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On Spaces of Periodic Functions with Wavelet Transforms

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ABSTRACT: Some boundedness results for the wavelet transform on $F_p([0,1]^n)$ and $F_p^*([0,1]^n)$, the spaces of periodic test functions, are obtained. The wavelet transform is also studied on generalized Sobolev space $B_p^\kappa([0,1]^n)$.

Key Words: Periodic functions, Wavelet transform, Sobolev space.

Contents

1. Introduction

In this paper, we have studied the wavelet transform, used by A. I. Zayed [11], of periodic functions on $\mathbb{R}^n$. Let us recall some definitions and properties which have been used in this paper.

Definition 1.1. *A function f defined on $\mathbb{R}^n$ is termed as periodic (or $\mathbb{Z}^n$-periodic) if $f(t) = f(t+k)$ for all $k \in \mathbb{Z}^n$.*

Definition 1.2 (The Fourier transform)**.** *The Fourier transform $\mathscr{F}(f) = \hat{f}$ of a function f on $\mathbb{R}^n$ is defined by*

$$\hat{f}(\xi) = (\mathscr{F}f)(\xi) = \int_{\mathbb{R}^n} f(t)\, e^{-i2\pi\langle t,\xi\rangle} dt,$$

provided the integral exists.

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Theorem 1.3. *[1, p.-12] The k^{th} Fourier coefficient of a $\mathbb{Z}^n$-periodic function $f \in L^2([0,1]^n)$ is given by*

$$a_k = \hat{f}(k) = \int_{[0,1]^n} f(t)\, e^{-i2\pi\langle t,k\rangle} dt,$$

and f can be represented into the Fourier series as

$$f(t) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, e^{i2\pi\langle t,k\rangle}. \tag{1.1}$$

Furthermore,

$$\int_{[0,1]^n} |f(t)|^2\, dt = \|f\|^2_{L^2([0,1]^n)} = \sum_{k\in\mathbb{Z}^n} \left|\hat{f}(k)\right|^2.$$

Definition 1.4. *In this paper, the periodization operator T^{per} is defined by*

$$T^{per}(f) = \tilde{f}(t) := \sum_{k\in\mathbb{Z}} f(t+k),$$

provided the series is convergent. The dilation operator D_a, for any function ψ, is defined as

$$D_a\psi(t) := \frac{1}{(a_1a_2a_3\cdots a_n)}\, \psi\left(\frac{t_1}{a_1}, \frac{t_2}{a_2}, \frac{t_3}{a_3}, \cdots, \frac{t_n}{a_n}\right),$$

where $a = (a_1, a_2, a_3, \cdots, a_n)$, $a_i > 0$, $i = 1, 2, 3, \cdots, n$, and the mother wavelet is defined by

$$\psi_{a,b}(t) := \frac{1}{(a_1a_2a_3\cdots a_n)}\, \psi\left(\frac{t_1-b_1}{a_1}, \frac{t_2-b_2}{a_2}, \frac{t_3-b_3}{a_3}, \cdots, \frac{t_n-b_n}{a_n}\right), \tag{1.2}$$

and

$$\tilde{\psi}_{a,b}(t) = \sum_{k\in\mathbb{Z}^n} \psi_{a,b}(t+k), \tag{1.3}$$

whenever the series is uniformly convergent.

Definition 1.5. *If ψ is a wavelet such that the series (1.3) is uniformly convergent, then the continuous wavelet transform [3, p.-28] of a $\mathbb{Z}^n$-periodic function $f \in L^2([0,1]^n)$ with respect to $\tilde{\psi}$ is given by*

$$(W_{\tilde{\psi}} f)(a,b) := \int_{[0,1]^n} f(t)\, \overline{\tilde{\psi}_{a,b}(t)}\, dt, \;\; a \in \mathbb{R}^n_+ := (0,\infty)^n \text{ and } b \in [0,1]^n. \tag{1.4}$$

