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# A New Construction on the Energy of Space Curves in Unit Vector Fields in Minkowski Space $\mathbb{E}_2^4$

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Abstract: In this paper, we firstly introduce kinematics properties of a moving particle lying in Minkowski space $\mathbb{E}_2^4$. We assume that the particle corresponds to different type of space curves such that they are characterized by Frenet frame equations. Guided by these, we present geometrical understanding of an energy and pseudo angle of the particle in each Frenet vector fields depending on the particle corresponds to a spacelike, timelike or lightlike curve in $\mathbb{E}_2^4$. Then we also determine the bending elastic energy functional for the same particle in $\mathbb{E}_2^4$ by assuming the particle has a bending feature of elastica. Finally, we prove that bending energy formula can be represented by the energy of the particle in each Frenet vector field.

Key Words: Energy, Pseudo Angle, Minkowski Space, Space Curves, Vector Field.

## Contents



## 1. Introduction

Minkowski 4-space with index 2; $\mathbb{E}_2^4$ is an extended version of Minkowski spacetime $\mathbb{E}_1^4$. Minkowski spacetime is significant to understand general relativity. Minkowski spacetime $\mathbb{E}_1^4$ introduces Lorentzian geometry, which is a space form having a constant sectional curvature. Thus, scientists are able to develop some up-to-date physical concepts by studying on $\mathbb{E}_1^4$ such as gravitational dilation of time, length contraction, cosmology, black holes, string theory. The innovation that emerged with Minkowski 4-space with index 2 is that it allows us to use tools of hyperbolic geometry. Thus, not only computations on the Lorentizan geometry become simpler but new investigations become available. For instance, complete

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solutions of field equations of Einstein for the empty universe with negative cosmological constant can be solved in Anti de Sitter space. It is a maximally symmetric and hyperquadric semi Euclidean space with index 2. Furthermore, current cosmological observations show that our expanding universe is an asymptotic. In this case, positive cosmological constant is used for the computation of Einstein equations in De-Sitter space. Therefore, it is meaningful to study in $\mathbb{E}_2^4$ since it has a key role to comprehend related concepts about general relativity, cosmology, and geometry [1].

A search of literature indicates that there is almost no concrete computations of entropy, laws of horizon dynamics and energy in the case of Minkowski spacetime $\mathbb{E}_2^4$. Various attempts are being made to describe the concept of the energy using quasi-local or local concepts. However, these definitions of energy do not agree with each other all the time and they are not applicable to the universes of De-Sitter and Anti De-Sitter type $[2-4]$. Therefore, we believe that we should start with using local approach to make any progress on the notion of energy in this spacetime. Thus, we consider that one of the most effective way to make this approach is to use intrinsic geometrical features of the moving particle in the Minkowski spacetime $\mathbb{E}_2^4$. To obtain these data and facts focusing on kinematics and dynamical aspect of the structure on the corresponding spacetime is crucial.

Motion of a particle in space is important due to wide range application of the subject. Motion of the particle in absolute space and time was defined firstly by Newtonian dynamics [5, 6]. Then, geometric generalization of the action, which includes terms belonging to curvature of the moving particle's trajectory in different space times are investigated [7].

The equations of the moving particle in the particular vector field are obtained by considering its generalized acceleration, velocity, and coordinate. Based on this, unit vector field's energy on a Riemannian manifold $M$ is described to be equal to the energy of the mapping $M \to T_1M$, where $T_1M$ is defined as unit tangent bundle equipped with Sasaki metric [8]. By similar argument volume of a unit vector field $X$ is described as the volume of the submanifold in the unit tangent bundle defined by $X(M)$ [9]. It is also investigated that computation of the energy of a particular particle in many spacetimes has various applications $[10-15]$.

This study organizes as follows. We firstly present fundamental definitions of Frenet frame equations for different type of space curves in Minkowski spacetime $\mathbb{E}_2^4$. Then we give geometrical interpretation of the energy for unit vector fields. In the following section, we set a connection between physical and geometrical understanding of the energy for a moving particle in Minkowski spacetime $\mathbb{E}_2^4$ considering dynamics of different type of space curves. Finally, we give some energy variation sketches for different cases.

## 2. Metric and Frenet Field in Minkowski Space $\mathbb{E}_2^4$

Let $\mathbf{y} = (y_1, y_2, y_3, y_4)$, $\mathbf{z} = (z_1, z_2, z_3, z_4)$ be two vectors in a 4-dimensional real vector space $\mathbb{R}^4$.

$$\mathbb{R}^4 = \{\mathbf{y} = (y_1, y_2, y_3, y_4) : y_i \in \mathbb{R} \ \ (i = 1, 2, 3, 4)\}.$$

Then pseudo scalar multiplication of $\mathbf{y}$ and $\mathbf{z}$ is defined as

$$\pi(\mathbf{y},\mathbf{z}) = -y_1z_1 - y_2z_2 + y_3z_3 + y_4z_4.$$

In this case, it is said that real vector space $\mathbb{R}^4$ with the above metric $\pi$ defines a new geometrical structure named as Minkowski 4-space with index 2. It is denoted by $\mathbb{E}_2^4$.

