Bol. Soc. Paran. Mat. (3s.) **v. 39** 2 (2021): 133–140.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.40302

Characterization of the $w-$Tempered Ultradistributions

Ibraheem Abu-Falahah and Hamed M. Obiedat

ABSTRACT: We use a previously obtained characterization of test functions of $w-$Tempered Ultradistributions to characterize the space $w-$Tempered Ultradistributions using Riesz representation theorem.

Key Words: Short-time Fourier transform, Tempered Ultradistributions, Structure Theorem.

Contents

1. Introduction

The Schwartz space $\mathcal{S}$, as defined by Laurent Schwartz (see [10]), consists of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|x^\alpha \partial^\beta \varphi\right\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$. The topological dual space of $\mathcal{S}$, is a space of generalized functions, called tempered distributions. Tempered distributions have essential connections with the Fourier transform and partial differential equations. Moreover, they fit in many ways to provide a satisfactory framework of mathematical analysis and mathematical physics.

In 1963, A. Beurling presented his generalization of tempered distributions. The aim of this generalization was to find an appropriate context for his work on pseudo-analytic extensions (see [2]).

In 1967 (see [3]), G. Björck studied and expanded the theory of Beurling on ultra distributions to extend the work of Hörmander on existence, nonexistence, and regularity of solutions of differential equations with constant coefficient and also consider equations which have no solutions. The Beurling-Björck space $\mathcal{S}_w$, as defined by G. Björck, consists all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw(x)} \partial^\beta \varphi\right\|_\infty < \infty$ and $\left\|e^{kw(x)} \partial^\beta \widehat{\varphi}\right\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$, where w is a subadditive weight function satisfying the classical Beurling conditions. The topological dual $\mathcal{S}'_w$ of $\mathcal{S}_w$ is a space of generalized functions, called $w-$tempered ultra distributions. When $w(x) = \log(1 + |x|)$ the Beurling- Björck space $\mathcal{S}_w$ becomes the Schwartz space $\mathcal{S}$ (see [1]).

2010 *Mathematics Subject Classification:* 46F05, 46F10, 46F20.
Submitted November 02, 2017. Published January 07, 2018

In this paper, We use the characterization of the space $\mathcal{S}_w$ of test functions of $w-$tempered ultradistribution in terms of their short-time Fourier transform to characterize $w-$tempered ultradistribution using Riesz representation theorem.

The symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \leq p \leq \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. Partial derivatives will be denoted ∂^α, where α is a multi-index $(\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. The Fourier transform of a function f will be denoted $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi ix\xi} f(x)\, dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm. The letter C will indicate a positive constant, that may be different at different occurrences.

2. Preliminary definitions and results

In this section, we start with the definition of the space of admissible functions $\mathcal{M}_c$ as they introduced by Björck.

Definition 2.1. *([3])With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where*

1. $\Omega : [0, \infty) \to [0, \infty)$ *is increasing, continuous and concave,*
2. $\Omega(0) = 0$,
3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,
4. $\Omega(t) \geq a + b \ln(1 + t)$ *for some* $a \in \mathbb{R}$ *and some* $b > 0$.

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p\ln(1 + |x|) \text{ for } p > 0.$$

Theorem 2.2. *([4]) The space $\mathcal{S}_w$ can be described as a set as well as topologically by*

$$\mathcal{S}_w = \left\{ \begin{array}{l} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \ p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\|e^{kw(x)}\varphi\right\|_\infty$, $\pi_{k,0}(\varphi) = \left\|e^{kw(\xi)}\widehat{\varphi}\right\|_\infty$. The space $\mathcal{S}_w$, equipped with the family of semi-norms

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Remark 2.3. *Let us observe for future use that if we take $N > \frac{n}{b}$ is an integer, then*

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)}dx < \infty, \text{ for all } w \in \mathcal{M}_c,$$

where b is the constant in Condition 4 of Definition 2.1. Moreover, property 1 in Definition 2.1 implies that $w(\cdot)$ is subaddative.