Thus, we have

$$
\begin{aligned}
(W_{\tilde{\psi}}f)(a,b) &= \int_{[0,1]^n} f(t)\,\overline{\tilde{\psi}_{a,b}(t)}\,dt \\
&= \int_{[0,1]^n} f(t) \sum_{k\in\mathbb{Z}^n} \overline{\psi_{a,b}(t+k)}\,dt \\
&= \sum_{k\in\mathbb{Z}^n} \int_{[0,1]^n} f(t)\,\overline{\psi_{a,b}(t+k)}\,dt \\
&\quad \textit{(series is uniformaly convergent)} \\
&= \sum_{k\in\mathbb{Z}^n} \int_{|t|\le 1} f(t) \\
&\qquad \times \frac{1}{(a_1a_2a_3\cdots a_n)}\,\overline{\psi\left(\frac{t_1+k_1-b_1}{a_1}, \frac{t_2+k_2-b_2}{a_2}, \cdots, \frac{t_n+k_n-b_n}{a_n}\right)}\,dt \\
&= \sum_{k\in\mathbb{Z}^n} \int_{|a\cdot y+b-k|\le 1} f(a\cdot y+b-k)\,\overline{\psi\,(y)}\,dy \\
&= \int_{\mathbb{R}^n} f(a\cdot y+b)\,\overline{\psi\,(y)}\,dy,
\end{aligned}
$$

where $a\cdot y$ *is defined as* $a\cdot y := (a_1y_1, a_2y_2, \cdots, a_ny_n)$, *and hence by using equation* (1.1), *we get*

$$
\begin{aligned}
(W_{\tilde{\psi}}f)(a,b) &= \int_{\mathbb{R}^n} \overline{\psi\,(y)} \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\,e^{i2\pi\langle a\cdot y+b,k\rangle}dy \\
&= \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\,e^{i2\pi\langle b,k\rangle} \int_{\mathbb{R}^n} \overline{\psi\,(y)}\,e^{i2\pi\langle a\cdot y,k\rangle}dy \\
&= \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\,e^{i2\pi\langle b,k\rangle}\,\overline{\hat{\psi}\,(a\cdot k)}.
\end{aligned} \tag{1.5}
$$

Furthermore,

$$
\mathscr{F}\left[W_{\tilde{\psi}}f(a,\cdot)\right](k) = \hat{f}(k)\,\overline{\hat{\psi}(a\cdot k)}. \tag{1.6}
$$

Theorem 1.6. *(**Parseval's relation**) If* $\psi \in L^2(\mathbb{R}^n)$ *is a wavelet and satisfies the admissibility condition*

$$
0 < C_\psi := \int_{\mathbb{R}^n_+} \frac{|\hat{\psi}(a\cdot k)|^2}{a_1a_2\cdots a_n}\,da < \infty, \quad \forall \;\; k \in \mathbb{N}_0^n, \tag{1.7}
$$

and $f, g \in L^2([0,1]^n)$ *are* $\mathbb{Z}^n$*-periodic functions, then*

$$
\int_{[0,1]^n}\int_{\mathbb{R}^n_+} (W_{\tilde{\psi}}f)(a,b)\overline{(W_{\tilde{\psi}}g)(a,b)}\,\frac{db\,da}{a_1a_2\cdots a_n} = C_\psi\,\langle f,g\rangle\,. \tag{1.8}
$$

This gives the Plancherel relation

$$
\int_{[0,1]^n}\int_{\mathbb{R}^n_+} \left|(W_{\tilde{\psi}}f)(a,b)\right|^2 \frac{dbda}{a_1a_2\cdots a_n} = C_\psi \int_{[0,1]^n} |f(t)|^2\,dt. \tag{1.9}
$$

