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Weakly μ-Compact Via a Hereditary Class*

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Abstract: The aim of this paper is to introduce and study some types of μ-compactness with respect to a hereditary class called weakly $\mu\mathcal{H}$-compact spaces and weakly $\mu\mathcal{H}$-compact subsets. We will provide several characterizations of weakly $\mu\mathcal{H}$-compact spaces and investigate their relationships with some other classes of generalized topological spaces.

Key Words: Generalized topology, Hereditary class, Weakly μ-compact, Weakly $\mu\mathcal{H}$-compact.

Contents

1. Introduction

The idea of generalized topology and hereditary classes was introduced and studied by Császár in [1] and [3], respectively. The purpose of the present paper is to introduce and investigate the class of a weakly $\mu\mathcal{H}$-compact space which is a natural generalization of a weakly compact space [10]. And also we define and investigate the notion of weakly $\mu\mathcal{H}$-compact subsets. The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as [3], [5], [7], and [12].

2. Preliminaries

Let X be a nonempty set and $p(X)$ the power set of X. A subfamily μ of $p(X)$ is called a generalized topology [1] if $\emptyset \in \mu$ and the arbitrary union of members of μ is again in μ. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained

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in A (see [1], [2]). A nonempty subcollection $\mathcal{H}$ of $p(X)$ is called a hereditary class (briefly HC) [3] if $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$. An HC $\mathcal{H}$ is called an ideal if $\mathcal{H}$ satisfies the additional condition: $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$ [6]. Some useful hereditary classes in X are: $p(A)$, where $A \subseteq X$ and $\mathcal{H}_f$, the HC of all finite subsets of X. A subset A of a GTS (X, μ) is s said to be weakly μ-compact [10] if any cover of A by μ-open sets of X has a finite subfamily, the union of the μ-closures of whose members covers A. If $A = X$, then (X, μ) is called a weakly μ-compact space. Given a generalized topological space (X, μ) with an HC $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [3] is defined as follows: $A^*(\mathcal{H}, \mu) = \{x \in X : U \cap A \notin \mathcal{H} \quad \text{for all} \quad U \in \mu_x\}$, where $\mu_x = \{U : x \in U \quad \text{and} \quad U \in \mu\}$. If there is no confusion, we simply write A^* instead of $A^*(\mathcal{H}, \mu)$. And for a subset A of X, $c^*_\mu(A)$ is defined by $c^*_\mu(A) = A \cup A^*$. The family $\mu^* = \{A \subset X : X \setminus A = c^*_\mu(X \setminus A)\}$ is a GT on X which is finer than μ [3]. The elements of μ^* are said to be μ^*-open and the complement of a μ^*-open set is called a μ^*-closed set. It is clear that a subset A is μ^*-closed if and only if $A^* \subset A$. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS. If $(X, \mu, \mathcal{H})$ is an HGTS, the set $\mathcal{B} = \{V \setminus H : V \in \mu \quad \text{and} \quad H \in \mathcal{H}\}$ is a base for a GT μ^*.

Next we recall some known definitions, lemmas and theorems which will be used in the work.

Theorem 2.1. *[3] Let (X, μ) be a GTS, $\mathcal{H}$ a hereditary class on X and A be a subset of X. If A is μ^*-open, then for each $x \in A$ there exist $U \in \mu_x$ and $H \in \mathcal{H}$ such that $x \in U \setminus H \subset A$.*

Definition 2.1. [1] Let (X, μ) and (Y, ν) be two GTS's, then a function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-continuous if $U \in \nu$ implies $f^{-1}(U) \in \mu$.

Definition 2.2. [11] Let (X, μ) and (Y, ν) be two GTS's. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-open (or μ-open) if $U \in \mu$ implies $f(U) \in \nu$.

Definition 2.3. [10] Let A be a subset of a space (X, μ). Then A is said to be:

1. μ-regular closed if $A = c_\mu(i_\mu(A))$,
2. μ-regular open if $X \setminus A$ is μ-regular closed.

Lemma 2.2. *[10] Let A be a subset of a GTS (X, μ). Then*

1. *A is μ-regular open if and only if $A = i_\mu(B)$ for some μ-closed set B,*
2. *A is μ-regular closed if and only if $A = c_\mu(B)$ for some μ-open set B.*

Definition 2.4. [10] A GTS (X, μ) is said to be μ-regular if for each μ-open subset U of X and each $x \in U$, there exist a μ-open subset V of X and a μ-closed subset F of X such that $x \in V \subset F \subset U$.

