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Coefficient Estimates for a New Subclass of Analytic and Bi-Univalent Functions by Hadamard Product

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Abstract: In this work, we introduce a new subclass of bi-univalent functions which is defined by Hadamard product and subordination in the open unit disk and find upper bounds for the second and third coefficients for functions in this new subclass. Further, we generalize and improve some of the previously published results.

Key Words: Bi-univalent functions, Deniz-Orhan operator, Hadamard product, Subordination.

Contents

1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. However, for each $f \in \mathcal{S}$, the Koebe one-quarter theorem [11] ensures that the image of $\mathbb{U}$ under f contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

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where $g = f^{-1}$ and

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

In 2010, Deniz and Orhan [10] introduced linear multiplier differential operator $D^m(\nu,\varphi)f$ defined by

$$\begin{aligned}
D^0(\nu,\varphi)f(z) &= f(z),\\
D^1(\nu,\varphi)f(z) &= D(\nu,\varphi)f(z) = \nu\varphi z^2(f(z))'' + (\nu-\varphi)z(f(z))'\\
&\quad +(1-\nu+\varphi)f(z),\\
D^2(\nu,\varphi)f(z) &= D(\nu,\varphi)(D^1(\nu,\varphi)f(z)),\\
&\ \vdots\\
D^m(\nu,\varphi)f(z) &= D(\nu,\varphi)(D^{m-1}(\nu,\lambda)f(z)),
\end{aligned}$$

where $\nu \geq \varphi \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If f is given by (1.1), then from the definition of the operator $D^m(\nu,\varphi)f(z)$ it is easy to see that

$$D^m(\nu,\varphi)f(z) = z + \sum_{n=2}^{\infty} \Phi_n^m(\nu,\varphi)a_nz^n, \tag{1.3}$$

where $\Phi_n^m(\nu,\varphi) = [1 + (\nu\varphi n + \nu - \varphi)(n-1)]^m$.

Determination of coefficient estimates $|a_n|$ $(n \in \mathbb{N})$ is an important problem in geometric function theory as they give information about the geometric properties of these functions. Recently, many researchers introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, see, for example, [3,5,6,7,13,14,15,16,17,23,24,27]. But The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2,3\})$ for each $f \in \Sigma$ is still an open problem.

In this work, we introduce a new subclass of bi-univalent functions which is defined by Hadamard product and find upper bounds for the second and third coefficients for functions in this new subclass. Besides, the estimates on the coefficients $|a_2|$ and $|a_3|$ presented in this work would generalize and improve some of results of Aouf et al. [4], Bulut [8], Çağlar et al. [9], El-Ashwah [12], Frasin and Aouf [14], Murugusundaramoorthy [18], Orhan et al. [19], Porwal and Darus [20], Prema and Keerthi [21], Srivastava et al. [24], Srivastava et al. [25] and related works in this literature.

2. Preliminaries

In this section, we recall some definitions and lemmas that used in this work.

Definition 2.1. *For $f(z)$ defined by (1.1) and $\Theta(z)$ defined by*

$$\Theta(z) = z + \sum_{n=2}^{\infty} c_nz^n, \tag{2.1}$$

the Hadamard product $(f * \Theta)(z)$ *of the functions* $f(z)$ *and* $\Theta(z)$ *defined by*

$$(f * \Theta)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n.$$

Definition 2.2 ([11])**.** *An analytic function* f *is said to be subordinate to another analytic function* g*, written as*

$$f(z) \prec g(z) \ (z \in \mathbb{U}),$$

if there exists a Schwarz function w*, which is analytic in* $\mathbb{U}$ *with*

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in \mathbb{U}),$$

such that $f(z) = g(w(z))$.

In particular, if the function g is univalent in $\mathbb{U}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Lemma 2.3 ([11,22])**.** *Let the function* $\phi(z)$ *given by* $\phi(z) = z + \sum\limits_{n=2}^{\infty} B_n z^n$*, be convex in* $\mathbb{U}$*. Suppose that the function* $h(z)$ *given by* $h(z) = z + \sum\limits_{n=2}^{\infty} h_n z^n$*, is holomorphic in* $\mathbb{U}$*. If* $h(z) \prec \phi(z)$*,* $(z \in \mathbb{U})$ *then* $|h_n| \leq |B_1|$*,* $(n \in \mathbb{N})$*.*

Now, we generalize and extend the Deniz-Orhan operator as follow.
For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $\Theta(z) = z + \sum_{n=2}^{\infty} c_n z^n$ and $\Omega(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product $(\Omega * (f * \Theta))(z)$ of the functions $f(z)$ and $\Theta(z)$, $\Omega(z)$ defined by

$$(\Omega * (f * \Theta))(z) = z + \sum_{n=2}^{\infty} a_n b_n c_n z^n.$$

We define operator $D^m(\nu, \varphi)(\Omega * (f * \Theta))(z)$ as follows:

$$D^m(\nu, \varphi)(\Omega * (f * \Theta))(z) = z + \sum_{n=2}^{\infty} \Phi_n^m(\nu, \varphi) a_n b_n c_n z^n, \tag{2.2}$$

where $\Phi_n^m(\nu, \varphi)$ is as (1.3).

By taking special cases for the parameters $\Omega(z)$ and $\Theta(z)$, the operator

$$D^m(\nu, \varphi)(\Omega * (f * \Theta))(z)$$

reduce to the well known differential operators.