Proof. From equation (1.5), we have

$$\begin{aligned}
\left\langle W_{\tilde{\psi}} f, W_{\tilde{\psi}} g \right\rangle &= \int_{[0,1]^n} \int_{\mathbb{R}^n_+} \left(\sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, \overline{\hat{\psi}(a\cdot k)}\, e^{i2\pi\langle b,k\rangle} \right) \\
&\quad \times \left(\sum_{m\in\mathbb{Z}^n} \overline{\hat{g}(m)}\, \hat{\psi}(a\cdot m)\, e^{-i2\pi\langle b,m\rangle} \right) \frac{db\, da}{a_1 a_2 \cdots a_n} \\
&= \sum_{k\in\mathbb{Z}^n} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n_+} \hat{f}(k)\, \overline{\hat{\psi}(a\cdot k)}\, \overline{\hat{g}(m)}\, \hat{\psi}(a\cdot m) \\
&\quad \times \left(\int_{[0,1]^n} e^{-i2\pi\langle b,m-k\rangle}\, db \right) \frac{da}{a_1 a_2 \cdots a_n} \\
&= \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, \overline{\hat{g}(k)} \int_{\mathbb{R}^n_+} \frac{\left|\hat{\psi}(a\cdot k)\right|^2}{a_1 a_2 \cdots a_n} da \\
&= C_\psi \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, \overline{\hat{g}(k)} \\
&= C_\psi \left\langle f, g \right\rangle .
\end{aligned}$$

□

Definition 1.7. ***(Inversion formula)*** *Let $f \in L^2([0,1]^n)$ be a $\mathbb{Z}^n$- periodic function and ψ be a wavelet that satisfies (1.7). Then the inversion formula for* (1.4) *is given by*

$$f(t) = \int_{[0,1]^n} \int_{\mathbb{R}^n_+} (W_{\tilde{\psi}} f)(a,b)\, \tilde{\psi}_{a,b}(t) \frac{db\, da}{a_1 a_2 \cdots a_n}.$$

2. The wavelet transform on test function space $F_p([0,1]^n)$

Let us recall the definitions of test function spaces $F_p([0,1]^n)$ and $G_p([0,1]^n)$, the spaces of periodic test functions, defined by Pathak and Singh [6].

Definition 2.1. *The test function space $F_p([0,1]^n)$; $1 \le p < \infty$, is defined as*

$$F_p([0,1]^n) := \left\{ f \quad : f \in C^\infty([0,1]^n) \text{ and } t^\alpha \frac{d^\beta f(t)}{dt^\beta} \in L^p([0,1]^n),\ \forall\, \alpha,\, \beta \in \mathbb{N}_0^n \right\}.$$

The space $F_p([0,1]^n)$ is a complex linear space with respect to usual point wise operation of addition and scalar multiplication. If we define $\gamma^p_{\alpha,\beta}(f)$ as

$$\gamma^p_{\alpha,\beta}(f) := \left\{ \left\| t^\alpha \frac{d^\beta f(t)}{dt^\beta} \right\|_{L^p} : \text{for } f \in F_p \text{ and } \alpha, \beta \in \mathbb{N}_0^n \right\},$$

then the collection

$$M_p = \left\{ \gamma^p_{\alpha,\beta}(f) : \alpha, \beta \in \mathbb{N}_0^n \right\}$$

is a countable multinorm and $F_p(\mathbb{R}^n)$ becomes a countably multinormed space with the topology generated by M_p.

Definition 2.2. *For $1 \le p < \infty$, $b \in [0,1]^n$ and $a \in \mathbb{R}^n_+$, the test function space $G_p(\mathbb{R}^n_+ \times [0,1]^n)$ is defined by*

$$\begin{aligned} & G_p(\mathbb{R}^n_+ \times [0,1]^n) \\ & := \left\{ \phi \in C^\infty(\mathbb{R}^n_+ \times [0,1]^n);\ \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n_+} a^\delta b^\alpha D_a^\gamma D_b^\beta \phi(a,b)da \right|^p db \right)^{1/p} < \infty \right\}, \end{aligned}$$

for all $\delta, \alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $\beta_i + \gamma_i \ge \delta_i + 1,\ i = 1, 2, \cdots, n$.

