Residual Power Series Method for Solving Nonlinear Reaction-diffusion-convection Problems

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ABSTRACT: In this paper, the residual power series method (RPSM) is applied to one of the most frequently used models in engineering and science, a nonlinear reaction diffusion convection initial value problems. The approximate solutions using the RPSM were compared to the exact solutions and to the approximate solutions using the homotopy analysis method.

Key Words: Nonlinear reaction-diffusion-convection equation, Residual power series method, Homotopy perturbation method, Absolute error.

Contents

1. Introduction

We consider the nonlinear reaction diffusion convection model, which is a mathematical model that describes how the concentration of the substance distributed in the medium changes under the influence of the three processes (i.e convection, diffusion and reaction). That is given by

$$u_t = (a(u)u_x)_x + b(u)u_x + c(u), \tag{1.1}$$

where $u = u(x,t)$ is an unknown function, and the arbitrary smooth functions $a(u)$, $b(u)$ and $c(u)$ denote the diffusion term, the convection term and the reaction term respectively. The reaction diffusion convection equations are widely used in many areas in science such as biology modeling, physics, chemistry, astrophysics, medicine and engineering. For example, heat conduction [4], [5] and [20], haemodynamics [9], [23] and [22], dynamics of blood coagulation [6] and [25], cardiac arrhythmias [8] and [24] and atherosclerosis [16] and [15].

The following equation is a special case of the reaction diffusion convection equations, the Murray equation [18] and [19]

$$u_t = u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, \tag{1.2}$$

where λ_1, λ_2, and λ_3 are real numbers. Whereas, the generalized Burgers equation [26] and [7]

$$u_t + (u^2/2)_x = f(t)u_{xx} \qquad x \in R, \quad t > 0, \tag{1.3}$$

this equation is the mathematical model of the propagation of the finite-amplitude sound waves in variable–area ducts, where u is an acoustic variable, with the linear effects of changes in the duct area taken out, and the coefficient $f(t)$ is a positive function that depends on the particular duct chosen. Also, the Fisher-Kolmogoroff equation was suggested by Fisher [12] and [18] in (1937), is a simplest case of the nonlinear reaction diffusion given by

$$u_t = Du_{xx} + ku(1 - u), \tag{1.4}$$

where k and D are positive parameters. It was used to determine the stochastic model for the spatial spread of a favoured gene in population. Both equations are special cases of the Murray equation.

There are several procedures to find the analytical approximate solution of the nonlinear partial differential equations such as the homotopy analysis method HAM presented by Liao [17], variational iteration method VIM presented by He [13], Homotopy perturbation method [14] and the residual power series solution presented by Abu Arqub [1] this powerful method is an accurate, efficient, straightforward and simple technique to solve nonlinear equations by constructing a power series solution subjected to a given initial condition. This method calculates the coefficients of the power series by a sequence of algebraic equations of one or more variables then we obtain the series solution, in particular a truncated series solution.

The idea behind the residual power series method [1] is to express the solution of the IVP

$$u_t = (a(u)u_x)_x + b(u)u_x + c(u), \tag{1.5}$$

completed by the initial data

$$u(x, 0) = f_0(x), \tag{1.6}$$

as a power series about the initial point $t = t_0$. To do so the approximate solution will be presented as

$$u(x, t) = \sum_{m=0}^{\infty} f_m(x)(t - t_0)^m. \tag{1.7}$$

Clearly, when $m = 0$, $u_0(x, t)$ satisfies the initial condition (1.6), where $u_0(x, t)$ is the initial guess approximation of $u(x, t)$. Hence, $f_0(x) = u_0(x, t) = u(x, 0)$. The kth truncated solution $u_k(x, t)$ for $k = 1, 2, 3, \ldots$ approximate the solution $u(x, t)$ for the system (1.5)-(1.6) which is defined as follows:

$$u_k(x, t) = \sum_{m=0}^{k} f_m(x)(t - t_0)^m. \tag{1.8}$$

Next, we rewrite the IVP (1.5) in the following way

$$u_t - (a(u)u_x)_x - b(u)u_x - c(u) = 0. \tag{1.9}$$

Also, we define the residual function for equation (1.5) by

$$Res(x, t) = u_t - (a(u)u_x)_x - b(u)u_x - c(u). \tag{1.10}$$

And the kth truncated residual can be defined as follows:

$$Res_k(x, t) = (u_k)_t - (a(u_{k-1})(u_k)_x)_x - b(u_{k-1})(u_k)_x - c(u_{k-1}). \tag{1.11}$$

