)$ as the product of three expressions as follows:

The first expression P_1 consists of contributions to $\Pi_1^*(G)$ of edges from G_1,

$$P_1 = \prod_{ab \in E(G_1); a,b \neq v_1} \big(d_{G_1}(a) + d_{G_1}(b)\big) \prod_{av_1 \in E(G_1)} \big(d_{G_1}(a) + (\delta_1 + 1)\big)$$

$$= \frac{\Pi_1^*(G_1)}{\prod_{av_1 \in E(G_1)} \big(d_{G_1}(a) + \delta_1\big)} \prod_{av_1 \in E(G_1)} \big(d_{G_1}(a) + (\delta_1 + 1)\big)$$

$$= \Pi_1^*(G_1) \prod_{a \in N_{G_1}(v_1)} \frac{d_{G_1}(a) + \delta_1 + 1}{d_{G_1}(a) + \delta_1}$$

$$= \Pi_1^*(G_1) \prod_{a \in N_{G_1}(v_1)} \big(1 + \frac{1}{d_{G_1}(a) + \delta_1}\big).$$

The second expression P_2 consists of contributions to $\Pi_1^*(G)$ of edges from G_2. Similar to the previous case, we obtain

$$P_2 = \Pi_1^*(G_2) \prod_{a \in N_{G_2}(v_2)} \big(1 + \frac{1}{d_{G_2}(a) + \delta_2}\big).$$

The third one P_3 consists of contributions to $\Pi_1^*(G)$ of the edge v_1v_2 of G,

$$P_3 = (\delta_1 + 1) + (\delta_2 + 1) = \delta_1 + \delta_2 + 2.$$

Now Eq. (3.2) is obtained by multiplying P_1, P_2, and P_3. $\square$

Suppose that v is a vertex of a graph G and let $G_1 = G_2 = G$ and $v_1 = v_2 = v$. Using Theorem 3.3, we get the following result.

Corollary 3.4. *The multiplicative sum Zagreb index of the link graph* $(G \sim G)(v, v)$ *is given by*

$$\Pi_1^*\big((G \sim G)(v, v)\big) = 2(\delta + 1)\ \Pi_1^*(G)^2 \prod_{a \in N_G(v)} \big(1 + \frac{1}{d_G(a) + \delta}\big)^2,$$

where $\delta = d_G(v)$.

In the following theorem, the multiplicative sum Zagreb index of the bridge graphs including at least three components is computed.

Theorem 3.5. *The multiplicative sum Zagreb index of the bridge graph* $B = B(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$, $d \geq 3$, *is given by*

$$
\begin{aligned}
\Pi_1^*(B) =& (\delta_1 + \delta_2 + 3)(\delta_{d-1} + \delta_d + 3) \prod_{i=2}^{d-2} (\delta_i + \delta_{i+1} + 4) \prod_{i=1}^{d} \Pi_1^*(G_i) \\
& \prod_{i \in \{1,d\}} \prod_{a \in N_{G_i}(v_i)} \big(1 + \frac{1}{d_{G_i}(a) + \delta_i}\big) \prod_{i=2}^{d-1} \prod_{a \in N_{G_i}(v_i)} \big(1 + \frac{2}{d_{G_i}(a) + \delta_i}\big).
\end{aligned}
\tag{3.3}
$$

Proof: By definition of the multiplicative sum Zagreb index,

$$\Pi_1^*(B) = \prod_{ab \in E(B)} \big(d_B(a) + d_B(b)\big).$$

By definition 2.2,

$$E(B) = E(G_1) \cup E(G_2) \cup ... \cup E(G_d) \cup \{v_i v_{i+1} \,|\, 1 \leq i \leq d - 1\}.$$

So, we can consider $\Pi_1^*(B)$ as the product of four expressions as follows:

The first expression P_1 consists of contributions to $\Pi_1^*(B)$ of edges from G_1 and the second one P_2 consists of contributions to $\Pi_1^*(B)$ of edges from G_d. Using the same argument as in the proof of Theorem 3.3, we obtain

$$P_1 = \Pi_1^*(G_1) \prod_{a \in N_{G_1}(v_1)} \big(1 + \frac{1}{d_{G_1}(a) + \delta_1}\big),$$

$$P_2 = \Pi_1^*(G_d) \prod_{a \in N_{G_d}(v_d)} \big(1 + \frac{1}{d_{G_d}(a) + \delta_d}\big).$$