Lemma 2.3. *[10] Let A and B be subsets of a GTS (X, μ) such that $A \subset B$. Then $c_{\mu_B}(A) = c_\mu(A) \cap B$.*

Definition 2.5. [10] Let A be a subset of a GTS (X, μ). A point $x \in X$ is called a θ_μ-accumulation point of A if $c_\mu(V) \cap A \neq \emptyset$ for every μ-open subset V of X that contains x. The set of all θ_μ-accumulation points of A is called the θ_μ-closure of A and is denoted by $(c_\mu)_\theta(A)$. A is called μ_θ-closed if $(c_\mu)_\theta(A) = A$. The complement of a μ_θ-closed set is said to be μ_θ-open.

It is clear that A is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set V such that $x \in V \subset c_\mu(V) \subset A$.

Lemma 2.4. *[10] Let A be a subset of a GTS (X, μ). Then:*

1. *if A is μ_θ-open, then A is the union of μ-regular open sets,*
2. *(X, μ) is μ-regular if and only if every μ-open subset of X is μ_θ-open,*
3. *if A is μ-clopen, i.e. μ-open and μ-closed, then A is μ_θ-closed,*
4. *$c_\mu(A) \subset (c_\mu)_\theta(A)$,*
5. *if A is μ-open, then $c_\mu(A) = (c_\mu)_\theta(A)$.*

Lemma 2.5. *[10] Let $f : (X, \mu) \to (Y, \nu)$ be a function. Then the following are equivalent:*

1. *f is (μ, ν)-continuous;*
2. *for every $x \in X$ and every ν-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subset V$;*
3. *$f(c_\mu(A)) \subset c_\nu(f(A))$ for every subset A of X;*
4. *$c_\mu(f^{-1}(B)) \subset f^{-1}(c_\nu(B))$ for every subset B of Y.*

3. Weakly $\mu\mathcal{H}$-Compact Spaces

We recall that A subset A of X is said to be $\mu\mathcal{H}$-compact [4] if for every cover $\{U_\lambda : \alpha \in \Lambda\}$ of A by μ-open sets, there exists a finite subset Λ_0 of Λ such that $A \setminus \cup\{U_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. If $A = X$, then (X, μ) is called a $\mu\mathcal{H}$-compact space.

Definition 3.1. Let (X, μ) be a GTS with HC. An HGTS $(X, \mu, \mathcal{H})$ is said to be weakly $\mu\mathcal{H}$-compact if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of X by μ-open sets in X, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{c_\mu(V_\alpha) : \alpha \in \Lambda_0\} \in \mathcal{H}$.

Remark 3.1. *The following properties are obvious from Definition 3.1:*

1. *If (X, μ) is weakly μ-compact then it is weakly $\mu\mathcal{H}$-compact;*
2. *(X, μ) is weakly μ-compact if and only if $(X, \mu, \{\emptyset\})$ is weakly $\mu\{\emptyset\}$-compact;*
3. *If $(X, \mu, \mathcal{H}_f)$ is weakly $\mu\mathcal{H}_f$-compact then it is weakly compact;*
4. *The following implications hold.*

$$\begin{array}{ccc} \mu\text{−compact} & \Rightarrow & \mu\mathcal{H}-\text{compact} \\ \Downarrow & & \Downarrow \\ \text{weakly } \mu\text{−compact} & \Rightarrow & \text{weakly } \mu\mathcal{H}-\text{compact} \end{array}$$

Example 3.1. *Let μ be the Khalimsky topology, i.e., the topology on the set of integers $\mathbb{Z}$ generated by the set of all triplets of the form $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$ as subbase and the hereditary class $\mathcal{H} = p(\mathbb{Z})$. Now it is clear that $(\mathbb{Z}, \mu)$ is not weakly μ-compact but it is evidently weakly $\mu\mathcal{H}$-compact (resp. $\mu\mathcal{H}$-compact) because for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open subsets of $\mathbb{Z}$ and for any finite subset Λ_0 of Λ we have $\mathbb{Z} \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$ (resp. $\mathbb{Z} \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$).*

Example 3.2. *Consider $X = (0, 1)$, $\mu = \{\emptyset, V_n : n \in \mathbb{Z}^+\}$ where $V_n = (\frac{1}{n}, 1)$ and $\mathcal{H} = \mathcal{H}_f$. Then (X, μ) is weakly μ-compact but not $\mu\mathcal{H}$-compact since every proper μ-open set is μ-dense.*

The notions of weakly μ-compact and $\mu\mathcal{H}$-compact are independent of each other as shown by Examples 3.1 and 3.2.