Example 2.4. *From Theorem 2.2, it is clear that the Gaussian $f(x) = e^{-\pi|x|^2}$ belongs to $\mathcal{S}_w$ for all w in $\mathcal{M}_c$.*

It is well known that Fourier series are a good tool to represent periodic functions. However, they fail to represent non-periodic functions accurately. To solve this problem, the short-time Fourier transform was introduced by D. Gabor [5]. The short-time Fourier transform works by first cutting off the function by multiplying it by another function called window then apply the Fourier transform. This technique maps a function of time x into a function of time x and frequency ξ .

Definition 2.5. *([6], [7])The short-time Fourier transform (STFT) of a function or distribution f on $\mathbb{R}^n$ with respect to a non-zero window function g is formally defined as*

$$\nu_g f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t.\xi}dt = \widehat{(f\overline{T_x g})}(\xi) =< f, M_\xi T_x g > .$$

where $T_x g(t) = g(t-x)$ is the translation operator and $M_\xi g(t) = e^{2\pi i t.\xi}g(t)$ is the modulation operator.

The composition of T_x and M_ξ is the time-frequency shift

$$(M_\xi T_x g)(t) = e^{2\pi i x.\xi} g(t-x),$$

and its Fourier transform is given by

$$\widehat{M_\xi T_x} g = e^{2\pi i x.\xi} M_{-x} T_\xi \widehat{g}.$$

The main properties of the short-time Fourier transform is given in the following lemma.

Lemma 2.6. *([6], [7])For $f, g \in \mathcal{S}_w$, the STFT has the following properties.*

1. (Inversion formula)

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \nu_g f(x,\xi)(M_\xi T_x g)(t)dxd\xi = \|g\|_2^2 f. \quad (2.1)$$

2. (STFT of the Fourier transforms)

$$\nu_{\widehat{g}}\widehat{f}(x,\xi) = e^{-2\pi i x.\xi}\nu_g f(-\xi, x).$$

3. (Fourier transform of the STFT)

$$\widehat{\nu_g f}(x,\xi) = e^{2\pi i x.\xi} f(-\xi)\overline{\widehat{g}(x)}. \tag{2.2}$$

Remark 2.7. *The space $\nu_g(\mathcal{S}_w) = \{\nu_g f : f \in \mathcal{S}_w\}$ has no functions with compact support.*

Now we will introduce two auxiliary results that we will use in the proof of the topological characterization of the space $\mathcal{S}_w$ via the short-time Fourier transform.

Lemma 2.8. *([7])Let f and g be two nonnegative measurable functions. If $N > n$, there exists $C > 0$ such that*

$$\left\| e^{kw(\cdot)}(f * g)\right\|_\infty \le C \left\| e^{2(N+k)w(\cdot)} f\right\|_\infty \left\| e^{2(N+k)w(\cdot)} g\right\|_\infty,$$

for all $k = 0, 1, 2, \dots$. The constant C does not depend on k.

In the following lemma, we include a proof using the topological characterization of $\mathcal{S}_w$ given in Theorem 2.2 which imposes no conditions on the derivative. Our proof is an adaptation of the proof of (Proposition 2.6 stated in [7]).

Lemma 2.9. *Let $w \in \mathcal{M}_c$ and $g \in \mathcal{S}_w$ be fixed and assume that $F : \mathbb{R}^{2n} \to \mathbb{C}$ is a measurable function that has a subexponential decay, i.e. such that for each $k = 0, 1, 2, \dots$, there is a constant $C = C_k > 0$ satisfying*

$$|F(x,\xi)| \le C e^{-k(w(x)+w(\xi))}.$$

Then the integral

$$f(t) = \int\int_{\mathbb{R}^{2n}} F(x,\xi)(M_\xi T_x g)(t)\,dx d\xi$$

defines a function in $\mathcal{S}_w$.