Let $\mathbf{y}$ be a vector in $\mathbb{E}_2^4 - \{0\}$. Then it has different characteristics with respect to the value of the given metric. That is,

$$\begin{cases} \mathbf{y} \text{ is spacelike, if } \pi(\mathbf{y},\mathbf{y}) \text{ is positive,} \\ \mathbf{y} \text{ is timelike, if } \pi(\mathbf{y},\mathbf{y}) \text{ is negative,} \\ \mathbf{y} \text{ is lightlike, if } \pi(\mathbf{y},\mathbf{y}) \text{ is zero.} \end{cases}$$

We can also describe new type of lightlike vectors such as pseudo lightlike vector, Cartan lightlike vector, and partially lightlike vector in case necessary conditions are held. These definitions can be extended to curves in $\mathbb{E}_2^4$, naturally. That is, let $\Gamma$ be a particle moving in a space $\mathbb{E}_2^4$ such that the precise location of the particle is specified by $\Gamma = \Gamma(t)$, where $t$ is a time parameter. Changing time parameter describes the motion. Hence, the trajectory of the particle corresponds to a curve $\mathcal{K}$ in the space for a moving particle. It is convenient to remind the arc-length parameter $s$ is used to compute the distance traveled by a particle along its trajectory. It is defined by

$$\frac{ds}{dt} = \|\mathbf{v}\|,$$

where $\mathbf{v} = \mathbf{v}(t) = \frac{d\zeta}{dt}$ is the velocity vector and $\frac{d\zeta}{dt} \neq 0$. In particle dynamics, the arc-length parameter $s$ is considered as a function of $t$. Thanks to the arc-length, it is also described Serret-Frenet frame, which allows us determining characterization of the intrinsic geometrical features of the regular curve. This coordinate system is constructed by four pseudo-orthonormal vectors assuming the curve is sufficiently smooth at each point. Frenet equations for different type of curves can be given as the following.

**Case 1.** Let $\mathcal{K}$ be a unit speed spacelike or timelike curve in $\mathbb{E}_2^4$, then Frenet equations are stated as the following [16].

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \sigma_2 k_1 \mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\sigma_1 k_1 \mathbf{T} + \sigma_3 k_2 \mathbf{B}_1, \\ \nabla_{\mathbf{T}}\mathbf{B}_1 &= -\sigma_2 k_2 \mathbf{N} + \sigma_1\sigma_2\sigma_3 k_3 \mathbf{B}_2, \\ \nabla_{\mathbf{T}}\mathbf{B}_2 &= \sigma_3 k_3 \mathbf{B}_1, \end{aligned} \tag{2.1}$$

where $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$ and only two of the $\sigma_i = 1$ $(i = 1,2,3,4)$. Moreover, we have

$$\pi(\mathbf{T},\mathbf{T}) = \sigma_1, \quad \pi(\mathbf{N},\mathbf{N}) = \sigma_2, \quad \pi(\mathbf{B}_1,\mathbf{B}_1) = \sigma_3, \quad \pi(\mathbf{B}_2,\mathbf{B}_2) = \sigma_4,$$

$$\pi(\mathbf{T},\mathbf{N}) = \pi(\mathbf{T},\mathbf{B}_1) = \pi(\mathbf{T},\mathbf{B}_2) = \pi(\mathbf{N},\mathbf{B}_1) = \pi(\mathbf{N},\mathbf{B}_2) = \pi(\mathbf{B}_1,\mathbf{B}_2) = 0.$$

**Case 2.** Let $\mathcal{K}$ be a unit speed pseudo lightlike curve, then Frenet equations are stated as the following [17].

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{N}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= k_2\mathbf{B}_1, \\
\nabla_{\mathbf{T}}\mathbf{B}_1 &= k_3\mathbf{N}-\sigma_2 k_2\mathbf{B}_2, \\
\nabla_{\mathbf{T}}\mathbf{B}_2 &= -\sigma_1 k_1\mathbf{T}-\sigma_2 k_3\mathbf{B}_1,
\end{aligned} \tag{2.2}$$

where $\sigma_1\sigma_2 = -\sigma_1,\ \sigma_2 \in \{-1, 1\}$. Here we also have

$$\begin{cases} k_1 = 0,\ \text{if } \mathcal{K} \text{ is straight line,} \\ \quad k_1 = 1,\ \text{otherwise.} \end{cases}$$

Moreover, we have

$$\pi(\mathbf{T},\mathbf{T}) = \sigma_1, \quad \pi(\mathbf{B}_1,\mathbf{B}_1) = \sigma_2, \quad \pi(\mathbf{N},\mathbf{B}_2) = 1,$$

$$\pi(\mathbf{T},\mathbf{N}) = \pi(\mathbf{T},\mathbf{B}_1) = \pi(\mathbf{T},\mathbf{B}_2) = \pi(\mathbf{N},\mathbf{B}_1) = \pi(\mathbf{B}_1,\mathbf{B}_2) = 0.$$

**Case 3.** Let $\mathcal{K}$ be a unit speed Cartan lightlike curve, then Frenet equations are stated as the following [18, 19].