Theorem 3.2. *An HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact if and only if for any μ-regular open cover $\{V_\alpha : \alpha \in \Lambda\}$ of X, there exits a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$.*

Proof. As necessity is clear, we prove only sufficiency. Suppose that $\{V_\alpha : \alpha \in \Lambda\}$ is a cover of X by μ-open sets. Now the family $\{i_\mu(c_\mu(V_\alpha)) : \alpha \in \Lambda\}$ is a cover of X by μ-regular open sets. Thus by assumption, there exists a finites subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(i_\mu(c_\mu(V_\alpha))) \in \mathcal{H}$. But $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \subseteq X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(i_\mu(c(V_\alpha)))$ which implies $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact. □

Proposition 3.3. *For an HGTS $(X, \mu, \mathcal{H})$, the following are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact;*
2. *If $\{F_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed sets having the property that for any finite subset Λ_0 of Λ $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \notin \mathcal{H}$, then $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$.*

Proof. (1)⇒(2): Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of μ-closed sets having the property that for any finite subset Λ_0 of Λ $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \notin \mathcal{H}$.
Suppose that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$. Then $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is an μ-open cover for X. Indeed, $\bigcup_{\alpha\in\Lambda} X \setminus F_\alpha = X \setminus \bigcap_{\alpha\in\Lambda} F_\alpha = X \setminus \emptyset = X$. By using (1), there is a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(X \setminus F_\alpha) \in \mathcal{H}$. It follows that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(X - F_\alpha) = \bigcap_{\alpha\in\Lambda_0} X - c_\mu(X\ F_\alpha) = \bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}$ which is contrary

to the (2). Thus $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$.
(2)⇒(1): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Suppose that for any a finite subset Λ_0 of Λ, $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \notin \mathcal{H}$. Then $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) = \bigcap_{\alpha\in\Lambda_0} X \setminus c_\mu(V_\alpha) = \bigcap_{\alpha\in\Lambda_0} i_\mu(X \setminus V_\alpha) \notin \mathcal{H}$. By (2) $\emptyset \neq \bigcap_{\alpha\in\Lambda} X \setminus V_\alpha = X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = \emptyset$. Hence, we get a contradiction to the fact that $\{V_\alpha : \alpha \in \Lambda\}$ is a μ-open cover of X. Therefore, X is weakly $\mu\mathcal{H}$-compact. □

Theorem 3.4. *For an HGTS $(X, \mu, \mathcal{H})$, the following conditions are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact;*
2. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}$;*
3. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-regular closed subsets of X such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}$.*

Proof. (1) ⇒ (2): Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of μ-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} = \phi$. Then $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is a μ-open cover of X. Since $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(X \setminus F_\alpha) \in \mathcal{H}$. Now we have

$$\begin{aligned} X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(X \setminus F_\alpha) &= \bigcap_{\alpha\in\Lambda_0} X \setminus c_\mu(X \setminus F_\alpha) \\ &= \bigcap_{\alpha\in\Lambda_0} i_\mu(X \setminus (X \setminus F_\alpha)) = \bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}. \end{aligned}$$

(2) ⇒ (3): It is obvious
(3) ⇒ (1): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Since $V_\alpha \subseteq i_\mu(c_\mu(V_\alpha))$ for all $\alpha \in \Lambda$, $\{i_\mu(c_\mu(V_\alpha)) : \alpha \in \Lambda\}$ is a cover of X by μ-regular open sets. Now $\{X \setminus i_\mu(c_\mu(V_\alpha)) : \alpha \in \Lambda\}$ is a family of μ-regular closed sets and

$$\bigcap_{\alpha\in\Lambda} X \setminus i_\mu(c_\mu(V_\alpha)) = \bigcap_{\alpha\in\Lambda} c_\mu(i_\mu(X \setminus V_\alpha)) = \emptyset.$$

Thus by assumption there exists a finite subset Λ_0 of Λ such that

$$\bigcap_{\alpha\in\Lambda_0} i_\mu(c_\mu(i_\mu(X \setminus V_\alpha))) \in \mathcal{H}.$$

Now

$$\bigcap_{\alpha\in\Lambda_0} i_\mu(c_\mu(i_\mu(X\backslash V_\alpha))) \supset \bigcap_{\alpha\in\Lambda_0} i_\mu(X\backslash V_\alpha) = \bigcap_{\alpha\in\Lambda_0} (X\backslash c_\mu(V_\alpha)) = X\backslash \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha).$$

Therefore, $X\backslash \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact. □

Theorem 3.5. *If a HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact, then for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of X by μ_θ-open sets there exits a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$.*