Clearly, $Res(x, t) = 0$ for each t. In order to approximate the solution, substitute the kth truncated series $u_k(x, t)$ as given in (1.8) into equation (1.11) so we have

$$Res_k(x, t) = \sum_{m=1}^{k} m f_m(x)(t - t_0)^{m-1} - \left(a(u_{k-1}) \sum_{m=0}^{k} \frac{df_m(x)}{dx}(t - t_0)^m \right)_x$$
$$- b(u_{k-1}) \sum_{m=0}^{k} \frac{df_m(x)}{dx}(t - t_0)^m - c(u_{k-1}). \tag{1.12}$$

Define $Res_\infty(x, t) = \lim_{k \to \infty} Res_k(x, t)$, we may also notice that $Res_\infty(x, t) = 0$ for each t. This concludes that the $Res_\infty(x, t)$ is infinitely differentiable at $t = 0$. Therefore, as an essential rule of residual power series method [1]

$$\frac{\partial^{s-1} Res_\infty(x, 0)}{\partial t^{s-1}} = \frac{\partial^{s-1} Res_k(x, 0)}{\partial t^{s-1}} = 0, \text{ for } s = 1, 2, 3, \ldots, k.$$

To find the first approximate solution, we let $k = 1$ in equation (1.12) then using the fact that $Res_\infty(x, 0) = Res_1(x, 0) = 0$, we conclude that

$$f_1(x) = \frac{d}{dx} \left\{ a(u_0) \frac{d}{dx} f_0(x) \right\} - b(u_0) \frac{d}{dx} f_0(x) + c(u_0).$$

Hence, using the first truncated series the first approximate solution for the IVP (1.5)-(1.6) can be written as

$$u_1(x, t) = u_0 + \left[\frac{d}{dx} \left\{ a(u_0) \frac{d}{dx} f_0(x) \right\} - b(u_0) \frac{d}{dx} f_0(x) + c(u_0) \right] t.$$

Furthermore, to find the second approximate solution we use $k = 2$, so we achieve

$$u_2(x, t) = \sum_{m=0}^{2} f_m(x) t^m,$$

then we substitute $u_2(x, t)$ into equation (1.12) and different both sides of the resulting equation with respect to t and substituting $t = 0$, as a result we will be able to find $f_2(x)$. Hence, the second approximate solution can be represented as:

$$u_2(x, t) = f_0(x) + f_1(x)t + f_2(x)t^2.$$

We can repeat this procedure as many times, allowing us to reach the arbitrary approximate solution. In [1] Abu Arqub used the residual power series method to find the approximate solution of fussy differential equations, he proved the convergence of the method. So, we may generalize his results to our problem nonlinear-reaction-diffusion-convection IVP given by (1.5)-(1.6).

Theorem 1.1. *[1] Suppose that $u(x,t)$ is the exact solution for the initial value problem (1.5)-(1.6). Then, the approximate solution obtained by the residual power series method is just the Taylor expansion of $u(x,t)$.*

Corollary 1.2. *[1] If $u(x,t)$ is a polynomial, then the exact solution will be obtained using the residual power series method.*

In this paper we use the residual power series method to get the numerical solution for nonlinear reaction diffusion convection problems.

This paper is organized as follows in Section 2 the residual power series method is described and applied to two nonlinear reaction-diffusion-convection problems. In section 3 the numerical results and the graphs are presented. Finally, in section 4 we conclude that RPSM is better than the HAM method.

2. Numerical applications of the residual power series method

In this section, we apply the residual power series method RPSM to solve some nonlinear reaction-diffusion-convection problem.

Example 2.1. *Consider the IVP*

$$\begin{cases} u_t = u_{xx} + uu_x + u - u^2, \\ u(x,0) = 1 + e^x, \qquad -\infty \le x \le \infty, \end{cases} \tag{2.1}$$

such that the exact solution for (2.1) is $u(x,t) = 2e^{2t}\sqrt{e^x - e^{-4x}}$.