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{N}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= -\sigma_1 k_2\mathbf{T}-\sigma_1 k_1\mathbf{B}_1, \\
\nabla_{\mathbf{T}}\mathbf{B}_1 &= k_2\mathbf{N}+k_3\mathbf{B}_2, \\
\nabla_{\mathbf{T}}\mathbf{B}_2 &= -\sigma_2 k_3\mathbf{T},
\end{aligned} \tag{2.3}$$

where $\sigma_1\sigma_2 = -\sigma_1,\ \sigma_2 \in \{-1, 1\}$. Here we also have

$$\begin{cases} k_1 = 0,\ \text{if } \mathcal{K} \text{ is straight line,} \\ \quad k_1 = 1,\ \text{otherwise.} \end{cases}$$

Moreover, we have

$$\pi(\mathbf{N},\mathbf{N}) = \sigma_1, \quad \pi(\mathbf{B}_2,\mathbf{B}_2) = \sigma_2, \quad \pi(\mathbf{T},\mathbf{B}_1) = 1,$$

$$\pi(\mathbf{T},\mathbf{N}) = \pi(\mathbf{T},\mathbf{B}_2) = \pi(\mathbf{N},\mathbf{B}_1) = \pi(\mathbf{N},\mathbf{B}_2) = \pi(\mathbf{B}_1,\mathbf{B}_2) = 0.$$

**Case 4.** Let $\mathcal{K}$ be a unit speed partially lightlike curve $\mathcal{K}$, then Frenet equations are stated as the following [17].

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{N}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= k_1\mathbf{T}+k_2\mathbf{B}_1, \\
\nabla_{\mathbf{T}}\mathbf{B}_1 &= k_3\mathbf{B}_1, \\
\nabla_{\mathbf{T}}\mathbf{B}_2 &= -\sigma_2 k_2\mathbf{N} - k_3\mathbf{B}_2,
\end{aligned} \tag{2.4}$$

where $\sigma_1\sigma_2 = -\sigma_1,\ \sigma_2 \in \{-1, 1\}$. Here we also have $k_3 = 0$, for every case. Moreover, we have

$$\pi(\mathbf{T},\mathbf{T}) = \sigma_1, \quad \pi(\mathbf{N},\mathbf{N}) = \sigma_2, \quad \pi(\mathbf{B}_1,\mathbf{B}_2) = 1,$$

$$\pi(\mathbf{T},\mathbf{N}) = \pi(\mathbf{T},\mathbf{B}_1) = \pi(\mathbf{T},\mathbf{B}_2) = \pi(\mathbf{N},\mathbf{B}_1) = \pi(\mathbf{N},\mathbf{B}_2) = 0.$$

## 3. Energy and Pseudo Angle of Unit Vector Fields

**Definition 3.1.** *Let $(M, \rho)$ and $(N, h)$ be two Riemannian manifolds. Then energy of a differentiable map $f : (M, \rho) \to (N, h)$ can be defined as*

$$\varepsilon nergy\,(f) = \frac{1}{2} \int_M \sum_{a=1}^{n} h\left(df\left(e_a\right), df\left(e_a\right)\right) v, \tag{3.1}$$

*where $\{e_a\}$ is a local basis of the tangent space and $v$ is the canonical volume form in $M$,* [8].

**Proposition 3.2.** *Let $Q : T\left(T^1M\right) \to T^1M$ be the connection map. Then following two conditions hold:*

***i)*** *$\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \tilde{\omega}$, where $\tilde{\omega} : T\left(T^1M\right) \to T^1M$ is the tangent bundle projection;*

***ii)*** *for $\varrho \in T_xM$ and a section $\xi : M \to T^1M$; we have*

$$Q\left(d\xi\left(\varrho\right)\right) = \nabla_\varrho \xi, \tag{3.2}$$

*where $\nabla$ is the Levi-Civita covariant derivative,* [8].

**Definition 3.3.** *Let $\varsigma_1, \varsigma_2 \in T_\xi\left(T^1M\right)$, then we define*

$$\rho_S\left(\varsigma_1, \varsigma_2\right) = \rho\left(d\omega\left(\varsigma_1\right), d\omega\left(\varsigma_2\right)\right) + \rho\left(Q\left(\varsigma_1\right), Q\left(\varsigma_2\right)\right). \tag{3.3}$$

*This yields a Riemannian metric on $TM$. As we know $\rho_S$ is called the Sasaki metric that also makes the projection $\omega : T^1M \to M$ a Riemannian submersion.*

**Definition 3.4.** *Pseudo angle is known as the angle between arbitrary Frenet vectors for any given curve $\mathcal{K}$. For an initial point the pseudo angle between Frenet vectors can be stated with the help of the curvature function of the curve $\mathcal{K}$ thanks to the following formula*

$$\mathcal{A}_i = \int_\vartheta^s \left\| \frac{dV_i}{du} \right\| du. \tag{3.4}$$

*where $V_i$ represents Frenet vector,* [20].

## 4. Energy of a Particle in Frenent Vector Fields in the Space $\mathbb{E}_2^4$

In the theory of relativity, all the energy moving through an object contributes to the total mass of the body that measures how much it can resist to acceleration. Each kinetic and potential energy makes a highly proportional contribution to the mass [21]. In this study not only we compute the energy and pseudo angle of Frenet vectors but we also investigate its close correlation with bending energy of elastica which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744. Euler elastica of bending energy formula for an elastic curve in Minkowski Space $\mathbb{E}_2^4$ is given by Frenet curvature along the curve

$$H_B = \frac{1}{2} \int \left\| \nabla_{\mathbf{T}} \mathbf{T} \right\|^2 ds, \tag{4.1}$$

where $s$ is an arclength and $\kappa^2 = \|\nabla_{\mathbf{T}}\mathbf{T}\|^2$ [22].
**Case 1**. Let $\mathcal{K}$ be a spacelike or timelike curve in $\mathbb{E}_2^4$.