Proof. Suppose that $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact and let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of X by μ_θ-open sets. Then for each $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in V_{\alpha_x}$. Since V_{α_x} is μ_θ-open, there exists a μ-open set U_{α_x} such that $x \in U_{\alpha_x} \subset c_\mu(U_{\alpha_x}) \subset V_{\alpha_x}$. Then $\{U_{\alpha_x} : x \in X\}$ is a μ-open cover of X. Since X is $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha_x\in\Lambda_0} c_\mu(U_{\alpha_x}) \in \mathcal{H}$. Observe that $X \setminus \bigcup_{\alpha_x\in\Lambda_0} V_{\alpha_x} \subset X \setminus \bigcup_{\alpha_x\in\Lambda_0} c_\mu(U_{\alpha_x})$ and therefore, $X \setminus \bigcup_{\alpha_x\in\Lambda_0} V_{\alpha_x} \in \mathcal{H}$. □

Theorem 3.6. *Let (X, μ) be a μ-regular GTS. Then the following are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact;*
2. *$(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact.*

Proof. (1) ⇒ (2): Suppose X is μ-regular, weakly $\mu\mathcal{H}$-compact and $\{V_\alpha : \alpha \in \Lambda\}$ is a μ-open cover of X. Then for each $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in V_{\alpha_x}$. Since X is μ-regular, there exists a μ-open set U_x containing x such that $U_x \subset c_\mu(U_x) \subset V_{\alpha_x}$. The family $\{U_x : x \in X\}$ is a μ-open cover of X. Since X is weakly $\mu\mathcal{H}$-compact, there exists a finite finite subset X_0 of X such that $X \setminus \bigcup_{x\in X_0} c_\mu (U_x) \in \mathcal{H}$. Since $X \setminus \bigcup_{x\in X_0} V_{\alpha_x} \subset X \setminus \bigcup_{x\in X_0} c_\mu (U_x)$ which implies that $X \setminus \bigcup_{x\in X_0} V_{\alpha_x} \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact.
(2) ⇒ (1): It is obvious. □

Definition 3.2. Let (X, μ) be a GTS with HC. A filter base $\mathcal{F} = P(X) - \mathcal{H}$ is said to θ_μ-converge to a point $x \in X$ if for each μ-open subset U of X such that $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subset c_\mu(U)$ and denoted by $\mathcal{F}\theta_\mu$-converge. $\mathcal{F}$ is said to θ_μ-accumulate at $x \in X$ if $(c_\mu(U)) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and every μ-open subset U of X such that $x \in U$ and denoted by $\mathcal{F}\theta_\mu$-accumulate

Note that if a filter base $\mathcal{F}\theta_\mu$-converges to a point $x \in X$, then $\mathcal{F}$ θ_μ-accumulates at x. On the other hand, it is easy to see that a maximal filter base $\mathcal{F}$ θ_μ-converges to a point $x \in X$ if and only if $\mathcal{F}$ θ_μ-accumulates at x.

Theorem 3.7. *For a HGTS $(X, \mu, \mathcal{H})$, the following are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact;*
2. *Every maximal filter base $\mathcal{F}$ on $p(X) \setminus \mathcal{H}$ μ_θ-converges to some point of X;*
3. *Every filter base $\mathcal{F}$ on $p(X) \setminus \mathcal{H}$ μ_θ-accumulates at some point of X;*
4. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}$.*

Proof. (1)⇒(2): Let $\mathcal{F}$ be a maximal filter base on $p(X) \setminus \mathcal{H}$. Suppose that $\mathcal{F}$ does not μ_θ-converge at any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not μ_θ-accumulate at any point of X. For each $x \in X$, there exist $F_x \in \mathcal{F}$ and a μ-open subset V_x of X such that $x \in V_x$ and $c_\mu(V_x) \cap F_x = \emptyset$. Now the family $\{V_x : x \in X\}$ is a cover of X by μ-open sets of X. Since X is weakly $\mu\mathcal{H}$-compact, there exists a finite subfamily $\{V_{x_i} : i \in \{1, 2, 3, ..., n\}\}$ such that $X \setminus \bigcup_{i=1}^{n} c_\mu(V_{x_i}) \in \mathcal{H}$. Let $X \setminus \bigcup_{i=1}^{n} c_\mu(V_{x_i}) = H$, where $H \in \mathcal{H}$. Since $\mathcal{F}$ is a filter base, there exists a $F_0 \in \mathcal{F}$ such that $F_0 \subset \bigcap_{i=1}^{n} F_{x_i}$, hence $F_0 \subset F_{x_i}$ for all i. Now $F_0 \cap c_\mu(V_{x_i}) \subset F_{x_i} \cap c_\mu(V_{x_i})$ which implies that $F_0 \cap c_\mu(V_{x_i}) = \emptyset$. Also, $F_0 = F_0 \cap X \subset F_0 \cap (\cup c_\mu(V_{x_i}) \cup H) = (F_0 \cap (\cup c_\mu(V_{x_i}))) \cup (F_0 \cap H) = \emptyset \cup (F_0 \cap H) = F_0 \cap H \subset H$ which implies that $F_0 \in \mathcal{H}$ which is contrary to the fact that $F_0 \in p(X) \setminus \mathcal{H}$. Hence $\mathcal{F}$ is μ_θ-converges to some point of X.