The third expression P_3 consists of contributions to $\Pi_1^*(B)$ of edges from $G_2, G_3, ...,$
G_{d-1},

$$
\begin{aligned}
P_3 &= \prod_{i=2}^{d-1} \prod_{ab \in E(G_i)} \big(d_B(a) + d_B(b) \big) \\
&= \prod_{i=2}^{d-1} \Big[\prod_{ab \in E(G_i); a,b \neq v_i} \big(d_{G_i}(a) + d_{G_i}(b) \big) \prod_{av_i \in E(G_i)} \big(d_{G_i}(a) + (\delta_i + 2) \big) \Big] \\
&= \prod_{i=2}^{d-1} \Pi_1^*(G_i) \prod_{i=2}^{d-1} \prod_{a \in N_{G_i}(v_i)} \Big(1 + \frac{2}{d_{G_i}(a) + \delta_i} \Big).
\end{aligned}
$$

The last expression P_4 is taken over all edges $v_i v_{i+1}$ for all $1 \leq i \leq d-1$,

$$
\begin{aligned}
P_4 &= \big((\delta_1 + 1) + (\delta_2 + 2) \big) \prod_{i=2}^{d-2} \big((\delta_i + 2) + (\delta_{i+1} + 2) \big) \big((\delta_{d-1} + 2) + (\delta_d + 1) \big) \\
&= (\delta_1 + \delta_2 + 3)(\delta_{d-1} + \delta_d + 3) \prod_{i=2}^{d-2} (\delta_i + \delta_{i+1} + 4).
\end{aligned}
$$

Now Eq. (3.3) is obtained by multiplying P_1, P_2, P_3, and P_4. □

Suppose that v is a vertex of a graph G and let $G_i = G$ and $v_i = v$ for all $1 \leq i \leq d$. Using Theorem 3.5, we get the following result.

Corollary 3.6. *The multiplicative sum Zagreb index of the bridge graph $B = B(G, G, ..., G; v, v, ..., v)$, $(d \geq 3$ times), is given by*

$$
\Pi_1^*(B) = (2\delta + 3)^2 \, (2\delta + 4)^{d-3} \, \Pi_1^*(G)^d
$$
$$
\prod_{a \in N_G(v)} \Big[\big(1 + \frac{1}{d_G(a) + \delta} \big)^2 \big(1 + \frac{2}{d_G(a) + \delta} \big)^{d-2} \Big],
$$

where $\delta = d_G(v)$.

3.3. Bridge-cycle graphs

In this subsection, we determine the multiplicative sum Zagreb index of the bridge-cycle graph $BC = BC(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ in terms of the multiplicative sum Zagreb indices of the graphs $G_1, G_2, ..., G_d$.

Theorem 3.7. *The multiplicative sum Zagreb index of the bridge-cycle graph $BC = BC(G_1, G_2, ..., G_d; v_1, v_2, ..., v_d)$ is given by*

$$
\Pi_1^*(BC) = \prod_{i=1}^{d} \Big[(\delta_i + \delta_{i+1} + 4) \, \Pi_1^*(G_i) \prod_{a \in N_{G_i}(v_i)} \big(1 + \frac{2}{d_{G_i}(a) + \delta_i} \big) \Big], \qquad (3.4)
$$

where $\delta_{d+1} = \delta_1$.

Proof: Using the same argument as in the proof of Theorem 3.5, we can get Eq. (3.4).   □

Suppose that v is a vertex of a graph G and let $G_i = G$ and $v_i = v$ for all $1 \leq i \leq d$. Using Theorem 3.7, we get the following result.

Corollary 3.8. *The multiplicative sum Zagreb index of the bridge-cycle graph* $BC = BC(G, G, ..., G; v, v, ..., v)$, *(d times), is given by*

$$\Pi_1^*(BC) = \left[(2\delta + 4) \; \Pi_1^*(G) \prod_{a \in N_G(v)} \left(1 + \frac{2}{d_G(a) + \delta} \right) \right]^d,$$

where $\delta = d_G(v)$.

4. Examples and applications

In this section, we apply the results obtained in the previous section to compute the multiplicative sum Zagreb index of several classes of chemical graphs and nanostructures.

Let P_n, C_n, and S_n denote the n-vertex path, cycle, and star, respectively. It is easy to see that,

$$\Pi_1^*(P_n) = \begin{cases} 2 \; n = 2, \\ 9 \times 4^{n-3} \; n \geq 3, \end{cases} \quad \Pi_1^*(C_n) = 4^n, \; \Pi_1^*(S_n) = n^{n-1}.$$

As the first example, consider the bridge graph $G = B(C_n, C_n, ..., C_n; v, v, ..., v)$, (*d* times), where v is an arbitrary vertex of C_n. Application of Corollary 3.4 for the case $d = 2$, and Corollary 3.6 for $d \geq 3$ yields,

Corollary 4.1.

$$\Pi_1^*(G) = \begin{cases} 1875 \times 2^{4n-7} & d = 2, \\ 30625 \times 9^{d-2} \times 2^{(2n+1)d-13} & d \geq 3. \end{cases}$$