Theorem 2.3. *Let $f \in F_p([0,1]^n)$ be a $\mathbb{Z}^n$-periodic function and $\psi \in \mathscr{S}(\mathbb{R}^n)$ be a wavelet such that*

$$0 < C_\psi = \left| \int_{\mathbb{R}^n} \xi^\delta D_\xi^\gamma \hat{\psi}(\xi) d\xi \right| < \infty,$$

where $\delta, \gamma \in \mathbb{N}_0^n$. Then the wavelet transform $W_{\tilde{\psi}} : F_p([0,1]^n) \to G_p(\mathbb{R}^n_+ \times [0,1]^n)$ is continuous linear operator.

Proof. If $\delta, \alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $\beta_i + \gamma_i \ge \delta_i + 1$, then

$$\begin{aligned} I &= a^\delta b^\alpha D_a^\gamma D_b^\beta (W_{\tilde{\psi}} f)(a,b) \\ &= a^\delta b^\alpha D_a^\gamma (W_{\tilde{\psi}} D^\beta f)(a,b) \\ &= a^\delta b^\alpha D_a^\gamma \sum_{k \in \mathbb{Z}^n} \mathscr{F}[D^\beta f](k)\, \overline{\hat{\psi}(a \cdot k)}\, e^{i2\pi\langle b,k\rangle} \\ &= a^\delta b^\alpha D_a^\gamma \sum_{k \in \mathbb{Z}^n} (i2\pi k)^\beta \hat{f}(k)\, \overline{\hat{\psi}(a \cdot k)}\, e^{i2\pi\langle b,k\rangle}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n_+} I da &= b^\alpha \sum_{k \in \mathbb{Z}^n} e^{i2\pi\langle b,k\rangle} (i2\pi k)^\beta \hat{f}(k) \int_{\mathbb{R}^n_+} a^\delta D_a^\gamma \overline{\hat{\psi}(a \cdot k)} da \\ &= b^\alpha \sum_{k \in \mathbb{Z}^n} e^{i2\pi\langle b,k\rangle} (i2\pi k)^\beta \hat{f}(k) \int_{\mathbb{R}^n_+} u^\delta k^{\gamma-\delta-\mathbf{1}} D_u^\gamma \overline{\hat{\psi}(u)} du \\ &= b^\alpha \sum_{k \in \mathbb{Z}^n} e^{i2\pi\langle b,k\rangle} (i2\pi k)^{\beta+\gamma-\delta-\mathbf{1}} (i2\pi)^{|\delta-\gamma+\mathbf{1}|} \hat{f}(k) \int_{\mathbb{R}^n_+} u^\delta D_u^\gamma \overline{\hat{\psi}(u)} du. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\int_{[0,1]^n} \left| \int_{\mathbb{R}^n_+} I da \right|^p db \\
&= \int_{[0,1]^n} \left| b^\alpha \sum_{k\in\mathbb{Z}^n} e^{i2\pi\langle b,k\rangle}(i2\pi k)^{\beta+\gamma-\delta-\mathbf{1}}(i2\pi)^{|\delta-\gamma+\mathbf{1}|}\hat{f}(k)\int_{\mathbb{R}^n_+} u^\delta D_u^\gamma \overline{\hat{\psi}(u)}du \right|^p db \\
&\leq \int_{[0,1]^n} \left| b^\alpha \sum_{k\in\mathbb{Z}^n} e^{i2\pi\langle b,k\rangle}(i2\pi k)^{\beta+\gamma-\delta-\mathbf{1}}(i2\pi)^{|\delta-\gamma+\mathbf{1}|}\hat{f}(k) \right|^p C_\psi^p\, db \\
&= C_\psi^p \int_{[0,1]^n} \left| b^\alpha (i2\pi)^{|\delta-\gamma+\mathbf{1}|} \sum_{k\in\mathbb{Z}^n} e^{i2\pi\langle b,k\rangle}(i2\pi k)^{\beta+\gamma-\delta-\mathbf{1}}\hat{f}(k) \right|^p db \\
&= C_\psi^p \int_{[0,1]^n} \left| b^\alpha (i2\pi)^{|\delta-\gamma+\mathbf{1}|} \sum_{k\in\mathbb{Z}^n} e^{i2\pi\langle b,k\rangle}\mathscr{F}[D^{\gamma+\beta-\delta-\mathbf{1}}f(x)](k) \right|^p db \\
&= (2\pi)^{|\delta-\gamma+\mathbf{1}|} C_\psi^p \int_{[0,1]^n} \left| b^\alpha D_b^{\gamma+\beta-\delta-\mathbf{1}} f(b) \right|^p db \\
&= (2\pi)^{|\delta-\gamma+\mathbf{1}|} C_\psi^p \left\| b^\alpha D_b^{\gamma+\beta-\delta-\mathbf{1}} f(b) \right\|^p_{L^p([0,1]^n)}.
\end{aligned}
$$