(2)⇒(3): Suppose that every maximal filter base filter base on $p(X) \setminus \mathcal{H}$ μ_θ-converges to some point of X. Let $\mathcal{F}$ be any filter base on $p(X) \setminus \mathcal{H}$. Since each filter base is contained in a maximal filter base, there exists a maximal filter base $\mathcal{G}$ on $p(X) \setminus \mathcal{H}$ such that $\mathcal{F} \subset \mathcal{G}$ which implies that $\mathcal{F} \cap \mathcal{G} = \mathcal{F}$. By hypothesis, $\mathcal{G}$ μ_θ-converges to some point $x \in X$. Therefore, for every μ-open set V containing x, there exists $G \in \mathcal{G}$ such that $G \subset c_\mu(V)$ which implies that $G \cap F \subset c_\mu(V) \cap F$ for every $F \in \mathcal{F}$ which in turn implies that $c_\mu(V) \cap F \neq \emptyset$. Hence $\mathcal{F}$ μ_θ-accumulates at some point of X.

(3)⇒(4): Let $\{F_\alpha : \alpha \in \Lambda\}$ be a fimily of μ-closed sets such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$. Suppose that for every finite subset Λ_0 of Λ, $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \notin \mathcal{H}$. If $\mathcal{F} = \{i_\mu(F_\alpha) : \alpha \in \Lambda_0\}$, then $\mathcal{F}$ forms a filter base on $p(X) \setminus \mathcal{H}$. Since $\{F_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed subsets of X, $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of X. By hypothesis, $\mathcal{F}$ μ_θ-accumulates at some point of X. Therefore, $x \in X \setminus F_{\alpha_x}$ for some $\alpha_x \in \Lambda$. Then $X \setminus F_{\alpha_x}$ is an μ-open set containing x and $i_\mu(F_{\alpha_x}) \in \mathcal{F}$. Hence $(X \setminus i_\mu(F_{\alpha_x})) \cap i_\mu(F_{\alpha_x})) = \emptyset$ which implies that $c_\mu(X \setminus F_{\alpha_x}) \cap i_\mu(F_{\alpha_x})) = \emptyset$. Which is a contradiction. Therefore, $\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha) \in \mathcal{H}$.

(4)⇒(1): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Then there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} i_\mu(X \setminus V_\alpha) \in \mathcal{H}$ implies that $\bigcap_{\alpha\in\Lambda_0} X \setminus c_\mu(V_\alpha) \in \mathcal{H}$

which impels that $X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$. Hence $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact. □

4. Weakly $\mu\mathcal{H}$-Compact Subsets

Definition 4.1. A subset A of an HGTS $(X, \mu, \mathcal{H})$ is said to be weakly $\mu\mathcal{H}$-compact if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by μ-open subsets of X there exits a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$.

Corollary 4.1. *For a subset A of X, the following are equivalent:*

1. *A is weakly $\mu\mathcal{H}_f$-compact in X;*
2. *A is weakly $\mu\mathcal{H}$-compact in X;*
3. *A is weakly $\mu\{\emptyset\}$-compact in X.*

We observe that every $\mu\mathcal{H}$-compact subset of a HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact.