Suppose we define the solution

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)t^m, \ t \in [0,R), \ x \in I. \tag{2.2}$$

Then, we define $u_k(x,t)$ to be the kth truncated series of $u(x,t)$ given by

$$u_k(x,t) = \sum_{m=0}^{k} f_m(x)t^m, \ t \in [0,R), \ x \in I, \tag{2.3}$$

with

$$u_0(x,t) = f_0(x) = u(x,0).$$

Hence, we can write equation (2.3) as follows:

$$u_k(x,t) = f_0(x) + \sum_{m=1}^{k} f_m(x)t^m, \ t \in [0,R), \ x \in I, \ k \in \{1, 2, \cdots\}. \tag{2.4}$$

Now, we need to evaluate the coefficients $f_m(x)$, where $m = 1, 2, \cdots, k$ in the series expansion of equation (2.4) above. To do so we define the residual function $Res(x, t)$, of equation (2.1), as follows:

$$Res(x, t) = u_t - u_{xx} - uu_x - u + u^2,$$

while, we define the kth residual function $Res_k(x, t)$, as follows:

$$Res_k(x, t) = (u_k)_t - (u_k)_{xx} - u_k(u_k)_x - u_k + u_k^2, \qquad k = 1, 2, 3, \cdots. \qquad (2.5)$$

As in Abu Arqub and partners [1], [10] it is clear that the $Res(x, t) = 0$ and $\lim_{k \to \infty} Res_k(x, t) = Res(x, t)$ for $x \in I$ and $t \geq 0$. Then $\dfrac{\partial^{s-1}}{\partial t^{s-1}} Res_s(x, t) = 0$ for $s = 1, 2, \cdots, k$, for $t = 0$.

Now, to determine $f_1(x)$, we compute $Res_1(x, t)$ supposing that $k = 1$ in (2.5) to get

$$Res_1(x, t) = (u_1)_t - (u_1)_{xx} - u_1(u_1)_x - u_1 + u_1^2, \qquad (2.6)$$

where

$$u_1(x, t) = f_0(x) + f_1(x) \; t, \qquad (2.7)$$

with

$$u_0(x, t) = f_0(x) = u(x, 0) = 1 + e^x.$$

Substituting equation (2.7) into equation (2.6) and using the fact that $Res_1(x, t) = 0$ for $t = 0$. Yields,

$$f_1(x) = e^x. \qquad (2.8)$$

Therefore, the first residual power series (RPS) approximate solution is

$$u_1(x, t) = 1 + e^x + e^x \; t. \qquad (2.9)$$

Similarly, to find the second unknown function $f_2(x)$, we write

$$u_2(x, t) = f_0(x) + f_1(x) \; t + f_2(x) \; t^2,$$

then we substitute $u_2(x, t)$ in $Res_2(x, t)$, with the condition $\partial Res_2(x, t)/\partial t = 0$ for $t = 0$, we get

$$f_2(x) = \frac{e^x}{2}. \qquad (2.10)$$

Therefore, the second term of RPS approximate solution is

$$u_2(x, t) = 1 + e^x + e^x \; t + \frac{e^x}{2} \; t^2. \qquad (2.11)$$

Proceeding to the third iteration $u_3(x, t)$, to find $f_3(x)$, we write

$$u_3(x, t) = f_0(x) + f_1(x) \; t + f_2(x) \; t^2 + f_3(x) \; t^3,$$

next, we substitute $u_3(x,t)$ in $Res_3(x,t)$, with the condition $\partial^2 Res_3(x,t)/\partial t^2 = 0$ for $t = 0$, we obtain that

$$f_3(x) = \frac{e^x}{6}, \qquad (2.12)$$

and the third term of RPS approximate solution is

$$u_3(x,t) = 1 + e^x + e^x\ t + \frac{e^x}{2}\ t^2 + \frac{e^x}{6}\ t^3. \qquad (2.13)$$

To compute the fourth iteration we set

$$u_4(x,t) = f_0(x) + f_1(x)\ t + f_2(x)\ t^2 + f_3(x)\ t^3 + f_4(x)\ t^4,$$

then we substitute $u_4(x,t)$ in $Res_4(x,t)$, with the condition $\partial^3 Res_4(x,t)/\partial t^3 = 0$ for $t = 0$, to get

$$f_4(x) = \frac{e^x}{24}, \qquad (2.14)$$

hence, the fourth term of RPS approximate solution is

$$\begin{aligned}
u_4(x,t) &= 1 + e^x + e^x\ t + \frac{e^x}{2}\ t^2 + \frac{e^x}{6}\ t^3 + \frac{e^x}{24}\ t^4 \\
&= 1 + e^x\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right].
\end{aligned} \qquad (2.15)$$