Therefore, the wavelet transform $W_{\tilde{\psi}} : F_p([0,1]^n) \to G_p(\mathbb{R}^n_+ \times [0,1]^n)$ is continuous and linear map. □

3. The wavelet transform on test function space $F_p^*([0,1]^n)$

Definition 3.1. *For $1 \leq p < \infty, \alpha \in \mathbb{N}_0^n$, the test function space $F_p^*([0,1]^n)$ is defined by*

$$
F_p^*([0,1]^n) := \left\{ f \ : f \in \mathbb{C}^\infty([0,1]^n) \text{ and } \prod_{j=1}^n \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \in L^p([0,1]^n) \right\},
$$

and the corresponding space of distributions is denoted by $(F_p^*)'$.

Definition 3.2. *The test function space $G_p^*(\mathbb{R}^n_+ \times [0,1]^n)$ is defined, for $1 \leq p < \infty, \alpha \in \mathbb{N}_0^n, a \in \mathbb{R}^n_+$ and $b \in [0,1]^n$, by*

$$
G_p^*(\mathbb{R}^n_+ \times [0,1]^n) := \Bigg\{ \phi \in \mathbb{C}^\infty(\mathbb{R}^n_+ \times [0,1]^n) :
\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n_+} \left| \prod_{j=1}^n \left(\alpha_j \frac{\partial}{\partial \alpha_j} + \beta_j \frac{\partial}{\partial \beta_j} \right)^{\alpha_j} \phi(a,b) \right|^p \frac{da\, db}{a_1 a_2 \cdots a_n} \right)^{1/p} < \infty \Bigg\}.
$$

The corresponding space of distributions is denoted by $(G_p^*)'$.

Note that any differentiable function ψ satisfies the following partial differential equations given in [4].

(i) For $a, b, t \in \mathbb{R}$,

$$\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right)\frac{1}{a}\psi\left(\frac{t-b}{a}\right) = -\left(t\frac{\partial}{\partial t} + 1\right)\frac{1}{a}\psi\left(\frac{t-b}{a}\right). \tag{3.1}$$

(ii) Furthermore, for $a, b, t \in \mathbb{R}$ and $m \in \mathbb{N}_0$, we have

$$\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right)^m \frac{1}{a}\psi\left(\frac{t-b}{a}\right) = (-1)^m\left(t\frac{\partial}{\partial t} + 1\right)^m \frac{1}{a}\psi\left(\frac{t-b}{a}\right). \tag{3.2}$$

(iii) In the n-dimensional case, this takes the form

$$\begin{aligned}&\prod_{j=1}^{n}\left(a_j\frac{\partial}{\partial a_j} + b_j\frac{\partial}{\partial b_j}\right)^{\alpha_j}\frac{1}{a_1a_2a_3\cdots a_n}\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2},\frac{t_3-b_3}{a_3},\cdots,\frac{t_n-b_n}{a_n}\right)\\ &= (-1)^{|\alpha|}\prod_{j=1}^{n}\left(t_j\frac{\partial}{\partial t_j}+1\right)^{\alpha_j}\frac{1}{a_1a_2a_3\cdots a_n}\psi\left(\frac{t_1-b_1}{a_1},\frac{t_2-b_2}{a_2},\frac{t_3-b_3}{a_3},\cdots,\frac{t_n-b_n}{a_n}\right),\end{aligned} \tag{3.3}$$

for $a, b, t \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, , \cdots, \alpha_n) \in \mathbb{N}_0^n$.