By Corollary 3.8 for the bridge-cycle graph $H = BC(C_n, C_n, ..., C_n; v, v, ..., v)$, (*d* times), we obtain

Corollary 4.2.

$$\Pi_1^*(H) = (9 \times 2^{2n+1})^d.$$

Consider the *square comb lattice graph* $C_q(N)$ with open ends, where $N = n^2$ is the number of its vertices. This graph can be seen as the bridge graph $B(P_n, P_n, ..., P_n; v, v, ..., v)$, (*n* times), where v is one of the pendent vertices (vertices of degree one) of P_n. Using Corollary 3.4 for the case $n = 2$, and Corollary 3.6 for $n \geq 3$, we obtain

Corollary 4.3.

$$\Pi_1^*(C_q(N)) = \begin{cases} 36 & n = 2, \\ 2^{2n^2-5n+1} \times 3^{2n-3} \times 5^n & n \geq 3. \end{cases}$$

Consider the *van Hove comb lattice graph* $CvH(N)$ with open ends, where $N = n^2$ is the number of its vertices. In the case $n = 2$, this graph is isomorphic to the star graph S_4. So, $\Pi_1^*(CvH(4)) = 64$. For $n \geq 3$, this graph can be represented as the bridge graph,

$$B(S_3, P_3, ..., P_{n-1}, P_n, P_{n-1}, ..., P_3, S_3; u, v_{1,3}, ..., v_{1,n-1}, v_{1,n}, v_{1,n-1}, ..., v_{1,3}, u),$$

where for $3 \leq i \leq n$, $v_{1,i}$ is one of the pendent vertices of P_i and u is the vertex of degree two of S_3. Using Theorem 3.5, we arrive at:

Corollary 4.4.

$$\Pi_1^*\big(CvH(N)\big) = 3^{4n-9} \times 4^{n^2-5n+11} \times 5^{2n-5}.$$

A *caterpillar* or *caterpillar tree* is a tree in which all vertices are within distance 1 from a central path. Let $P_n^*(p_1, p_2, ..., p_n)$ denote a caterpillar tree obtained by attaching p_i pendent vertices to the i-th vertex of the path P_n for $1 \leq i \leq n$. This graph can be considered as the bridge graph $B(S_{p_1+1}, S_{p_2+1}, ..., S_{p_n+1}; v_1, v_2, ..., v_n)$, where for $1 \leq i \leq n$, v_i is the vertex of degree p_i of S_{p_i+1}. Let $p_1, p_2, ..., p_n$ be positive. Application of Theorem 3.3 for the case $n=2$, and Theorem 3.5 for $n \geq 3$ yields,

Corollary 4.5.

$$\Pi_1^*\big(P_n^*(p_1, p_2, ..., p_n)\big) = \begin{cases} (p_1+2)^{p_1}(p_2+2)^{p_2}(p_1+p_2+2) & n=2, \\ (p_1+2)^{p_1}(p_n+2)^{p_n}(p_1+p_2+3) & n \geq 3 \\ (p_{n-1}+p_n+3)\prod_{i=2}^{n-1}(p_i+3)^{p_i} \\ \prod_{i=2}^{n-2}(p_i+p_{i+1}+4). \end{cases}$$

A unicyclic graph is called *cycle-caterpillar* if deleting all its pendent vertices will reduce it to a cycle. Let $C_n^*(p_1, p_2, ..., p_n)$ denote a cycle-caterpillar obtained by attaching p_i pendent vertices to the i-th vertex of the cycle C_n for $1 \leq i \leq n$. This graph can be considered as the bridge-cycle graph $BC(S_{p_1+1}, S_{p_2+1}, ..., S_{p_n+1}; v_1, v_2, ..., v_n)$, where for $1 \leq i \leq n$, v_i is the vertex of degree p_i of S_{p_i+1}. By Theorem 3.7, we arrive at:

Corollary 4.6. *Let $p_1, p_2, ..., p_n$ be positive. Then*

$$\Pi_1^*\big(C_n^*(p_1, p_2, ..., p_n)\big) = \prod_{i=1}^{n}\big((p_i+3)^{p_i}(p_i+p_{i+1}+4)\big),$$

where $p_{n+1} = p_1$.