Theorem 4.2. *A subset A of an HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact if and only if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by μ-regular open subsets of X, there exits a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}$*

Proposition 4.3. *For a subset A of an HGTS $(X, \mu, \mathcal{H})$, the following are equivalent:*

1. *A is weakly $\mu\mathcal{H}$-compact;*
2. *If $\{F_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed sets having the property that for any finite subset Λ_0 of Λ $[\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha)] \cap A \notin \mathcal{H}$, then $[\bigcap_{\alpha\in\Lambda} F_\alpha] \cap A \neq \emptyset$.*

Theorem 4.4. *For a subset A of an HGTS $(X, \mu, \mathcal{H})$, the following are equivalent:*

1. *A is weakly $\mu\mathcal{H}$-compact;*
2. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $[\bigcap_{\alpha\in\Lambda} F_\alpha] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha)] \cap A \in \mathcal{H}$;*
3. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-regular closed subsets of X such that $[\bigcap_{\alpha\in\Lambda} F_\alpha] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha)] \cap A \in \mathcal{H}$.*

Theorem 4.5. *Let A be a weakly $\mu\mathcal{H}$-compact subset of an HGTS $(X, \mu, \mathcal{H})$. Then for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by μ_θ-open subsets of X there exits a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$.*

Proposition 4.6. *Let (X, μ) be a μ-regular GTS. Then a subset A of $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact if and only if A is $\mu\mathcal{H}$-compact.*

Theorem 4.7. *For a subset A of an HGTS $(X, \mu, \mathcal{H})$, the following are equivalent:*

1. *A is weakly $\mu\mathcal{H}$-compact;*
2. *Every maximal filter base $\mathcal{F}$ on $p(X)\setminus\mathcal{H}$ which meets A μ_θ-converges to some point of A;*
3. *Every filter base $\mathcal{F}$ on on $p(X)\setminus\mathcal{H}$ which meets A μ_θ-accumulates at some point of A;*
4. *For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $[\bigcap_{\alpha\in\Lambda} F_\alpha] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap_{\alpha\in\Lambda_0} i_\mu(F_\alpha)] \cap A \in \mathcal{H}$.*

Proposition 4.8. *If A is μ_θ-closed and B is weakly $\mu\mathcal{H}$-compact, then $A \cap B$ is weakly $\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $A \cap B$ by μ-open subsets of X. Then $\{V_\alpha : \alpha \in \Lambda\} \cup \{X \setminus A\}$ is a μ-open cover of B. Since $X \setminus A$ is μ_θ-open, for each $x \in X \setminus A$, there exists a μ-open set V_x such that $x \in V_x \subset c_\mu(V_x) \subset X \setminus A$. Thus $\{V_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X \setminus A\}$ is a cover of B by μ-open sets. Since B is weakly $\mu\mathcal{H}$-compact, there exist a finite a finite subset Λ_0 of Λ and finite points, says, $x_1, x_2, ..., x_n \in X \setminus A$ such that $B \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))] \in \mathcal{H}$. Now $A \cap B \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))] \subset B \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))]$ which implies $A \cap B \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))] \in \mathcal{H}$. Hence $A \cap B$ is weakly $\mu\mathcal{H}$-compact □

Theorem 4.9. *Every μ_θ-closed subset of a weakly $\mu\mathcal{H}$-compact space $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact.*

Proof. Let F be a μ_θ-closed subset of X. Suppose $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of F by μ-open sets of X. Since $X \setminus F$ is μ_θ-open, for each $x \in X \setminus F$, there exists a μ-open set V_x such that $x \in V_x \subset c_\mu(V_x) \subset X \setminus F$. Then the collection $\{V_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X \setminus F\}$ forms a μ-open cover of X. Since X is weakly $\mu\mathcal{H}$-compact, there exist a finite subset Λ_0 of Λ and finite points, says, $x_1, x_2, ..., x_n \in X \setminus F$ such that $X \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))] \in \mathcal{H}$. Then

$$\begin{aligned} X \setminus [(\bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cup (\bigcup_{i=1}^{n} c_\mu(V_{x_i}))] &= (X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cap (X \setminus \bigcup_{i=1}^{n} c_\mu(V_{x_i})) \\ &\supset (X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cap X \setminus (X \setminus F) \\ &= (X \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha)) \cap F = F \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha), \end{aligned}$$

which implies $F \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu\left(V_\alpha\right) \in \mathcal{H}$. Therefore, F is weakly $\mu\mathcal{H}$-compact. □

Let $(X, \mu, \mathcal{H})$ be an HGTS and let $A \subseteq X$, $A \neq \phi$. We denote by $\mathcal{H}_A$ the collection $\{H \cap A : H \in \mathcal{H}\}$ and by (A, μ_A) the subspace of (X, μ) on A. It is clear that the collection μ_A is a generalized topology on A and the collection $\mathcal{H}_A$ is a hereditary class of subsets in A.