3. Main results

In this section, by using the operator $D^m(\nu,\varphi)(\Omega*(f*\Theta))$ which defined by (2.2), we introduce the following class of functions.

Definition 3.1. *Let $h: U\to\mathbb{C}$ be a convex univalent function such that*

$$h(0)=1,\ h(\bar{z})=\overline{h(z)}\ (z\in\mathbb{U};\ \Re(h(z))>0).$$

A function $f\in\Sigma$ defined by (1.1), is said to be in the class $\mathcal{NP}_\Sigma^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ with $\Theta\in\Sigma$ if the following conditions are satisfied:

$$e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu}+\lambda\left(D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)\right)'\right.$$
$$\left.\times\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu-1}\right]\prec h(z)\cos\beta+i\sin\beta, \tag{3.1}$$

and

$$e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu}\right. \tag{3.2}$$
$$\left.+\lambda\left(D^m(\nu,\varphi)\Pi(w)\right)'\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu-1}\right]\prec h(w)\cos\beta+i\sin\beta,$$

where $\Pi=(\Omega*(f*\Theta)^{-1})$, $z,w\in\mathbb{U}$, $\lambda\geq 1$, $\mu\geq 0$, $\beta\in(-\pi/2,\pi/2)$ and the functions $\Omega*(f*\Theta)$ and $\Omega*(f*\Theta)^{-1}$ are given

$$(\Omega*(f*\Theta))(z)=z+\sum_{n=2}^{\infty}a_nb_nc_nz^n\ (c_n,b_n>0),$$

and

$$\Pi(w)=(\Omega*(f*\Theta)^{-1})(w)\ =\ w-a_2b_2c_2w^2+b_3(2a_2^2c_2^2-a_3c_3)w^3$$
$$-b_4(5a_2^3c_2^3-5a_2a_3c_2c_3+a_4c_4)w^4+\cdots,$$

respectively.

By specializing the parameters of $\mathcal{NP}_\Sigma^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ we get the following subclasses.

Remark 3.2. *If we set*

$$h(z)=\frac{1+Az}{1+Bz},\ -1\leq B<A\leq 1,$$

in Definition 3.1, then the class $\mathcal{NP}_\Sigma^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ reduces to the class denoted by $\mathcal{NP}_\Sigma^{\lambda,\mu}(m,\beta,\nu,\varphi,A,B;\Theta,\Omega)$ which is defined as $f\in\Sigma$,

$$e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu}+\lambda\left(D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)\right)'\right.$$
$$\left.\times\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu-1}\right]\prec\frac{1+Az}{1+Bz}\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\Bigg[(1-\lambda)\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu}$$
$$+\lambda\left(D^m(\nu,\varphi)\Pi(w)\right)'\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu-1}\Bigg] \prec \frac{1+Aw}{1+Bw}\cos\beta+i\sin\beta.$$

Remark 3.3. *If we set*

$$h(z)=\frac{1+(1-2\gamma)z}{1-z},\;\; 0\le\gamma<1,$$

in Definition 3.1, then the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduces to the class denoted by* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi,\gamma;\Theta,\Omega)$ *which is defined as* $f\in\Sigma$*,*

$$\mathfrak{Re}\Bigg\{e^{i\beta}\Bigg[(1-\lambda)\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu}+\lambda\left(D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)\right)'$$
$$\times\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu-1}\Bigg]\Bigg\}>\gamma\cos\beta,$$

and

$$\mathfrak{Re}\Bigg\{e^{i\beta}\Bigg[(1-\lambda)\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu}$$
$$+\lambda\left(D^m(\nu,\varphi)\Pi(w)\right)'\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu-1}\Bigg]\Bigg\}>\gamma\cos\beta.$$

Remark 3.4. *If we set* $\mu=1$ *and* $\Theta(z)=\Omega(z)=\frac{z}{1-z}$ *in Definition 3.1, then the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduces to the class denoted by* $\mathcal{NP}_{\Sigma}^{\lambda}(m,\beta,\nu,\varphi;h)$ *which is defined as* $f\in\Sigma$*,*

$$e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)f(z)}{z}\right)+\lambda\left(D^m(\nu,\varphi)f(z)\right)'\right]\prec h(z)\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)g(w)}{w}\right)+\lambda\left(D^m(\nu,\varphi)g(w)\right)'\right]\prec h(w)\cos\beta+i\sin\beta.$$

Remark 3.5. *If we set* $\lambda=1$ *and* $\Theta(z)=\Omega(z)=\frac{z}{1-z}$ *in Definition 3.1, then the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduces to the class denoted by* $\mathcal{NP}_{\Sigma}^{\mu}(m,\beta,\nu,\varphi;h)$ *which is defined as* $f\in\Sigma$*,*

$$e^{i\beta}\left[\left(D^m(\nu,\varphi)f(z)\right)'\left(\frac{D^m(\nu,\varphi)f(z)}{z}\right)^{\mu-1}\right]\prec h(z)\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\left[\left(D^m(\nu,\varphi)g(w)\right)'\left(\frac{D^m(\nu,\varphi)g(w)}{w}\right)^{\mu-1}\right]\prec h(w)\cos\beta+i\sin\beta.$$