**Theorem 4.1.** *Let $\Gamma$ be a moving particle such that it corresponds to a spacelike or timelike curve $\mathcal{K}$ in $E_2^4$. Then energy on the particle in tangent, normal, first binormal, and second binormal vector field is stated by using Sasaki metric as the following, respectively.*

$$\begin{aligned} \varepsilon nergy\,(\mathbf{T}) &= \frac{1}{2}(s\sigma_1 + \int_0^s \sigma_2 k_1^2 ds), \\ \varepsilon nergy\,(\mathbf{N}) &= \frac{1}{2}(s\sigma_1 + \int_0^s \left(\sigma_1 k_1^2 + \sigma_3 k_2^2\right) ds), \\ \varepsilon nergy\,(\mathbf{B}_1) &= \frac{1}{2}(s\sigma_1 + \int_0^s \left(\sigma_2 k_2^2 + \sigma_4 k_3^2\right) ds), \\ \varepsilon nergy\,(\mathbf{B}_2) &= \frac{1}{2}(s\sigma_1 + \int_0^s \sigma_3 k_3^2 ds). \end{aligned} \tag{4.2}$$

**Proof:** We prove only energy on the particle in tangent vector field. The rest of the proof can be completed by using the similar argument. From Eqs. 3.1 and 3.2, we know

$$\varepsilon nergy\,(\mathbf{T}) = \frac{1}{2}\int_0^s \rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) ds.$$

Using the Eq. 3.3, we have

$$\rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) = \rho(d\omega(\mathbf{T}(\mathbf{T})), d\omega(\mathbf{T}(\mathbf{T}))) + \rho(Q(\mathbf{T}(\mathbf{T})), Q(\mathbf{T}(\mathbf{T}))).$$

Since $\mathbf{T}$ is a section, we get

$$d(\omega) \circ d(\mathbf{T}) = d(\omega \circ \mathbf{T}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathbf{T}(\mathbf{T})) = \nabla_{\mathbf{T}}\mathbf{T} = \sigma_2 k_1 \mathbf{N}.$$

Thus, we find from the Eq. 2.1

$$\begin{aligned} \rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) &= \rho\left(\mathbf{T}, \mathbf{T}\right) + \rho\left(\nabla_{\mathbf{T}}\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right) \\ &= \sigma_1 + k_1^2 \sigma_2. \end{aligned}$$

So we can easily obtain

$$\varepsilon nergy\,(\mathbf{T}) = \frac{1}{2}(s\sigma_1 + \int_0^s \sigma_2 k_1^2 ds).$$

This completes the proof. □

**Corollary 4.2.** *Let energy on the particle* $\Gamma$ *in each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_1^2 = -\sigma_1\sigma_2, \quad \tfrac{k_2^2}{1+k_1^2} = -\sigma_1\sigma_3, \quad \sigma_2 k_2^2 + \sigma_4 k_3^2 = -\sigma_1, \quad k_3^2 = -\sigma_1\sigma_3.$$

**Proof:** Let $\varepsilon nergy\,(\mathbf{T})$ be a constant. Then we have $(\varepsilon nergy\,(\mathbf{T}))' = 0$. Using the equality given at the Eq. 4.2, we obtain

$$\sigma_1 + \sigma_2 k_1^2 = 0.$$

Thus, we get $k_1^2 = -\sigma_1\sigma_2$. The rest of the proof can be done similarly. □

**Theorem 4.3.** *Pseudo angle between tangent, normal, first binormal, and second binormal vector fields can be given as the following, respectively.*

$$\begin{aligned} \mathcal{A}_1 &= \int_0^s \left|\sigma_2 k_1\left(\gamma\left(u\right)\right)\right| du, \\ \mathcal{A}_2 &= \int_0^s \sqrt{\left|\sigma_1 k_1^2\left(\gamma\left(u\right)\right) + \sigma_3 k_2^2\left(\gamma\left(u\right)\right)\right|} du, \\ \mathcal{A}_3 &= \int_0^s \sqrt{\left|\sigma_2 k_2^2\left(\gamma\left(u\right)\right) + \sigma_4 k_3^2\left(\gamma\left(u\right)\right)\right|} du, \\ \mathcal{A}_4 &= \int_0^s \left|\sigma_3 k_3\left(\gamma\left(u\right)\right)\right| du. \end{aligned} \tag{4.3}$$

**Proof:** If we use Eqs. 2.1 and 3.4, it is obvious. □

**Corollary 4.4.** *Let pseudo angle of each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_1 = 0, \quad \tfrac{k_2^2}{k_1^2} = -\sigma_1\sigma_3, \quad \tfrac{k_3^2}{k_2^2} = -\sigma_2\sigma_4, \quad k_3 = 0.$$