Proposition 4.10. *Let A and B be subsets of an HGTS $(X, \mu, \mathcal{H})$ such that $A \subset B$. If A is weakly $\mu_B\mathcal{H}_B$-compact, then A is weakly $\mu\mathcal{H}$-compact.*

Proof. Suppose that A is weakly $\mu_B\mathcal{H}_B$-compact and let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of A by μ-open sets of X. Then $\{V_\alpha \cap B : \alpha \in \Lambda\}$ is a cover of A by μ_B-open sets and A is weakly $\mu_B\mathcal{H}_B$-compact. Hence there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha\in\Lambda_0} c_{\mu_B}(V_\alpha \cap B) \in \mathcal{H}_B$. By Lemma 2.2 $c_{\mu_B}(V_\alpha \cap B) = c_\mu(V_\alpha \cap B) \cap B \subset c_\mu(V_\alpha)$ and $A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \subseteq A \setminus \bigcup_{\alpha\in\Lambda_0} c_{\mu_B}(V_\alpha \cap B)$ which impels $A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu(V_\alpha) \in \mathcal{H}_B = \mathcal{H} \bigcap B \subset \mathcal{H}$. Hence, A is weakly $\mu\mathcal{H}$-compact. □

Corollary 4.11. *Let A be a subset of an HGTS $(X, \mu, \mathcal{H})$, then the following hold.*

1. *If A is weakly $\mu_A\mathcal{H}_A$-compact, then A is weakly $\mu\mathcal{H}$-compact.*
2. *If A is a μ-clopen subset of a weakly $\mu\mathcal{H}$-compact $(X, \mu, \mathcal{H})$, then A is weakly $\mu\mathcal{H}$-compact.*

Proof. (1) This follows from Proposition 4.4.
(2) This follows from Lemma 2.3 (3) and Theorem 4.5. □

Theorem 4.12. *Let $(X, \mu, \mathcal{H})$ be a HGTS and $\mathcal{H}$ be an ideal on X, then the union of two weakly $\mu\mathcal{H}$-compact sets is a weakly $\mu\mathcal{H}$-compact set.*

Proof. Let A, B be two weakly $\mu\mathcal{H}$-compact sets of X and let $\{V_\alpha : \alpha \in \Lambda\}$ be any cover of $A \cup B$ by μ-open sets of X. Then there exist two finite subsets $\Lambda_0, \Lambda_1 \subset \Lambda$ such that: $A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu\left(V_\alpha\right) \in \mathcal{H}$ *and* $B \setminus \bigcup_{\alpha\in\Lambda_1} c_\mu\left(V_\alpha\right) \in \mathcal{H}$. Observe that $(A \cup B) \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} c_\mu\left(V_\alpha\right) \subset (A \setminus \bigcup_{\alpha\in\Lambda_0} c_\mu\left(V_\alpha\right)) \cup (B \setminus \bigcup_{\alpha\in\Lambda_1} c_\mu\left(V_\alpha\right))$. Since $\Lambda_0 \cup \Lambda_1$ is a finite subset of Λ and $\mathcal{H}$ is an ideal on X, it follows that $(A \cup B) \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} c_\mu\left(V_\alpha\right) \in \mathcal{H}$. In consequence $A \cup B$ is a weakly $\mu\mathcal{H}$-compact subset of X. □

The following example shows that if the class $\mathcal{H}$ is not an ideal then the union of two weakly $\mu\mathcal{H}$-compact subsets is not necessary weakly $\mu\mathcal{H}$-compact.

Example 4.1. *Let $\mathbb{R}$ be the set of real numbers, μ the usual topology and the hereditary class $\mathcal{H} = \{A \subset \mathbb{R} : A \subset (0,1) \quad or \quad A \subset (1,2)\}$. Observe that $A = (0,1)$ and $B = (1,2)$ are weakly $\mu\mathcal{H}$-compact sets. But $A \cup B$ is not weakly $\mu\mathcal{H}$-compact. The family $\{(\frac{1}{n}, 2 - \frac{1}{n}) : n \in \mathbb{Z}^+\}$ is a cover of $A \cup B$ by μ-open sets. Let $\{n_1, n_2, ..., n_k\}$ be any finite subset of the positive integer $\mathbb{Z}^+$ and let $N = max\{n_1, n_2, ..., n_k\}$. Then*

$$(A \cup B) \setminus \bigcup_{i=1}^{k} c_\mu(\tfrac{1}{n_i},\ 2 - \tfrac{1}{n_i}) = A \cup B \setminus \bigcup_{i=1}^{k} [\tfrac{1}{n_i},\ 2 - \tfrac{1}{n_i}]$$
$$= (A \cup B) \setminus \big[\tfrac{1}{N},\ 2 - \tfrac{1}{N}\big] = \big(0, \tfrac{1}{N}\big) \cup \big(2 - \tfrac{1}{N}, 2\big) \notin \mathcal{H}.$$