Therefore, the kth iteration can be written as follows:

$$u_k(x,t) = 1 + e^x \sum_{m=0}^{k} \frac{t^m}{m!}. \qquad (2.16)$$

Since, $u(x,t) = \lim_{k\to\infty} u_k(x,t)$, then we conclude that

$$u(x,t) = 1 + e^{x+t}, \qquad (2.17)$$

which is the exact solution.

Example 2.2. *Consider the IVP*

$$\begin{cases} u_t = (uu_x)_x + 3uu_x + 2(u - u^2), \\ u(x,0) = 2\sqrt{e^x - e^{-4x}}, \qquad -\infty \leq x \leq \infty, \end{cases} \qquad (2.18)$$

such that the exact solution for (2.18) is $u(x,t) = 2e^{2t}\sqrt{e^x - e^{-4x}}$.

We define the residual function $Res(x,t)$, for equation (2.18), as follows:

$$Res(x,t) = u_t - (uu_x)_x - 3uu_x - 2(u - u^2),$$

while, we define the kth residual function $Res_k(x,t)$, as follows:

$$Res_k(x,t) = (u_k)_t - (u_k(u_k)_x)_x - 3u_k(u_k)_x - 2(u_k - u_k^2), \qquad k = 1, 2, 3, \cdots. \quad (2.19)$$

To determine $f_1(x)$, we compute $Res_1(x,t)$ by letting $k = 1$ in (2.5) to get

$$Res_1(x,t) = (u_1)_t - (u_1(u_1)_x)_x - 3u_1(u_1)_x - 2(u_1 - u_1^2), \qquad (2.20)$$

where

$$u_1(x,t) = f_0(x) + f_1(x)\ t, \qquad (2.21)$$

such that

$$u_0(x,t) = f_0(x) = u(x,0) = 2\sqrt{e^x - e^{-4x}}.$$

Substituting equation (2.21) into equation (2.20) and using the fact that $Res_1(x,t) = 0$ for $t = 0$. Yields

$$f_1(x) = 4\sqrt{e^x - e^{-4x}}. \qquad (2.22)$$

Therefore, the first residual power series (RPS) approximate solution is

$$u_1(x,t) = 2\sqrt{e^x - e^{-4x}} + 4\sqrt{e^x - e^{-4x}}\ t. \qquad (2.23)$$

Similarly, to find the second unknown function $f_2(x)$, we write

$$u_2(x,t) = f_0(x) + f_1(x)\ t + f_2(x)\ t^2,$$

then we substitute $u_2(x,t)$ in $Res_2(x,t)$, with the condition $\partial Res_2(x,t)/\partial t = 0$ for $t = 0$, we get

$$f_2(x) = 4\sqrt{e^x - e^{-4x}}. \qquad (2.24)$$

Therefore, the second term of RPS approximate solution is

$$u_2(x,t) = 2\sqrt{e^x - e^{-4x}} + 4\sqrt{e^x - e^{-4x}}\ t + 4\sqrt{e^x - e^{-4x}}\ t^2. \qquad (2.25)$$

Proceeding to the third iteration $u_3(x,t)$, to find $f_3(x)$ we write

$$u_3(x,t) = f_0(x) + f_1(x)\ t + f_2(x)\ t^2 + f_3(x)\ t^3,$$

next, we substitute u_3 in $Res_3(x,t)$, with the condition $\partial^2 Res_3(x,t)/\partial t^2 = 0$ for $t = 0$, we obtain that

$$f_3(x) = 8\sqrt{e^x - e^{-4x}}, \qquad (2.26)$$

and the third term of RPS approximate solution is

$$u_3(x,t) = 2\sqrt{e^x - e^{-4x}} + 4\sqrt{e^x - e^{-4x}}\ t + 4\sqrt{e^x - e^{-4x}}\ t^2 + \frac{8}{3}\sqrt{e^x - e^{-4x}}\ t^3. \qquad (2.27)$$