**Proof:** Let $\mathcal{A}_1$ be a constant. Then we have $(\mathcal{A}_1)' = 0$. Using the equality given at the Eq. 4.3, we obtain

$$|\sigma_2 k_1| = 0.$$

Thus, we get $k_1 = 0$. The rest of the proof can be done similarly. □

**Remark 4.5.** *Let* $\mathcal{K}$ *be a spacelike or timelike curve in* $E_2^4$ *such that it has a bending feature of elastica.Then we have following relations between energy on the particle in Frenet vector fields and bending energy functional of elastica in* $E_2^4$.

$$\begin{aligned} H_B &= \varepsilon nergy\,(\mathbf{T}) - \frac{1}{2} s\sigma_1, \\ H_B &= \varepsilon nergy\,(\mathbf{N}) - \frac{1}{2}\left(s\sigma_1 + \int_0^s \sigma_3 k_2^2 ds\right). \end{aligned}$$

**Proof:** It is obvious from Eqs. 4.1 and 4.2. □

**Case 2**. Let $\mathcal{K}$ be a pseudo lightlike curve in $\mathbb{E}_2^4$.

**Theorem 4.6.** *Let* $\Gamma$ *be a moving particle such that it corresponds to a pseudo lightlike curve* $\mathcal{K}$ *in* $E_2^4$*. Then energy on the particle in tangent, normal, first binormal, and second binormal vector field is stated by using Sasaki metric as the following, respectively.*

$$\begin{aligned} \varepsilon nergy\,(\mathbf{T}) &= \frac{\sigma_1}{2}s, \\ \varepsilon nergy\,(\mathbf{N}) &= \frac{1}{2}(\sigma_1 s + \int_0^s \sigma_2 k_2^2 ds), \\ \varepsilon nergy\,(\mathbf{B}_1) &= \frac{1}{2}(\sigma_1 s - 2\int_0^s \sigma_2 k_2 k_3 ds), \\ \varepsilon nergy\,(\mathbf{B}_2) &= \frac{1}{2}(\sigma_1 s + \int_0^s \left(\sigma_1 k_1^2 + \sigma_2 k_3^2\right) ds). \end{aligned} \tag{4.4}$$

**Proof:** Here, we prove only energy on the particle in tangent vector field. The rest of the proof can be completed by using similar argument. From Eqs. 3.1 and 3.2 we know that

$$\varepsilon nergy\,(\mathbf{T}) = \frac{1}{2}\int_0^s \rho_S\,(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T}))\,ds.$$

Then using the Eq. 3.3 and knowing $\mathbf{T}$ is a section, we obtain

$$\rho_S\,(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})) = \rho(d\omega(\mathbf{T}(\mathbf{T})), d\omega(\mathbf{T}(\mathbf{T}))) + \rho(Q(\mathbf{T}(\mathbf{T})), Q(\mathbf{T}(\mathbf{T}))),$$

and

$$d(\omega)\circ d(\mathbf{T}) = d(\omega\circ\mathbf{T}) = d(id_C) = id_{TC}.$$

It is also true that

$$Q(\mathbf{T}(\mathbf{T})) = \bigtriangledown_{\mathbf{T}}\mathbf{T} = k_1\mathbf{N}.$$

Moreover, we find from the Eq. 2.2

$$\begin{aligned} \rho_S\,(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})) &= \rho\,(\mathbf{T},\mathbf{T}) + \rho\,(\nabla_{\mathbf{T}}\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}) \\ &= \sigma_1. \end{aligned}$$

Thus, we can easily obtain

$$\varepsilon nergy\,(\mathbf{T}) = \frac{\sigma_1}{2}s.$$

□

**Corollary 4.7.** *Let energy on the particle* $\Gamma$ *in each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_2 = k_3 = \sigma_1 = 0.$$

**Proof:** Let $\varepsilon nergy\,(\mathbf{T})$ be a constant. Then we have $(\varepsilon nergy\,(\mathbf{T}))' = 0$. Using the equality given at the Eq. 4.4, we obtain

$$\frac{\sigma_1}{2} = 0.$$

Thus, we get $0 = \sigma_1$. Also we know from the assumption $(\varepsilon nergy\,(\mathbf{N}))' = 0$. Then we have

$$0 = \sigma_1 + \sigma_2 k_2^2.$$

From the Eq. 2.2, we know that $\sigma_2 \in \{-1, 1\}$. Thus $k_2 = 0$. Similarly,

$$(\varepsilon nergy\,(\mathbf{B}_1))' = 0$$

implies that

$$\sigma_1 + \sigma_1 k_1^2 + \sigma_2 k_3^2 = 0.$$

Hence, $k_3 = 0$. This completes the proof. □

**Theorem 4.8.** *Pseudo angle between tangent, normal, first binormal, and second binormal vector fields can be given as the following, respectively.*

$$\begin{aligned} \mathcal{A}_1 &= 0, \\ \mathcal{A}_2 &= \int_0^s |\sigma_2 k_2 (\gamma(u))|\, du, \\ \mathcal{A}_3 &= \int_0^s \sqrt{|2\sigma_2 k_2 k_3 (\gamma(u))|}\, du, \\ \mathcal{A}_4 &= \int_0^s \sqrt{|(\sigma_1 k_1^2 (\gamma(u)) + \sigma_2 k_3^2 (\gamma(u)))|}\, du. \end{aligned} \tag{4.5}$$