Remark 3.6. *If we set* $m=0,\ \Omega(z)=\frac{z}{1-z}$ *and*

$$h(z)=\frac{1+(1-2\gamma)z}{1-z},\ \ 0\le\gamma<1,$$

in Definition 3.1, then the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduces to the class denoted by* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(\beta,\gamma;\Theta)$ *which is defined as* $f\in\Sigma$,

$$\mathfrak{Re}\left[e^{i\beta}(1-\lambda)\left(\frac{(f*\Theta)(z)}{z}\right)^{\mu}+\lambda\left((f*\Theta)(z)\right)'\left(\frac{(f*\Theta)(z)}{z}\right)^{\mu-1}\right]>\gamma\cos\beta,$$

and

$$\mathfrak{Re}\left[e^{i\beta}(1-\lambda)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu}+\lambda\left((f*\Theta)^{-1}(w)\right)'\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu-1}\right]>\gamma\cos\beta.$$

Remark 3.7. *For* $m=0$ *and* $\Omega(z)=\Theta(z)=\frac{z}{1-z}$ *the class*

$$\mathcal{NP}_{\Sigma}^{\lambda,\mu}(n,\beta,m,\nu,\varphi;h,\Theta,\Omega)$$

reduce to a class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(\beta;h)$ *which defined by Orhan et al. [19, Definition 1.1].*

Remark 3.8. *For* $m=\beta=0,\ \mu=\lambda=1,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(n,\beta,m,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a class* $\mathcal{H}_{\Sigma}(\gamma)$ *which defined by Srivastava et al. [24, Definition 2].*

Remark 3.9. *For* $m=\beta=0,\ \mu=1,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a class* $\mathcal{B}_{\Sigma}(\gamma,\lambda)$ *which defined by Frasin and Aouf [14, Definition 3.1].*

Remark 3.10. *For* $m=\beta=0,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a class* $\mathcal{N}_{\Sigma}^{\mu}(\gamma,\lambda)$ *which defined by Çağlar et al. [9, Definition 3.1].*

Remark 3.11. *For* $m=\beta=0,\ \lambda=1,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a class* $P_{\Sigma}(\gamma,\lambda)$ *which defined by Prema and Keerthi [21, Definition 3.1].*

Remark 3.12. *For* $m=\beta=0,\ \lambda-1=\mu=0,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a well-known class* $\mathcal{S}_{\Sigma}^{*}(\gamma)$ *of bi-starlike functions of order* γ *which defined by Brannan and Taha [5].*

Remark 3.13. *For* $m=\beta=0,\ \mu=1,\ \Omega(z)=\frac{z}{1-z}$,

$$\Theta(z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_q)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}}\frac{z^n}{n!}=z+\sum_{n=2}^{\infty}\Gamma_{n-1}[a_1;b_1]z^n$$

and $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$, *the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to a class* $\mathcal{T}_{q,s}^{\Sigma}(a_1,b_1,\gamma,\lambda)$ *which defined by Aouf et al. [4, Definition 6].*

Remark 3.14. *For $m = \beta = 0$, $\mu = 1$, $\Omega(z) = \frac{z}{1-z}$ and $h(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$, the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$ reduce to a class $\mathcal{B}_{\Sigma}(\Theta, \gamma, \lambda)$ which defined by El-Ashwah [12, Definition 2].*

Remark 3.15. *For $\nu = 1$, $\varphi = \beta = 0$, $\mu = 1$, $\Omega(z) = \Theta(z) = \frac{z}{1-z}$ and $h(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$. Now if $f \in \mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$, then*

$$\mathfrak{Re}\left((1-\lambda)\frac{D^m f(z)}{z} + \lambda(D^m f(z))'\right) = \mathfrak{Re}\left(\frac{(1-\lambda)D^m f(z) + \lambda D^{m+1} f(z)}{z}\right) > \gamma,$$

and

$$\mathfrak{Re}\left(\frac{(1-\lambda)D^m g(w) + \lambda D^{m+1} g(w)}{w}\right) > \gamma,$$

where D^m is the Sălăgean differential operator and the function g is defined by (1.2). Therefore in this case, the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$ reduce to class $\mathcal{H}_{\Sigma}(m, \gamma, \lambda)$ which defined by Porwal and Darus [20, Definition 2.1].

Remark 3.16. *For $m = 0$, $\lambda = 1$, $\Theta(z) = \frac{z}{1-z}$ and $\Omega(z) = z + \sum_{n=2}^{\infty} C_n^c(b, \omega, k) z^n = z + \sum_{n=2}^{\infty} \Psi_n z^n$. Now if $f \in \mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$, then*

$$e^{i\beta}\left(\frac{z^{1-\mu}(\mathcal{J}_{\omega,b}^{c,k} f(z))'}{[\mathcal{J}_{\omega,b}^{c,k} f(z)]^{1-\mu}}\right) \prec h(z)\cos\beta + i\sin\beta,$$

and

$$e^{i\beta}\left(\frac{w^{1-\mu}(\mathcal{J}_{\omega,b}^{c,k} g(w))'}{[\mathcal{J}_{\omega,b}^{c,k} g(w)]^{1-\mu}}\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $\mathcal{J}_{\omega,b}^{c,k}$ is Murugusundaramoorthy generalized integral operator and the function g is defined by (1.2). Therefore in this case, the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$ reduce to class $\mathcal{B}_{\Sigma,\omega,b}^{c,k,\beta}(\mu, h)$ which defined by Murugusundaramoorthy [18, Definition 1.1].