Proof: To prove that $f \in \mathcal{S}_w$, we start with

$$\begin{aligned}
\left|(e^{kw(t)} f)(t)\right| &\le \int\int_{\mathbb{R}^{2n}} (F(x,\xi) e^{kw(t)} (M_\xi T_x g)(t))\,dx d\xi \\
&\le \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left| M_\xi T_x(e^{kw(t+x)} g))(t)\right| dx d\xi \\
&\le \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left| T_x(e^{kw(t+x)} g))(t)\right| dx d\xi \\
&\le \int\int_{\mathbb{R}^{2n}} e^{kw(x)} e^{Nw(\xi)} e^{-Nw(\xi)} |F(x,\xi)| \left\| e^{kw(\cdot)} g\right\|_\infty dx d\xi \\
&\le \int\int_{\mathbb{R}^{2n}} e^{(k+N)(w(x)+w(\xi))} e^{-N(w(x)+w(\xi))} |F(x,\xi)| \left\| e^{kw(\cdot)} g\right\|_\infty dx d\xi \\
&\le \left\| e^{kw(\cdot)} g\right\|_\infty \left\| e^{(N+k)(w(x)+w(\xi))} F\right\|_\infty \int\int_{\mathbb{R}^{2n}} e^{-N(w(x)+w(\xi))} dx d\xi \\
&\le C \left\| e^{(N+k)(w(x)+w(\xi))} F\right\|_\infty.
\end{aligned}$$

So,

$$\left\| e^{kw(\cdot)} f \right\|_\infty \le C \left\| e^{(N+k)(w(x)+w(\xi))} F \right\|_\infty . \tag{2.3}$$

This implies that $\left\| e^{kw(\cdot)} f \right\|_\infty < \infty$.

To show that $\left\| e^{kw(\cdot)} \widehat{f} \right\|_\infty < \infty$, we write

$$\widehat{f}(\tau) = \int\int_{\mathbb{R}^{2n}} (F(x,\xi)(M_{-x}T_\xi \widehat{g})(\tau)) e^{2\pi i x.\xi} dx d\xi,$$

using

$$(\widehat{M_\xi T_x g})(\tau) = (M_{-x}T_\xi \widehat{g})(\tau) e^{2\pi i x.\xi}.$$

Using an argument similar to the one leading to the proof of (2.3), we have

$$\left| e^{kw(t)} \widehat{f}(\tau) \right| \le C \left\| e^{(N+k)(w(x)+w(\xi))} F \right\|_\infty .$$

This completes the proof of Lemma 2.9.

Remark 2.10. *For the Gaussian $g(x) = e^{-\pi|x|^2}$ and f with $e^{-kw(x)} f \in L^1$ for some $k \in \mathbb{N}_0$, then $\nu_g f$ is well-defined and continuous. In fact,*

$$\begin{aligned} |\nu_g f(x,\xi)| &= \left| \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} dt \right| \\ &\le \int_{\mathbb{R}^n} \left| f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} \right| dt \\ &= \int_{\mathbb{R}^n} e^{-kw(x)} |f(t)| e^{kw(t)} \left| \overline{g(t-x)} \right| dt \\ &\le \int_{\mathbb{R}^n} e^{-kw(x)} |f(t)| e^{kw(t-x)} \left| \overline{g(t-x)} \right| e^{kw(x)} dt \\ &= \left\| e^{-kw(x)} f \right\|_1 \left\| e^{kw(x)} g \right\|_\infty e^{kw(x)}. \end{aligned}$$

This shows that $\nu_g f$ is well-defined. Moreover, if we fix $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and let (x_j, ξ_j) be any sequence in $\mathbb{R}^{2n}$ converging to (x_0, ξ_0) as $j \to \infty$, the function $f(t)\overline{g(t-x_j)} e^{-2\pi i t.\xi_j}$ converges to $f(t)\overline{g(t-x_0)} e^{-2\pi i t.\xi_0}$ pointwise as $j \to \infty$ and

$$\begin{aligned} \left| f(t)\overline{g(t-x_j)} e^{-2\pi i t.\xi_j} \right| &\le \left| e^{-kw(t)} f(t) e^{kw(t)} \overline{g(t-x_j)} e^{-2\pi i t.\xi_j} \right| \\ &\le \left| e^{-kw(t)} f(t) e^{kw(t-x)} \overline{g(t-x_j)} e^{kw(x_j)} \right| \\ &\le C \left| e^{-kw(t)} f(t) \right| \left\| e^{kw(\cdot)} g \right\|_\infty \\ &\le C \left| e^{-kw(t)} f(t) \right| . \end{aligned}$$