**Proof:** If we use Eqs. 2.2 and 3.4, then it is obvious. □

**Corollary 4.9.** *Let pseudo angle of each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_2 = 0, \quad \frac{k_3^2}{k_1^2} = \sigma_1.$$

**Proof:** Let $\mathcal{A}_2$ be a constant. Then we have $(\mathcal{A}_2)' = 0$. Using the equality given at the Eq. 4.5, we obtain

$$\sigma_2 k_2 = 0.$$

Thus, we get $k_2 = 0$. Similarly, we know that $(\mathcal{A}_4)' = 0$. Thus, it implies that

$$\left|\left(\sigma_1 k_1^2 + \sigma_2 k_3^2\right)\right| = 0.$$

Here, if we also use the Eq. 2.2 we obtain $\frac{k_3^2}{k_1^2} = \sigma_1$. □

**Case 3.** Let $\mathcal{K}$ be a Cartan lightlike curve in $\mathbb{E}_2^4$.

**Theorem 4.10.** *Let $\Gamma$ be a moving particle such that it corresponds to a Cartan lightlike curve $\mathcal{K}$ in $E_2^4$. Then energy on the particle in tangent, normal, first binormal, and second binormal vector field is stated by using Sasaki metric as the following, respectively.*

$$\begin{aligned} \varepsilon nergy\,(\mathbf{T}) &= \frac{1}{2}\int_0^s \sigma_1 k_1^2 ds, \\ \varepsilon nergy\,(\mathbf{N}) &= \int_0^s k_1 k_2 ds, \\ \varepsilon nergy\,(\mathbf{B}_1) &= \frac{1}{2}\int_0^s \left(\sigma_1 k_2^2 + \sigma_1 k_3^2\right) ds, \\ \varepsilon nergy\,(\mathbf{B}_2) &= 0. \end{aligned} \tag{4.6}$$

**Proof:** Here, we prove only energy on the particle in tangent vector field. The rest of the proof can be completed by using similar argument. From Eqs. 3.1 and 3.2 we know

$$\varepsilon nergy\,(\mathbf{T}) = \frac{1}{2}\int_0^s \rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) ds.$$

By using the Eq. 3.3, we have

$$\rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) = \rho(d\omega(\mathbf{T}(\mathbf{T})), d\omega(\mathbf{T}(\mathbf{T}))) + \rho(Q(\mathbf{T}(\mathbf{T})), Q(\mathbf{T}(\mathbf{T}))).$$

Since $\mathbf{T}$ is a section, we also get

$$d(\omega) \circ d(\mathbf{T}) = d(\omega \circ \mathbf{T}) = d(id_C) = id_{TC}.$$

Moreover, it is clear that

$$Q(\mathbf{T}(\mathbf{T})) = \nabla_{\mathbf{T}}\mathbf{T} = k_1 \mathbf{N}.$$

Thus, we find from the Eq. 2.3

$$\begin{aligned} \rho_S\left(d\mathbf{T}(\mathbf{T}), d\mathbf{T}(\mathbf{T})\right) &= \rho\left(\mathbf{T}, \mathbf{T}\right) + \rho\left(\nabla_{\mathbf{T}}\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right) \\ &= k_1^2 \sigma_1. \end{aligned}$$

and finally

$$\varepsilon nergy\,(\mathbf{T}) = \frac{1}{2}\int_0^s \sigma_1 k_1^2 ds.$$

□

**Corollary 4.11.** *Let energy on the particle $\Gamma$ in each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$\sigma_1 k_1^2 = 0, \quad k_1 k_2 = 0, \quad \sigma_1 (k_2^2 + k_3^2) = 0.$$

**Proof:** It is obvious from Eqs. 4.6 and 2.3. □

**Theorem 4.12.** *Pseudo angle between tangent, normal, first binormal, and second binormal vector fields can be given as the following, respectively.*

$$\begin{aligned}
\mathcal{A}_1 &= \int_0^s |\sigma_1 k_1(\gamma(u))|\, du, \\
\mathcal{A}_2 &= \int_0^s \sqrt{|k_1 k_2(\gamma(u))|}du, \\
\mathcal{A}_3 &= \int_0^s \sqrt{|(\sigma_1 k_2^2(\gamma(u)) + \sigma_1 k_3^2(\gamma(u)))|}du, \\
\mathcal{A}_4 &= 0.
\end{aligned} \tag{4.7}$$

**Proof:** If we use Eqs. 2.3 and 4.1, then it is obvious. □

**Corollary 4.13.** *Let pseudo angle of each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$\sigma_1 k_1 = 0, \quad k_1 k_2 = 0, \quad \sigma_1(k_2^2 + k_3^2) = 0.$$

**Proof:** If we use the Eq. 4.7, then it is obvious. □

**Remark 4.14.** *Let $\mathcal{K}$ be a Cartan lightlike curve in $E_2^4$ such that it has a bending feature of elastica.Then we have following relations between energy on the particle in Frenet vector fields and bending energy functional of elastica in $E_2^4$.*

$$\begin{aligned}
H_B &= \varepsilon nergy(\mathbf{T}), \\
H_B &= \varepsilon nergy(\mathbf{B}_1) - \frac{1}{2}\int_0^s \sigma_1 k_3^2 ds.
\end{aligned}$$