Remark 3.17. *For $m = 0$, $\lambda = 1$, $\Theta(z) = \frac{z}{1-z}$ and $\Omega(z) = z + \sum_{n=2}^{\infty} \Lambda_n z^n$. Now if $f \in \mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$. Then*

$$e^{i\beta}\left(\frac{z^{1-\mu}(\mathcal{I}_{a,b;c} f(z))'}{[\mathcal{I}_{a,b;c} f(z)]^{1-\mu}}\right) \prec h(z)\cos\beta + i\sin\beta,$$

and

$$e^{i\beta}\left(\frac{w^{1-\mu}(\mathcal{I}_{a,b;c} g(w))'}{[\mathcal{I}_{a,b;c} g(w)]^{1-\mu}}\right) \prec h(w)\cos\beta + i\sin\beta,$$

where $\mathcal{I}_{a,b;c}$ is the Hohlov operator and the function g is defined by (1.2). Therefore in this case, the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m, \beta, \nu, \varphi; h, \Theta, \Omega)$ reduce to class $\mathcal{B}_{\Sigma}^{a,b;c}(\beta, \mu; h)$ which defined by Srivastava et al. [25, Definition].

Remark 3.18. *For* $m=0,\ \mu=1,\ \Theta(z)=\frac{z}{1-z}$ *and* $\Omega(z)=z+\sum_{k=2}^{\infty}\varpi_k z^k$. *Now if* $f\in\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$, *then*

$$e^{i\beta}\left[(1-\lambda)\frac{\ell_{(\alpha_j),\ (\beta_j)}f(z)}{z}+\lambda(\ell_{(\alpha_j),\ (\beta_j)}f(z))'\right]\prec h(z)\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\left[(1-\lambda)\frac{\ell_{(\alpha_j),\ (\beta_j)}g(w)}{w}+\lambda(\ell_{(\alpha_j),\ (\beta_j)}g(w))'\right]\prec h(w)\cos\beta+i\sin\beta,$$

where $\ell_{(\alpha_j),\ (\beta_j)}$ *is the Alamoush-Darus operator and the function* g *is defined by* (1.2). *Therefore in this case, the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to class* $\hbar\ell_{\Sigma}^{(\alpha_j),\ (\beta_j)}(k,\beta;h)$ *which defined by Alamoush and Darus [2, Definition].*

Remark 3.19. *For* $\varphi=\mu-1=0,\ \Omega(z)=\Theta(z)=\frac{z}{1-z}$ *if*

$$f\in\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega),$$

then

$$e^{i\beta}\left[(1-\lambda)\frac{D_{\delta}^{m}f(z)}{z}+\lambda(D_{\delta}^{m}f(z))'\right]\prec h(z)\cos\beta+i\sin\beta,$$

and

$$e^{i\beta}\left[(1-\lambda)\frac{D_{\delta}^{m}g(w)}{w}+\lambda(D_{\delta}^{m}g(w))'\right]\prec h(w)\cos\beta+i\sin\beta,$$

where D_{δ}^{n} *is the Al-Oboudi differential operator and the function* g *is defined by* (1.2). *Therefore in this case, the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to class* $\mathcal{NP}_{\Sigma}^{\lambda,\delta}(m,\beta;h)$ *which defined by Bulut [8, Definition].*

Remark 3.20. *For* $\nu=1,\ \varphi=\beta=0,\ \mu=1,\ \Theta(z)=\frac{z}{1-z},\ \Omega(z)=\phi(\delta+1,1;z)=z+\sum_{n=2}^{\infty}C(\delta,n)a_n z^n$, *where* $C(\delta,n)=\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+1)}$ *and* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, *where* $0\le\gamma<1$. *Now if* $f\in\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$. *Then*

$$\mathfrak{Re}\left((1-\lambda)\frac{D_{n,\delta}^{m}f(z)}{z}+\lambda(D_{n,\delta}^{m}f(z))'\right)=\mathfrak{Re}\left(\frac{(1-\lambda)D_{n,\delta}^{m}f(z)+\lambda D_{n,\delta}^{m+1}f(z)}{z}\right)>\gamma$$

and

$$\mathfrak{Re}\left(\frac{(1-\lambda)D_{n,\delta}^{m}g(w)+\lambda D_{n,\delta}^{m+1}g(w)}{w}\right)>\gamma,$$

where $D_{n,\delta}^{m}$ *is is differential operator Alamoush-Darus and the function* g *is defined by* (1.2). *Therefore in this case, the class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *reduce to class* $\mathcal{H}_{\Sigma}(n,\delta,\gamma,\lambda)$ *which defined by Alamoush and Darus [1, Definition 2].*

Remark 3.21. *By specializing the parameters of* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *and setting* $h(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, *(or* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, $0\le\gamma<1$*), we get subclasses defined by Murugusundaramoorthy [18, Examples 1.1 to 1.7] and [18, Remark 1.8].*

Remark 3.22. *By specializing the parameters of* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *and setting* $h(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, *(or* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, $0\le\gamma<1$*), we get subclasses defined by Srivastava et al. [25, Examples 1 to 6] and [25, Remark 1].*

Remark 3.23. *By specializing the parameters of* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *and setting* $h(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, *(or* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, $0\le\gamma<1$*), we get subclasses defined by Bulut [8, Remark 2 to 6].*

Remark 3.24. *By specializing the parameters of* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$ *and setting* $h(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, *(or* $h(z)=\frac{1+(1-2\gamma)z}{1-z}$, $0\le\gamma<1$*), we get subclasses defined by Alamoush and Darus [2, Remark 1.1 to 1.4].*