Since the function $\left| e^{-kw(t)} f(t) \right| \in L^1$, we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\nu_g f(x_j, \xi_j) \to \nu_g f(x_0, \xi_0)$$

as $j \to \infty$. This shows the continuity of $\nu_g f$.

□

3. Characterization of the dual space $\mathcal{S}'_w$

The following topological characterization as stated in ([8]) imposes no conditions on the derivatives.

Theorem 3.1. *Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the Beurling-Björck space $\mathcal{S}_w$ can be described as a set as well as topologically by*

$$S_w = \left\{ \begin{array}{c} f : \mathbb{R}^n \to \mathbb{C}\text{: } e^{-mw(x)} f \in L^1 \text{ for some } m \in \mathbb{N}_0 \text{ and } \pi_k(f) < \infty \\ \text{for all } k \in \mathbb{N}_0 \end{array} \right\}, \quad (3.1)$$

where $\pi_k(f) = \left\| e^{k(w(x)+w(\xi))} \nu_g f \right\|_\infty$.

Theorem 3.2. *([9])Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ so that*

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

Theorem 3.3. *Let $w \in \mathcal{M}_c$ and $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then if $L : \mathcal{S}_w \to \mathbb{C}$, the following statements are equivalent: (i) $L \in \mathcal{S}'_w$ (ii) There exist a regular complex Borel measure μ of finite total variation and $k \in \mathbb{N}_0$ so that*

$$L = e^{k(w(x)+w(\xi))} \nu_g d\mu,$$

in the sense of $\mathcal{S}'_w$.

Proof: $(i) \Rightarrow (ii)$. Given $L \in \mathcal{S}'_w$, there exist k, C so that

$$L(\varphi) \le C \left\| e^{k(w(x)+w(\xi))} \nu_g \varphi \right\|_\infty$$

for all $\varphi \in \mathcal{S}_w$. Moreover, the map

$$\begin{aligned} s_w(\mathbb{R}^n) &\to \mathcal{C}_0(\mathbb{R}^{2n}) \\ \varphi &\to e^{k(w(x)+w(\xi))} \nu_g \varphi \end{aligned}$$

is well-defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1 \left(e^{k(w(x)+w(\xi))} \nu_g \varphi \right) = L(\varphi),$$

for a unique $\varphi \in \mathcal{S}_w$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn-Banach theorem, there exists a functional L_1 in the topological dual $\mathcal{C}_0'(\mathbb{R}^{2n})$ of $\mathcal{C}_0(\mathbb{R}^{2n})$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1. Using Theorem 3.2, there exist a regular complex Borel measure μ of finite total variation so that

$$L_1(f) = \int_{\mathbb{R}^{2n}} f d\mu$$

for all $f \in \mathcal{C}_0(\mathbb{R}^{2n})$. If $f \in \mathcal{R}$, we conclude

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))} \nu_g \varphi d\mu$$

for all $\varphi \in \mathcal{S}_w$. In the sense of $\mathcal{S}_w'$,

$$L = e^{k(w(x)+w(\xi))} \nu_g d\mu.$$

$(ii) \Rightarrow (i)$. If μ is a regular complex Borel measure satisfying (ii) and $\varphi \in \mathcal{S}_w$, then

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))} \nu_g \varphi d\mu.$$

This implies that

$$\begin{aligned} |L(\varphi)| &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))} \nu_g \varphi d\mu \right| \\ &\leq |\mu| (\mathbb{R}^{2n}) \left\| e^{k(w(x)+w(\xi))} \nu_g \varphi \right\|_\infty \\ &\leq C(\left\| e^{k(w(x)+w(\xi))} \nu_g \varphi \right\|_\infty). \end{aligned}$$