Theorem 3.3. *If $f \in F_2^*([0,1]^n)$ is a $\mathbb{Z}^n$ periodic function and $\psi \in \mathscr{S}(\mathbb{R}^n)$ is a wavelet, then the wavelet transform $W_{\tilde{\psi}} : F_2^*([0,1]^n) \to G_2^*(\mathbb{R}_+^n \times [0,1]^n)$ is one-one onto continuous linear map.*

Proof. Let $f \in F_2^*([0,1]^n) \subset L^2([0,1]^n)$ be a $\mathbb{Z}^n$- periodic function and the wavelet $\psi \in \mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Since

$$\begin{aligned}&(W_{\tilde{\psi}}f)(a,b)\\ &= \int_{[0,1]^n} f(t)\sum_{k\in\mathbb{Z}^n}\frac{1}{a_1a_2a_3\cdots a_n}\overline{\psi\left(\frac{t_1-b_1+k_1}{a_1},\frac{t_2-b_2+k_2}{a_2},\cdots,\frac{t_n-b_n+k_n}{a_n}\right)}dt\\ &= \sum_{k\in\mathbb{Z}^n}\int_{[0,1]^n} f(t)\frac{1}{a_1a_2a_3\cdots a_n}\overline{\psi\left(\frac{t_1-b_1+k_1}{a_1},\frac{t_2-b_2+k_2}{a_2},\cdots,\frac{t_n-b_n+k_n}{a_n}\right)}dt\\ &= \sum_{k\in\mathbb{Z}^n}\int_{|y-k|\le 1} f(y-k)\frac{1}{a_1a_2a_3\cdots a_n}\overline{\psi\left(\frac{y_1-b_1}{a_1},\frac{y_2-b_2}{a_2},\cdots,\frac{y_n-b_n}{a_n}\right)}dy\\ &= \int_{\mathbb{R}^n} f(y)\frac{1}{a_1a_2a_3\cdots a_n}\overline{\psi\left(\frac{y_1-b_1}{a_1},\frac{y_2-b_2}{a_2},\cdots,\frac{y_n-b_n}{a_n}\right)}dy.\end{aligned}$$

That is,

$$(W_{\tilde{\psi}}f)(a,b) = (W_{\psi}f)(a,b). \tag{3.4}$$

For $a, b, t \in \mathbb{R}$, using equation (3.1) and integrating by parts, we have

$$\begin{aligned}\left(a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}\right)(W_\psi f)(a,b) &= \int_{\mathbb{R}^n} f(t)\left(a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}\right)\frac{1}{a}\overline{\psi\left(\frac{t-b}{a}\right)}dt\\ &= -\int_{\mathbb{R}^n}\left(\left(t\frac{\partial}{\partial t}+1\right)\frac{1}{a}\overline{\psi\left(\frac{t-b}{a}\right)}\right)f(t)dt\\ &= \int_{\mathbb{R}^n}\left(\left(t\frac{\partial}{\partial t}\right)f(t)\right)\frac{1}{a}\overline{\psi\left(\frac{t-b}{a}\right)}dt\\ &= W_\psi\left(\left(t\frac{\partial}{\partial t}\right)f(t)\right)(a,b).\end{aligned}$$

In general, by induction on $m \in \mathbb{N}_0$,

$$\left(a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}\right)^m (W_\psi f)(a,b) = W_\psi\left(\left(t\frac{\partial}{\partial t}\right)^m f(t)\right)(a,b).$$