Remark 3.25. *The coefficients estimate obtained by Alamoush and Darus [2, Theorem 2.1] did not true and we correct them as following:*

$$|a_2|\le\sqrt{\frac{|B_1|\cos\beta}{(2\lambda+1)\varpi_3}},\quad |a_3|\le\frac{|B_1|^2\cos^2\beta}{(1+2\lambda)^2\varpi_2^2}+\frac{|B_1|\cos\beta}{(1+2\lambda)\varpi_3}$$

Remark 3.26. *The coefficients estimate obtained by Alamoush and Darus [1, Theorem 4] did not true and we correct them as following:*

$$|a_2|\le\sqrt{\frac{\Gamma(\delta+1)}{\Gamma(\delta+3)}\frac{4(1-\beta)}{[(1-\lambda)3^k+\lambda 3^{k+1}]}},$$

and

$$|a_3|\le\frac{\Gamma(\delta+1)}{\Gamma(\delta+3)}\frac{4(1-\beta)}{(1-\lambda)3^k+\lambda 3^{k+1}}+\left[\frac{\Gamma(\delta+1)}{\Gamma(\delta+2)}\right]^2\frac{4(1-\beta)^2}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2}.$$

Remark 3.27. *The coefficients estimate obtained by Orhan et al. [19, Theorem 2.1] did not true for* $|a_3|$ *and we correct them as following:*

$$|a_3|\le\begin{cases}\frac{2|B_1|\cos\beta}{(2\lambda+\mu)(1+\mu)}, & 0\le\mu<1\\[2ex] \frac{|B_1|\cos\beta}{(2\lambda+\mu)}, & \mu\ge 1.\end{cases}$$

Now, we obtain the following estimates for class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$.

Theorem 3.28. *Let the function* $f(z)$ *given given by the Taylor-Maclaurin series expansion (1.1) be in the function class* $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$, $\lambda\ge 1$, $\mu\ge 0$ *and* $\beta\in(-\pi/2,\pi/2)$ *with* $\Theta\in\Sigma$ *and*

$$\Omega(z)=z+\sum_{n=2}^{\infty}b_nz^n \text{ and } \Theta(z)=z+\sum_{n=2}^{\infty}c_nz^n\ (b_n,c_n>0). \tag{3.3}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots. \tag{3.4}$$

Then

$$|a_2| \leq \frac{1}{c_2} \min \left\{ \frac{|B_1| \cos\beta}{(\lambda+\mu)\Phi_2^m(\nu,\varphi) b_2}, \sqrt{\frac{2|B_1|\cos\beta}{|(\mu-1)(\Phi_2^m(\nu,\varphi))^2 b_2^2 + 2b_3\Phi_3^m(\nu,\varphi)|(2\lambda+\mu)}} \right\} \tag{3.5}$$

and

$$|a_3| \leq \begin{cases} \min\left\{ \frac{|B_1|\cos\beta}{c_3(2\lambda+\mu)} \frac{|4b_3\Phi_3^m(\nu,\varphi)+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2|+(1-\mu)b_2^2(\Phi_2^m(\nu,\varphi))^2}{2b_3\Phi_3^m(\nu,\varphi)\left|(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)\right|}, \right. & \\ \left. \frac{|B_1|^2\cos^2\beta}{(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2 b_2^2 c_3} + \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3} \right\}, & 0 \leq \mu < 1 \\ \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3}, & \mu \geq 1 \end{cases} \tag{3.6}$$

Proof: It follows from (3.1) and (3.2) that

$$\begin{aligned} & e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu}\right. \\ & \left. +\lambda\,(D^m(\nu,\varphi)(\Omega*(f*\Theta))(z))'\left(\frac{D^m(\nu,\varphi)(\Omega*(f*\Theta))(z)}{z}\right)^{\mu-1}\right] \\ = \;& p(z)\cos\beta + i\sin\beta \;\; (z\in\mathbb{U}), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & e^{i\beta}\left[(1-\lambda)\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu}\right. \\ & \left. +\lambda\,(D^m(\nu,\varphi)\Pi(w))'\left(\frac{D^m(\nu,\varphi)\Pi(w)}{w}\right)^{\mu-1}\right] = q(w)\cos\beta + i\sin\beta \;\; (w\in\mathbb{U}), \end{aligned} \tag{3.8}$$

where the function

$$p(z) \prec h(z) \;\; (z\in\mathbb{U}) \; \textit{and} \; q(w) \prec h(w) \;\; (w\in\mathbb{U}),$$

have the following Taylor-Maclaurin series expansions:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \tag{3.9}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots. \tag{3.10}$$

Using (3.9) and (3.10) in (3.7) and (3.8), respectively, we obtain

$$e^{i\beta}(\lambda+\mu)\Phi_2^m(\nu,\varphi)a_2b_2c_2 = p_1\cos\beta, \tag{3.11}$$

$$\begin{aligned} & e^{i\beta}[(2\lambda+\mu)\Phi_3^m(\nu,\varphi)a_3b_3c_3 \\ & +\frac{(\mu-1)(2\lambda+\mu)}{2}[\Phi_2^m(\nu,\varphi)]^2 a_2^2 b_2^2 c_2^2] = p_2\cos\beta \end{aligned} \tag{3.12}$$

$$-e^{i\beta}(\lambda+\mu)\Phi_2^m(\nu,\varphi)a_2b_2c_2 = q_1\cos\beta, \tag{3.13}$$

and

$$\begin{aligned} & e^{i\beta}[(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3(2a_2^2c_2^2-a_3c_3)+\frac{(\mu-1)(2\lambda+\mu)}{2}[\Phi_2^m(\nu,\varphi)]^2a_2^2b_2^2c_2^2] \\ = \; & q_2\cos\beta. \end{aligned} \tag{3.14}$$