The *sunlike graph* $C_n(k_1, k_2, ..., k_n)$ is the graph obtained by identifying a pendent vertex of the path P_{k_i} with the i-th vertex of C_n for $1 \leq i \leq n$. The sunlike graph $C_n(k_1, k_2, ..., k_n)$ can be viewed as the bridge-cycle graph $BC(P_{k_1}, P_{k_2}, ..., P_{k_n}; v_1, v_2, ..., v_n)$, where v_i is a pendent vertex of P_{k_i}, $1 \leq i \leq n$. By Theorem 3.7, we arrive at

Corollary 4.7. *Let* $k_1, k_2, ..., k_n > 1$, $I = \{i | 1 \le i \le n, k_i \ge 3\}$, *and* $|I| = t$. *Then*

$$\Pi_1^*(C_n(k_1, k_2, ..., k_n)) = 24^n \times 15^t \times 2^{-8t} \times \prod_{i \in I} 4^{k_i}. \qquad (4.1)$$

In particular, the square comb lattice graph $CC_q(N)$ with closed ends can be seen as the sunlike graph $C_n(n, n, ..., n)$, where $n \ge 3$. By Eq. (4.1),

Corollary 4.8.
$$\Pi_1^*\big(CCq(N)\big) = (45 \times 2^{2n-5})^n.$$

The level of a vertex in a rooted tree is one more than its distance from the root vertex. A *generalized Bethe tree* of k levels, $k > 1$, is a rooted tree in which vertices at the same level have the same degree. Let B_k be a generalized Bethe tree of k levels. For $1 \le i \le k$, we denote by d_{k-i+1} the degree of the vertices at the level i of B_k. Also, suppose $e_k = d_k$ and $e_i = d_i - 1$ for $1 \le i \le k - 1$. Thus, $d_1 = 1$ is the degree of the vertices at the level k (pendent vertices) and d_k is the degree of the root vertex. In the following theorem, the multiplicative sum Zagreb index of the generalized Bethe tree B_k is computed. The result follows easily from the definition of this graph and the proof of the theorem is therefore omitted.

Theorem 4.9.
$$\Pi_1^*(B_k) = \prod_{i=2}^{k} (d_i + d_{i-1})^{\Pi_{j=i}^{k} e_j}. \qquad (4.2)$$

The *ordinary Bethe tree* $B_{d,k}$ is a rooted tree of k levels whose root vertex has degree d, the vertices from levels 2 to $k - 1$ have degree $d + 1$, and the vertices at level k have degree 1. Using Eq. (4.2), we obtain

Corollary 4.10.

$$\Pi_1^*(B_{d,k}) = (2d + 1)^d (2d + 2)^{\frac{d^{k-1} - d^2}{d-1}} (d + 2)^{d^{k-1}}. \qquad (4.3)$$

A *dendrimer tree* $T_{d,k}$ is a rooted tree such that the degree of its non-pendent vertices is equal to d and the distance between the root (central) vertex and the pendent vertices is equal to k. So $T_{d,k}$ can be considered as a generalized Bethe tree with $k + 1$ levels, such that whose non-pendent vertices have equal degrees. Using Eq. (4.2), we obtain

Corollary 4.11.

$$\Pi_1^*(T_{d,k}) = (d + 1)^{d(d-1)^{k-1}} (2d)^{\frac{d(d-1)^{k-1} - d}{d-2}}.$$

Denote by $P(d, k, n)$, the chemical tree obtained by attaching the root vertex of $B_{d,k}$ to the vertices of the n-vertex path P_n. For more information about this graph, see [5,18]. The graph $P(d, k, n)$ can be considered as the bridge graph $B(B_{d,k}, ..., B_{d,k}; v, ..., v)$, ($n$ times), where v is the root vertex (vertex of degree d) of $B_{d,k}$. Applying Corollary 3.4 for the case $n = 2$, and Corollary 3.6 for $n \ge 3$ and then using Eq. (4.3), we obtain

Corollary 4.12.

$$\Pi_1^*\big(P(d,k,n)\big) = \begin{cases} (2d+2)^{\frac{2(d^{k-1}-d^2)}{d-1}+2d+1}(d+2)^{2d^{k-1}} & n=2, \\ 2^{n-3}(2d+2)^{\frac{n(d^{k-1}-d^2)}{d-1}+2d}(d+2)^{nd^{k-1}+n-3} & n\geq 3 \\ (2d+3)^{d(n-2)+2}. \end{cases}$$

Finally, let $C(d,k,n)$ denote the dendrimer graph obtained by attaching the root vertex of $B_{d,k}$ to the vertices of the n-cycle C_n. For more information about this graph, see [5,19]. The graph $C(d,k,n)$ can be viewed as the bridge-cycle graph $BC(B_{d,k},...,B_{d,k};v,...,v)$, ($n$ times), where v is the root vertex of $B_{d,k}$. Using Corollary 3.8 and then Eq. (4.3), we obtain

Corollary 4.13.

$$\Pi_1^*\big(C(d,k,n)\big) = \left(2(2d+2)^{\frac{d^{k-1}-d^2}{d-1}}(d+2)^{d^{k-1}+1}(2d+3)^d\right)^n.$$

Acknowledgments

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