It may be noted that μ, employed to obtain the above inequality, is of finite total variation. This completes the proof of Theorem 3.3. □

Corollary 3.4. *If $L \in \mathcal{S}_w'$ and $\varphi \in \mathcal{S}_w$, then the classical definition of the convolution $L * \varphi$ is defined by*

$$(L * \varphi, \psi) = (L_x, (\varphi_z, \psi(x+y)))$$

for all $\psi \in \mathcal{S}_w$. Moreover, the functional $L * \varphi$ coincides with the functional given by the integration against the function

$$f(y) = (L, \varphi(y - \cdot)).$$

Proof: *The inequality*

$$\begin{aligned} |f(y)| &= \left| \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))} \nu_g \varphi(y-x,\xi) d\mu \right| \\ &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(w(y-x)+w_2(\xi))} e^{kw(y)} \nu_g \varphi(y-x,\xi) d\mu \right| \\ &\leq C \left\| e^{k(w(x)+w(\xi))} \nu_g \varphi \right\|_\infty \end{aligned}$$

implies the boundedness of $f(y) = (L, \varphi(y - \cdot))$. *Then*

$$
\begin{aligned}
(L * \varphi, \psi) &= (L_x, (\varphi_z, \psi(x+y))) \\
&= \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))}\nu_g(\int_{\mathbb{R}^n} \varphi(y-x)\psi(y)dy)dxd\xi \\
&= \int_{\mathbb{R}^{2n}} e^{k(w(x)+w(\xi))}\nu_g(\psi * \varphi(x))dxd\xi \\
&= (e^{k(w(x)+w(\xi))}\nu_g, \psi * \varphi(x)) \\
&= (e^{k(w(x)+w(\xi))}\nu_g \varphi(y-x,\xi), \psi(y)) \\
&= ((e^{k(w(x)+w(\xi))}\nu_g, \varphi(y-x)), \psi(y)) \\
&= ((e^{k(w(x)+w(\xi))}\nu_g, (\varphi(y-x), \psi(y)) \\
&= (L_x, (\varphi(y-x), \psi(y))).
\end{aligned}
$$

□

References

1. J. Alvarez, H. Obiedat, Characterizations of the Schwartz space $\mathcal{S}$ and the Beurling-Björck space Sw, Cubo 6 (2004) 167-183.
2. A. Beurling, On quasi-analiticity and general distributions, Notes by P.L. Duren, lectures delivered at the AMS Summer Institute on Functional Analysis, Stanford University, August 1961.
3. G. Björck, Linear partial differential operators and generalized distributions,Ark. Mat. 6 (1965-1967) 351-407.
4. S.-Y. Chung, D. Kim, S. Lee. Characterizations for Beurling-Björck space and Schwartz space. Proc. Amer. Math. Soc. 125 (1997), 3229-3234.
5. D. Gabor. Theory of communication. J. IEE (London), 93(III): 429-457,November 19.
6. K. Gröchenig, G. Zimmermann. Hardy's theorem and the short-time Fourier transform of Schwartz functions. J. London Math. Soc. 63 (2001),205-214.
7. K. Gröchenig, G. Zimmermann. Spaces of test functions via the STFT.Function Spaces Appl. 2 (2004), 25-53.
8. H. Obiedat, A topological characterization of the Beurling- Björck space $\mathcal{S}_w$ using short-time Fourier transform,Cubo 8 (2006), 33-45.
9. W. Rudin, Functional Analysis, Second Edition, McGraw-Hill Inc., 1991.
10. L. Schwartz, Theorie des distributions, Hermann, Paris, 1966.

Ibraheem Abu-Falahah,
Department of Mathematics,
Hashemite University,
P.O.Box150459, Jordan.
E-mail address: iabufalahah@hu.edu.jo

and

Hamed M. Obiedat,
Department of Mathematics,
Hashemite University,
P.O.Box150459, Jordan.
E-mail address: hobiedat@hu.edu.jo