From (3.11) and (3.13), we get that

$$p_1=-q_1, \tag{3.15}$$

and

$$2e^{2i\beta}(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2a_2^2b_2^2c_2^2=(p_1^2+q_1^2)\cos^2\beta. \tag{3.16}$$

Adding (3.12) and (3.14), we obtain

$$a_2^2=\frac{e^{-i\beta}(p_2+q_2)\cos\beta}{[(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)]\,(2\lambda+\mu)c_2^2}. \tag{3.17}$$

Since, by definition, $p(z),\ q(w)\in h(\mathbb{U})$, according Lemma 2.3 we get that

$$|p_n|=\left|\frac{p^{(n)}(0)}{n!}\right|\le|B_1|\ (n\in\mathbb{N}) \tag{3.18}$$

and

$$|q_n|=\left|\frac{q^{(n)}(0)}{q!}\right|\le|B_1|\ (n\in\mathbb{N}). \tag{3.19}$$

Using (3.18), (3.19) and Lemma 2.3 for the coefficients $p_1,\ p_2,q_1$ and q_2 from the equalities (3.16) and (3.17), we immediately have

$$|a_2|^2\le\frac{|B_1|^2\cos^2\beta}{(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2b_2^2c_2^2}$$

and

$$|a_2|^2\le\frac{2|B_1|\cos\beta}{|(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)|\,(2\lambda+\mu)c_2^2},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (3.5).

Finally, in order to find the bound on the coefficient $|a_3|$, we subtract (3.14) from (3.12). We get

$$\begin{aligned} & 2(2\lambda+\mu)\Phi_3^m(\nu,\varphi)a_3b_3c_3-2a_2^2c_2^2b_3(2\lambda+\mu)\Phi_3^m(\nu,\varphi) \\ = \; & e^{-i\beta}(p_2-q_2)\cos\beta. \end{aligned} \tag{3.20}$$

Upon substituting the value of a_2^2 from (3.16) into (3.20), it follows that

$$a_3 = \frac{e^{-2i\beta}(p_1^2+q_1^2)\cos^2\beta}{2(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2 b_2^2 c_3} + \frac{e^{-i\beta}(p_2-q_2)\cos\beta}{2(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3}.$$

By (3.18), (3.19) and Lemma 2.3, we have

$$|a_3| \le \frac{|B_1|^2\cos^2\beta}{(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2 b_2^2 c_3} + \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3}. \tag{3.21}$$

On the other hand, upon substituting the value of a_2^2 from (3.17) into (3.20), it follows that

$$a_3 = \frac{e^{-i\beta}(p_2+q_2)\cos\beta}{[(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)]\,(2\lambda+\mu)c_3} + \frac{e^{-i\beta}(p_2-q_2)\cos\beta}{2(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3}.$$

Consequently,

$$a_3 = \frac{e^{-i\beta}\cos\beta}{c_3(2\lambda+\mu)}\,\frac{[4b_3\Phi_3^m(\nu,\varphi)+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2]p_2-(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2q_2}{2b_3\Phi_3^m(\nu,\varphi)\,[(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)]}.$$

By (3.18), (3.19) and Lemma 2.3, we have

$$|a_3| \le \frac{|B_1|\cos\beta}{c_3(2\lambda+\mu)}\,\frac{|4b_3\Phi_3^m(\nu,\varphi)+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2|+|\mu-1|b_2^2(\Phi_2^m(\nu,\varphi))^2}{2b_3\Phi_3^m(\nu,\varphi)\,|(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)|}.$$

Now, we obtain the bound on $|a_3|$ according to μ from the above inequality

Case 1. We suppose that let $0 \le \mu < 1$ thus we have

$$|a_3| \le \frac{|B_1|\cos\beta}{c_3(2\lambda+\mu)}\,\frac{|4b_3\Phi_3^m(\nu,\varphi)+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2|+(1-\mu)b_2^2(\Phi_2^m(\nu,\varphi))^2}{2b_3\Phi_3^m(\nu,\varphi)\,|(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)|}.$$

Case 2. We let $\mu \ge 1$ thus we have

$$\begin{aligned} |a_3| &\le \frac{|B_1|\cos\beta}{c_3(2\lambda+\mu)}\,\frac{4b_3\Phi_3^m(\nu,\varphi)+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2}{2b_3\Phi_3^m(\nu,\varphi)\,[(\mu-1)b_2^2(\Phi_2^m(\nu,\varphi))^2+2b_3\Phi_3^m(\nu,\varphi)]} \\ &= \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)b_3c_3}, \end{aligned}$$

which is the second part of assertion (3.21).
So, from (3.21) and two above case, we obtain the desired estimate on $|a_3|$ given in (3.6). This completes the proof. □

4. Corollaries and Consequences

By setting $m = 0$ and $\Omega(z) = \Theta(z) = \frac{z}{1-z}$ in Theorem 3.28, we obtain the following result which is an improvement of the estimates obtained for $|a_2|$ by Orhan et al. [19, Theorem 2.1].