To compute the fourth iteration we set

$$u_4(x,t) = f_0(x) + f_1(x)\ t + f_2(x)\ t^2 + f_3(x)\ t^3 + f_4(x)\ t^4,$$

then we substitute $u_4(x,t)$ in $Res_4(x,t)$, with the condition $\partial^3 Res_4(x,t)/\partial t^3 = 0$ for $t = 0$, to get

$$f_4(x) = \frac{4}{3}\sqrt{e^x - e^{-4x}}, \qquad (2.28)$$

hence, the fourth term of RPS approximate solution is

$$
\begin{aligned}
u_4(x,t) =& 2\sqrt{e^x - e^{-4x}} + 4\sqrt{e^x - e^{-4x}}\, t + 4\sqrt{e^x - e^{-4x}}\, t^2 \\
& + \frac{8}{3}\sqrt{e^x - e^{-4x}}\, t^3 + \frac{4}{3}\sqrt{e^x - e^{-4x}}\, t^4 \\
=& 2\sqrt{e^x - e^{-4x}}\left[1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!}\right].
\end{aligned}
\tag{2.29}
$$

Therefore the kth iteration can be written as follows:

$$
u_k(x,t) = 2\sqrt{e^x - e^{-4x}} \sum_{m=0}^{k} \frac{(2t)^m}{m!}.
\tag{2.30}
$$

Since, $u(x,t) = \lim_{k\to\infty} u_k(x,t)$, then we conclude that

$$
\begin{aligned}
u(x,t) &= 2\sqrt{e^x - e^{-4x}} \sum_{m=0}^{\infty} \frac{(2t)^m}{m!} \\
&= 2e^{2t}\sqrt{e^x - e^{-4x}},
\end{aligned}
\tag{2.31}
$$

which is the exact solution. Table (1) is used to compare the absolute error of HAM with our findings using RPSM. The latter is more accurate due to the fact that we obtained the exact solution.

Table 1: Comparison of the absolute errors for Example 2.2 with values of HAM taken from [21]

t	x	RPSM	HAM
0.1	0.5	0	0
	1.0	0	8.88178×10^{-16}
	1.5	0	1.77636×10^{-15}
	2.0	0	1.77636×10^{-15}
	4.0	0	3.55271×10^{-15}
	6.0	0	1.42109×10^{-11}
0.5	0.5	0	1.25233×10^{-13}
	1.0	0	1.68754×10^{-13}
	1.5	0	2.18492×10^{-13}
	2.0	0	2.75335×10^{-13}
	4.0	0	7.46070×10^{-13}
	6.0	0	2.06057×10^{-12}
0.5	0.5	0	8.72582×10^{-9}
	1.0	0	1.16550×10^{-8}
	1.5	0	1.50118×10^{-8}
	2.0	0	1.92804×10^{-8}
	4.0	0	5.24108×10^{-8}
	6.0	0	1.42467×10^{-7}

3. Discussion and Numerical Simulations

In this section the numerical results for both Examples 2.1 and 2.2 are presented. In table 1 we compare the absolute error only for Example 2.2 using both methods the residual power series method (RPSM) and the homotopy perturbation method (HAM) in which we found that the RPSM gives better estimates than the HAM. In figure 1 and 3 the comparison between the approximate and the exact solution for both examples 2.1 and 2.2 is presented, in which we found out that both are fairly similar in both examples. While, in figure 2 and 4 we compare the approximate solution $u_4(x,t)$ for different values of t, and also, the comparison for the approximate solutions $u_i(x,t)$ for $i = 1,\ 2,\ 3,\ 4$ when t is fixed.
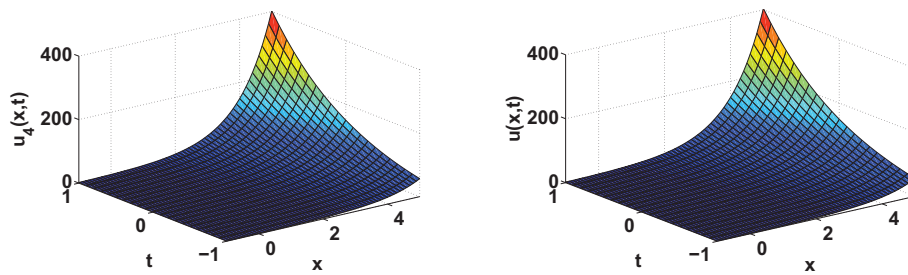