**Proof:** It is obvious from Eqs. 4.1 and 4.6. □

**Case 4.** Let $\mathcal{K}$ be a partially lightlike curve in $\mathbb{E}_2^4$.

**Theorem 4.15.** *Let $\Gamma$ be a moving particle such that it corresponds to a partially lightlike curve $\mathcal{K}$ in $E_2^4$. Then energy on the particle in tangent, normal, first binormal, and second binormal vector field is stated by using Sasaki metric as the following, respectively.*

$$\begin{aligned}
\varepsilon nergy(\mathbf{T}) &= \frac{1}{2}(\sigma_1 s + \int_0^s \sigma_2 k_1^2 ds), \\
\varepsilon nergy(\mathbf{N}) &= \frac{1}{2}(\sigma_1 s + \int_0^s \sigma_1 k_1^2 ds), \\
\varepsilon nergy(\mathbf{B}_1) &= \frac{\sigma_1}{2} s, \\
\varepsilon nergy(\mathbf{B}_2) &= \frac{1}{2}(\sigma_1 s + \int_0^s \sigma_2 k_2^2 ds).
\end{aligned} \tag{4.8}$$

**Proof:** We prove only energy of tangent vector, here. Energy of other vectors can be computed if the same method is followed. From Eqs. 3.1 and 3.2 we get

$$\varepsilon nergy\left(\mathbf{T}\right)=\frac{1}{2}\int_{0}^{s}\rho_{S}\left(d\mathbf{T}(\mathbf{T}),d\mathbf{T}(\mathbf{T})\right)ds.$$

By using the Eq. 3.3 we have

$$\rho_{S}\left(d\mathbf{T}(\mathbf{T}),d\mathbf{T}(\mathbf{T})\right)=\rho(d\omega(\mathbf{T}(\mathbf{T})),d\omega(\mathbf{T}(\mathbf{T})))+\rho(Q(\mathbf{T}(\mathbf{T})),Q(\mathbf{T}(\mathbf{T}))).$$

Since $\mathbf{T}$ is a section, we also get

$$d(\omega)\circ d(\mathbf{T})=d(\omega\circ\mathbf{T})=d(id_{C})=id_{TC}.$$

Moreover, it is clear that

$$Q(\mathbf{T}(\mathbf{T}))=\bigtriangledown_{\mathbf{T}}\mathbf{T}=k_{1}\mathbf{N}.$$

Thus, we find from the Eq. 2.4

$$\begin{aligned}\rho_{S}\left(d\mathbf{T}(\mathbf{T}),d\mathbf{T}(\mathbf{T})\right)&=\rho\left(\mathbf{T},\mathbf{T}\right)+\rho\left(\nabla_{\mathbf{T}}\mathbf{T},\nabla_{\mathbf{T}}\mathbf{T}\right)\\&=\sigma_{1}+k_{1}^{2}\sigma_{2}.\end{aligned}$$

and finally

$$\varepsilon nergy\left(\mathbf{T}\right)=\frac{1}{2}(\sigma_{1}s+\int_{0}^{s}\sigma_{2}k_{1}^{2}ds).$$

□

**Corollary 4.16.** *Let energy on the particle $\Gamma$ in each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_{1}=k_{2}=0.$$

**Proof:** Let $\varepsilon nergy\left(\mathbf{B}_{1}\right)$ be a constant. Then we have $(\varepsilon nergy\left(\mathbf{B}_{1}\right))'=0$. Using the equality given at the Eq. 4.8 we obtain

$$\frac{\sigma_{1}}{2}=0.$$

Thus, we get $\sigma_{1}=0$. Also we know from the assumption $(\varepsilon nergy\left(\mathbf{T}\right))'=0$. Then we have

$$0=\sigma_{1}+\sigma_{2}k_{1}^{2}.$$

From the Eq. 2.4, we know that $\sigma_{2}\in\{-1,1\}$. Thus $k_{1}=0$. Similarly,

$$(\varepsilon nergy\left(\mathbf{B2}\right))'=0$$

implies that

$$\sigma_{1}+\sigma_{2}k_{1}^{2}=0.$$

Hence, $k_{2}=0$. This completes the proof. Having constant energy in Frenet vector fields also implies that the particle follow a trajectory of a straight line, since $k_{1}=0$.

□

**Theorem 4.17.** *Pseudo angle between tangent, normal, first binormal, and second binormal vector fields can be given as the following, respectively.*

$$\begin{aligned} \mathcal{A}_1 &= \int_0^s |\sigma_2 k_1 (\gamma(u))| \, du, \\ \mathcal{A}_2 &= \int_0^s |\sigma_1 k_1 (\gamma(u))| \, du, \\ \mathcal{A}_3 &= 0, \\ \mathcal{A}_4 &= \int_0^s |\sigma_2 k_2 (\gamma(u))| \, du. \end{aligned} \tag{4.9}$$

**Proof:** If we use Eqs. 2.4 and 3.4, then it is obvious. □

**Corollary 4.18.** *Let pseudo angle of each tangent, normal, first binormal, and second binormal vector fields given above be constant, then we have following relations, respectively.*

$$k_1 = k_2 = 0.$$

**Proof:** It is obvious from the Eq. 4.9. □

## 5. Application

In this section, we draw energy variation graph for different type of curves given in $\mathbb{E}_2^4$ spacetime. By doing this practice, we have a chance to observe differentiation of the energy on the particle with respect to time and different curves. For the simplicity and convenience, we choose

$$\sigma_1 = 1, \ \sigma_2 = -1, \ \sigma_3 = -1, \ \sigma_4 = 1$$

and $k_1(s) = s, \ k_2(s) = \sin s$, and $k_3(s) = \sinh s$.