Corollary 4.1. *Let the function f be given by (1.1) in the class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(\beta;h)$. Then*

$$|a_2| \leq \min\left\{\frac{|B_1|\cos\beta}{(\lambda+\mu)}, \sqrt{\frac{2|B_1|\cos\beta}{(\mu+1)(2\lambda+\mu)}}\right\}.$$

By setting $m=0,\ \lambda=1,\ \Theta(z)=\frac{z}{1-z}$ and $\Omega(z)=z+\sum_{n=2}^{\infty}\Psi_n z^n$ in Theorem 3.28, we obtain the following result which is an improvement of the estimates obtained by Murugusundaramoorthy [18, Theorem 2.1].

Corollary 4.2. *Let the function f be given by (1.1) in the class $\mathcal{B}_{\Sigma,\omega,b}^{c,k,\beta}(\mu,h)$. Then*

$$|a_2| \leq \min\left\{\frac{|B_1|\cos\beta}{(1+\mu)\Psi_2}, \sqrt{\frac{2|B_1|\cos\beta}{|(\mu-1)(2+\mu)\Psi_2^2+2(2+\mu)\Psi_3|}}\right\},$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{|B_1|\cos\beta}{(2\lambda+\mu)}\frac{|4\Psi_3+(\mu-1)\Psi_2^2|+(1-\mu)\Psi_2^2}{2\Psi_3\left|(\mu-1)\Psi_2^2+2\Psi_3\right|},\right. \\ \left.\frac{|B_1|^2\cos^2\beta}{(\lambda+\mu)^2\Psi_2^2}+\frac{|B_1|\cos\beta}{(2\lambda+\mu)\Psi_3}\right\}, & 0\leq\mu<1 \\ \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Psi_3}, & \mu\geq 1. \end{cases}$$

By setting $m=0,\ \lambda=1,\ \Theta(z)=\frac{z}{1-z}$ and $\Theta(z)=z+\sum_{n=2}^{\infty}\Lambda_n z^n$ in Theorem 3.28, we obtain the following result which is an improvement of the estimates obtained by Srivastava et al. [25, Theorem 1].

Corollary 4.3. *Let the function f be given by (1.1) in the class $\mathcal{B}_{\Sigma}^{a,b;c}(\beta,\mu;h)$. Then*

$$|a_2| \leq \min\left\{\frac{|B_1|\cos\beta}{(1+\mu)\Lambda_2}, \sqrt{\frac{2|B_1|\cos\beta}{|(\mu-1)(2+\mu)\Lambda_2^2+2(2+\mu)\Lambda_3|}}\right\},$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{|B_1|\cos\beta}{(2\lambda+\mu)}\frac{|4\Lambda_3+(\mu-1)\Lambda_2^2|+(1-\mu)\Lambda_2^2}{2\Lambda_3\left|(\mu-1)\Lambda_2^2+2\Lambda_3\right|},\right. \\ \left.\frac{|B_1|^2\cos^2\beta}{(\lambda+\mu)^2\Lambda_2^2}+\frac{|B_1|\cos\beta}{(2\lambda+\mu)\Lambda_3}\right\}, & 0\leq\mu<1 \\ \frac{|B_1|\cos\beta}{(2\lambda+\mu)\Lambda_3}, & \mu\geq 1. \end{cases}$$

By setting $\varphi=0,\ \mu=1$ and $\Omega(z)=\Theta(z)=\frac{z}{1-z}$ in Theorem 3.28, we obtain the following result which is the estimates obtained by Bulut [8, Theorem 8].

Corollary 4.4. *Let the function f be given by (1.1) in the class $\mathcal{NP}_{\Sigma}^{\lambda,\delta}(m,\beta;h)$. Then*

$$|a_2| \leq \min\left\{\frac{|B_1|\cos\beta}{(1+\lambda)(1+\nu)^m}, \sqrt{\frac{|B_1|\cos\beta}{(2\lambda+1)(1+2\nu)^m}}\right\},$$

and

$$|a_3| \leq \frac{|B_1|\cos\beta}{(2\lambda+1)(1+2\nu)^m}.$$

By letting $\Omega(z) = \frac{z}{1-z}$ and

$$h(z) = \frac{1+(1-2\gamma)z}{1-z} \quad (0 \leq \gamma < 1,\ z \in \mathbb{U}),$$

in Theorem 3.28, we have the following corollary.

Corollary 4.5. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathcal{NP}_{\Sigma}^{\lambda,\mu}(m,\beta,\nu,\varphi;h,\Theta,\Omega)$, $\lambda \geq 1$, $\mu \geq 0$ and $\beta \in (-\pi/2,\pi/2)$ with (3.3) and (3.4). Then*

$$|a_2| \leq \frac{1}{c_2}\min\left\{\frac{2(1-\gamma)\cos\beta}{(\lambda+\mu)\Phi_2^m(\nu,\varphi)}, \sqrt{\frac{4(1-\gamma)\cos\beta}{|(\mu-1)(\Phi_2^m(\nu,\varphi))^2+2\Phi_3^m(\nu,\varphi)|\,(2\lambda+\mu)}}\right\},$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{2(1-\gamma)\cos\beta}{c_3(2\lambda+\mu)}\frac{|4\Phi_3^m(\nu,\varphi)+(\mu-1)(\Phi_2^m(\nu,\varphi))^2|+(1-\mu)(\Phi_2^m(\nu,\varphi))^2}{2\Phi_3^m(\nu,\varphi)\left|(\mu-1)(\Phi_2^m(\nu,\varphi))^2+2\Phi_3^m(\nu,\varphi)\right|},\right. \\ \left.\frac{4(1-\gamma)^2\cos^2\beta}{(\lambda+\mu)^2[\Phi_2^m(\nu,\varphi)]^2c_3} + \frac{2(1-\gamma)\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)c_3}\right\}, & 0 \leq \mu < 1 \\ \frac{2(1-\gamma)\cos\beta}{(2\lambda+\mu)\Phi_3^m(\nu,\varphi)c_3}, & \mu \geq 1. \end{cases}$$