Figure 1: The 3D plot of the fourth approximate solution of Example 2.1 as a function of x and t (Left). The 3D plot of the exact solution of Example 2.1 as a function of x and t (Right).
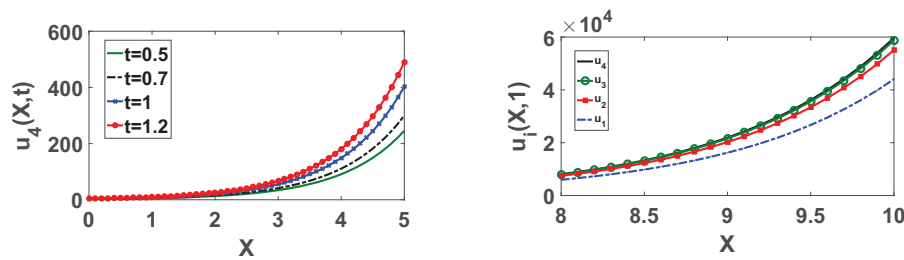

Figure 2: The Plot of the fourth approximate solution $u_4(x,t)$ for Example 2.1 as a function of x with $t = 0.5$ solid line, $t = 0.7$ dotted line, $t = 1 \times$ and $t = 1.2 \bullet$(Left). The plot of the four approximate solutions of Example 2.1, u_1 dotted line, u_2 ■, $u_3 \circ$ and u_4 solid line, as functions of x and $t = 1$ (Right).
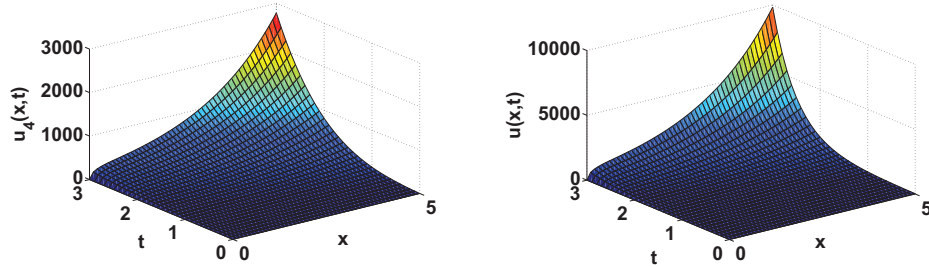
Figure 3: The 3D plot of the fourth approximate solution of Example 2.2 as a function of x and t (Left). The 3D plot of the exact solution of Example 2.2 as a function of x and t (Right).
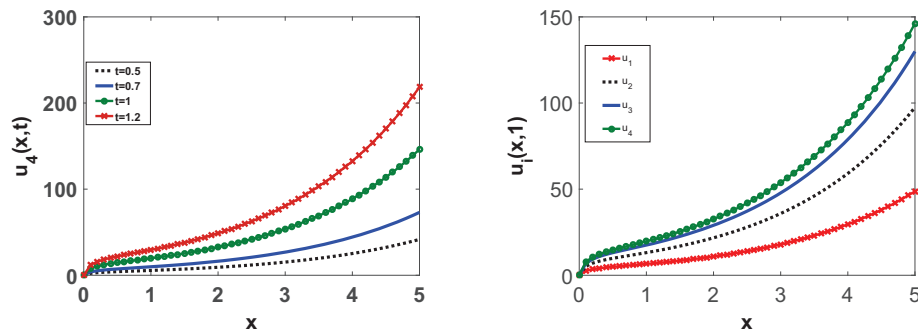

Figure 4: The Plot of the fourth approximate solution $u_4(x,t)$ for Example 2.2 as a function of x with $t = 0.5$ dotted line, $t = 0.7$ solid line, $t = 1$ • and $t = 1.2$ ×(Left). The plot of the four approximate solutions of Example 2.2, u_1 ×, u_2 dotted line, u_3 solid line and u_4 • as functions of x and $t = 1$ (Right).

4. Conclusion

The RPSM is applied to find the approximate solution for nonlinear reaction-diffusion-convection equations, the problems were already studied using the HAM method [21]. The results show that the RPSM is an efficient technique for solving nonlinear problems and gives better estimates than the HAM.

Acknowledgments

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