**Case 1**. Let $\Gamma$ be a moving particle such that it corresponds to a spacelike or timelike curve in $\mathbb{E}_2^4$ for given values. In this case, we find energy on the particle in Frenet vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ as follows, respectively.

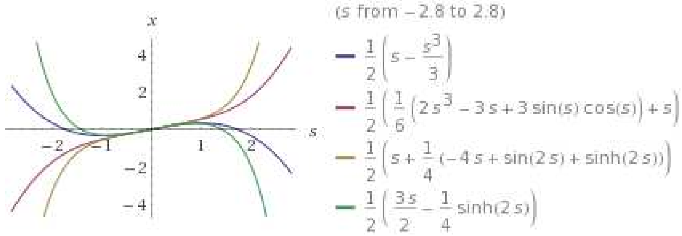


Figure 1: Energy of a spacelike or a timelike curve in Frenet vector fields

**Case 2**. Let $\Gamma$ be a moving particle such that it corresponds to a pseudo lightlike curve in $\mathbb{E}_2^4$ for given values. In this case, we find energy on the particle in Frenet vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ as follows, respectively.

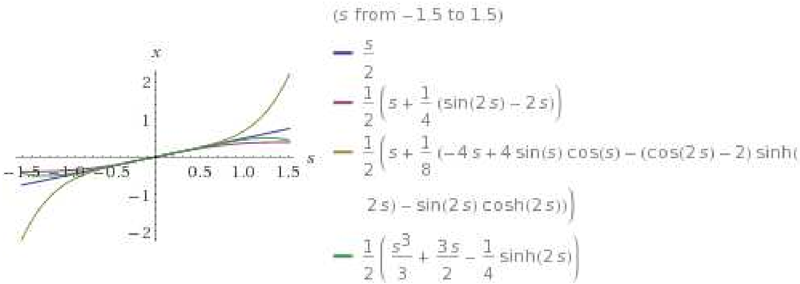


Figure 2: Energy of a pseudo lightlike curve in Frenet vector fields

**Case 3**. Let $\Gamma$ be a moving particle such that it corresponds to a Cartan lightlike curve in $\mathbb{E}_2^4$ for given values. In this case, we find energy on the particle in Frenet vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ as follows, respectively.

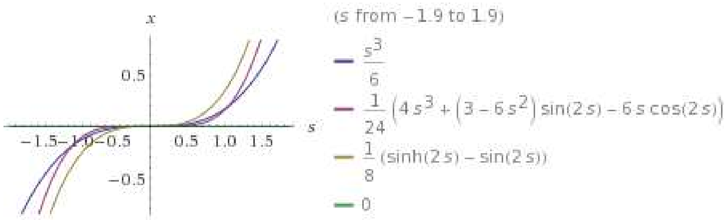


Figure 3: Energy of a Cartan lightlike curve in Frenet vector fields

**Case 4**. Let $\Gamma$ be a moving particle such that it corresponds to a partially lightlike curve in $\mathbb{E}_2^4$ for given values. In this case, we find energy on the particle in Frenet vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ as follows, respectively.

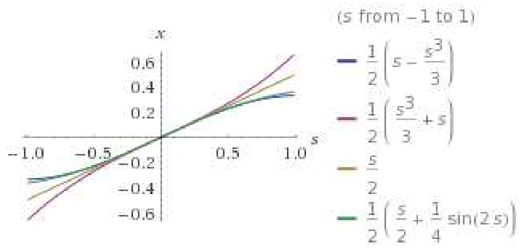


Figure 4: Energy of a partially lightlike curve in Frenet vector fields

## 6. Conclusion

In this study, we studied energy on the particle in the Frenet vector fields in 4-dimensional Minkowski spacetime with index 2. Furthermore, we set a connection between energy on the particle in these vector fields and elastica of bending energy functional. This is important for our future work since a simple characterization on the energy of a vector field can be described as it is up to constants, in other words, it is square $L_2$ norm of the vector field's covariant derivative. Thanks to this definition, we correlate the concept of the energy with a volume for the moving particle in these vector fields in space.

As is known, elastic energy may occur by applying different forces besides bending such as twisting and stretching. In our next studies, we also aim to determine the correlation between energy on the particle in each Frenet vector fields and stretching, twisting energy functional.

Computing the energy on the moving particle has a wide range of application in the theoretical and applied physics. Therefore, it will also be investigated the energy on the moving particle in different force fields thanks to the dynamics of the particle in space including work done and force acting on the particle. We believe that this study will also lead up to further research on the relativistic dynamics of the particle in different spacetimes in terms of computing the energy on a particle in different force fields.

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