By setting $m = \beta = 0$ and $\Theta(z) = \frac{z}{1-z}$ in Corollary 4.5, we obtain the following result which is the estimates obtained for $|a_3|$ by Çağlar et al. [9, Theorem 3.2].

Corollary 4.6. *Let the function f be given by (1.1) in the class $\mathcal{N}_{\Sigma}^{\mu}(\gamma,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\gamma)}{\lambda+\mu}, \sqrt{\frac{4(1-\gamma)}{(\mu+1)(2\lambda+\mu)}}\right\},$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\gamma)}{(2\lambda+\mu)(\mu+1)}, \frac{4(1-\gamma)^2}{(\lambda+\mu)^2} + \frac{2(1-\gamma)}{(2\lambda+\mu)}\right\}, & 0 \leq \mu < 1 \\ \frac{2(1-\gamma)}{(2\lambda+\mu)}, & \mu \geq 1. \end{cases}$$

By setting $m = \beta = 0,\ \mu = 1$ in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by El-Ashwah [12, Theorem 2].

Corollary 4.7. *Let the function f be given by (1.1) in the class $\mathcal{T}^{\Sigma}_{q,s}(a_1, b_1, \gamma, \lambda)$. Then*

$$|a_2| \le \frac{1}{c_2} \min\left\{ \frac{2(1-\gamma)}{\lambda+1}, \sqrt{\frac{2(1-\gamma)}{(2\lambda+1)}} \right\},$$

and

$$|a_3| \le \frac{2(1-\gamma)}{(2\lambda+1)c_3}.$$

By setting $m = \beta = 0,\ \mu = 1$ and $\Theta(z) = z + \sum_{n=2}^{\infty} \Gamma_{n-1}[a_1; b_1] z^n$ in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by Aouf et al. [4, Theorem 8].

Corollary 4.8. *Let the function f be given by (1.1) in the class $\mathcal{B}_{\Sigma}(\Theta, \gamma, \lambda)$. Then*

$$|a_2| \le \frac{1}{|\Gamma_1[a_1; b_1]|} \min\left\{ \frac{2(1-\gamma)}{\lambda+1}, \sqrt{\frac{2(1-\gamma)}{(2\lambda+1)}} \right\},$$

and

$$|a_3| \le \frac{2(1-\gamma)}{(2\lambda+1)|\Gamma_2[a_1; b_1]|}.$$

By setting $\nu = 1,\ \varphi = \beta = 0,\ \mu = 1,\ \Theta(z) = \frac{z}{1-z}$. in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by Porwal and Darus [20, Theorem 3.1].

Corollary 4.9. *Let the function f be given by (1.1) in the class $\mathcal{H}_{\Sigma}(m, \gamma, \lambda)$. Then*

$$|a_2| \le \min\left\{ \frac{2(1-\gamma)}{2^m(\lambda+1)}, \sqrt{\frac{2(1-\gamma)}{3^m(2\lambda+1)}} \right\},$$

and

$$|a_3| \le \frac{2(1-\gamma)}{(2\lambda+1)3^m}.$$

By setting $m = \beta = 0,\ \mu = 1$ and $\Theta(z) = \frac{z}{1-z}$ in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by Frasin and Aouf [14, Theorem 3.2].

Corollary 4.10. *Let the function f be given by (1.1) in the class $\mathcal{B}_\Sigma(\gamma,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\gamma)}{\lambda+1}, \sqrt{\frac{2(1-\gamma)}{(2\lambda+1)}}\right\},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{2\lambda+1}.$$

By setting $m=\beta=0,\ \mu=\lambda=1\ \Theta(z)=\frac{z}{1-z}$ in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by Srivastava et al. [24, Theorem 2].

Corollary 4.11. *Let the function f be given by (1.1) in the class $H_\Sigma(\gamma)$. Then*

$$|a_2| \leq \min\left\{1-\gamma, \sqrt{\frac{2(1-\gamma)}{3}}\right\},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{3}.$$

By setting $m=\beta=0,\ \lambda=1$ and $\Omega(z)=\Theta(z)=\frac{z}{1-z}$ in Corollary 4.5, we obtain the following result which is an improvement of the estimates obtained by Prema and Keerthi [21, Theorem 3.2].

Corollary 4.12. *Let the function f be given by (1.1) in the class $P_\Sigma(\gamma,\lambda)$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\gamma)}{(1+\mu)}, \sqrt{\frac{4(1-\gamma)}{(\mu+1)(2+\mu)}}\right\},$$

and

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\gamma)}{(2+\mu)(\mu+1)}, \frac{4(1-\gamma)^2}{(1+\mu)^2}+\frac{2(1-\gamma)}{(2+\mu)}\right\}, & 0\leq\mu<1 \\ \frac{2(1-\gamma)}{(2+\mu)}, & \mu\geq 1. \end{cases}$$

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