Theorem 4.13. *If every proper μ-regular closed subset of an HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact, then $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact.*

Proof. Suppose that $\{V_\alpha : \alpha \in \Lambda\}$ is a cover of X by μ-open sets. Choose and fix $\alpha_0 \in \Lambda$ such that $V_{\alpha_0} \neq \emptyset$. Then the set $X \setminus i_\mu(c_\mu(V_{\alpha_0}))$ is a proper μ-regular closed set. Thus by assumption, there exists a finite subset Λ_0 of Λ such that $(X \setminus i_\mu (c_\mu (V_{\alpha_0}))) \setminus \bigcup_{\alpha \in \Lambda_0 - \{\alpha_0\}} c_\mu (V_\alpha) \in \mathcal{H}$. Observe that

$$(X \setminus i_\mu (c_\mu (V_{\alpha_0}))) \setminus \bigcup_{\alpha \in \Lambda_0 - \{\alpha_0\}} c_\mu (V_\alpha) = X \setminus [(\bigcup_{\alpha \in \Lambda_0 - \{\alpha_0\}} c_\mu (V_\alpha)) \cup (i_\mu (c_\mu (V_{\alpha_0})))]$$
$$\supset X \setminus [(\bigcup_{\alpha \in \Lambda_0 - \{\alpha_0\}} c_\mu (V_\alpha)) \cup c_\mu (V_{\alpha_0})]$$
$$= X \setminus \bigcup_{\alpha \in \Lambda_0} c_\mu (V_\alpha).$$

Therefore, $X \setminus \bigcup_{\alpha \in \Lambda_0} c_\mu (V_\alpha) \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact. □

Lemma 4.14. *[4] Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is a hereditary class on Y.*

Theorem 4.15. *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous function. If A is a weakly $\mu\mathcal{H}$-compact subset of X, then $f(A)$ is weakly $\nu f(\mathcal{H})$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by ν-open sets. Since f is (μ, ν)-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a cover of A by μ-open sets and A is weakly $\mu\mathcal{H}$-compact, and hence exits a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha \in \Lambda_0} c_\mu \big(f^{-1}(V_\alpha)\big) \in \mathcal{H}$. Since f is (μ, ν)-continuous, it follows from Lemma 2.4 (4) that $c_\mu(f^{-1}(V_\alpha)) \subset f^{-1}(c_\nu(V_\alpha))$. Therefore, $A \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(c_\nu(V_\alpha)) \subset A \setminus \bigcup_{\alpha \in \Lambda_0} c_\mu(f^{-1}(V_\alpha)) \in \mathcal{H}$. Hence

$$A \setminus \bigcup_{\alpha \in \Lambda_0} f^{-1}(c_\nu(V_\alpha)) = A \setminus f^{-1}(\bigcup_{\alpha \in \Lambda_0} c_\nu(V_\alpha))$$
$$= A \cap f^{-1}(Y \setminus \bigcup_{\alpha \in \Lambda_0} c_\nu(V_\alpha)) \in \mathcal{H},$$

and hence

$$f(A \cap f^{-1}(Y \setminus \bigcup_{\alpha \in \Lambda_0} c_\nu(V_\alpha))) = f(A) \cap (Y \setminus \bigcup_{\alpha \in \Lambda_0} c_\nu(V_\alpha))$$
$$= f(A) \setminus \bigcup_{\alpha \in \Lambda_0} c_\nu(V_\alpha) \in f(\mathcal{H}).$$

Hence $f(A)$ is weakly $\nu f(\mathcal{H})$-compact. □

Corollary 4.16. *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous surjection. If $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact, then (Y, ν) is weakly $\nu f(\mathcal{H})$-compact.*

Corollary 4.17. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ be a (μ, ν)-open bijective function. If $(Y, \nu, \mathcal{G})$ is weakly $\nu\mathcal{G}$-compact, then (X, μ) is weakly $\mu f^{-1}(\mathcal{G})$-compact.*

Proof. Since $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ is a μ-open bijection, $f^{-1} : (Y, \nu, \mathcal{G}) \to (X, \mu)$ is a (ν, μ)-continuous surjection. Since $(Y, \nu, \mathcal{G})$ is weakly $\nu\mathcal{G}$-compact, by Theorem 4.8 we obtain that $(X, \mu, f^{-1}(\mathcal{G}))$ is weakly $\mu f^{-1}(\mathcal{G})$-compact. □

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