Now, generalising for the n-dimensional case, we have

$$\prod_{j=1}^{n}\left(a_j\frac{\partial}{\partial a_j}+b_j\frac{\partial}{\partial b_j}\right)^{\alpha_j}(W_\psi f)(a,b) = W_\psi\left(\prod_{j=1}^{n}\left(t_j\frac{\partial}{\partial t_j}\right)^{\alpha_j} f(t)\right)(a,b), \quad (3.5)$$

where $a \in \mathbb{R}^n_+$, $b, t \in [0,1]^n$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n) \in \mathbb{N}^n_0$. Using equation (3.4), (3.5) and Plancherel formula, we have

$$\begin{aligned}&\left(\frac{1}{C_\psi}\int_{[0,1]^n}\int_{\mathbb{R}^n_+}\left|\prod_{j=1}^{n}\left(a_j\frac{\partial}{\partial a_j}+b_j\frac{\partial}{\partial b_j}\right)^{\alpha_j}\left(W_{\tilde{\psi}} f\right)(a,b)\right|^2\frac{dbda}{a_1a_2\cdots a_n}\right)^{1/2}\\ &= \left(\frac{1}{C_\psi}\int_{[0,1]^n}\int_{\mathbb{R}^n_+}\left|\prod_{j=1}^{n}\left(a_j\frac{\partial}{\partial a_j}+b_j\frac{\partial}{\partial b_j}\right)^{\alpha_j}(W_\psi f)(a,b)\right|^2\frac{dbda}{a_1a_2\cdots a_n}\right)^{1/2}\\ &= \left(\frac{1}{C_\psi}\int_{[0,1]^n}\int_{\mathbb{R}^n_+}\left|W_\psi\left(\prod_{j=1}^{n}\left(t_j\frac{\partial}{\partial t_j}\right)^{\alpha_j} f(t)\right)(a,b)\right|^2\frac{dbda}{a_1a_2\cdots a_n}\right)^{1/2}\\ &= \left(\int_{[0,1]^n}\left|\prod_{j=1}^{n}\left(t_j\frac{\partial}{\partial t_j}\right)^{\alpha_j} f(t)\right|^2 dt\right)^{1/2}\\ &= \left\|\prod_{j=1}^{n}\left(t_j\frac{\partial}{\partial t_j}\right)^{\alpha_j} f(t)\right\|_{L^2([0,1]^n)} < \infty.\end{aligned}$$

Thus, $W_{\tilde{\psi}} f : F_2^*([0,1]^n) \to G_2^*(\mathbb{R}_+^n \times [0,1]^n)$ is one-one onto continuous linear map. If $W_{\tilde{\psi}}^{-1}$ denotes the inverse of $W_{\tilde{\psi}}$, then

$$\left(\int_{[0,1]^n} \left| \prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} W_{\tilde{\psi}}^{-1} \left(W_{\tilde{\psi}} f \right) \right|^2 dt \right)^{1/2}$$

$$= \left(\int_{[0,1]^n} \left| \prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right|^2 dt \right)^{1/2}$$

$$= \left(\frac{1}{C_\psi} \int_{[0,1]^n} \int_{\mathbb{R}_+^n} \left| W_\psi \left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right) (a,b) \right|^2 \frac{dbda}{a_1 a_2 \cdots a_n} \right)^{1/2}$$

$$= \left(\frac{1}{C_\psi} \int_{[0,1]^n} \int_{\mathbb{R}_+^n} \left| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} (W_\psi f)(a,b) \right|^2 \frac{dbda}{a_1 a_2 \cdots a_n} \right)^{1/2}$$

$$= \left(\frac{1}{C_\psi} \int_{[0,1]^n} \int_{\mathbb{R}_+^n} \left| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \left(W_{\tilde{\psi}} f \right)(a,b) \right|^2 \frac{dbda}{a_1 a_2 \cdots a_n} \right)^{1/2} < \infty.$$

Thus, $W_{\tilde{\psi}}^{-1}$ is also a continuous linear map from $G_2^*(\mathbb{R}_+^n \times [0,1]^n)$ onto $F_2^*([0,1]^n)$.
□

4. The wavelet transform on Generalized Sobolev space $B_\kappa^p([0,1]^n)$

Let us recall the definitions of generalised Sobolev spaces defined in [6].

Definition 4.1 (**Temperate weight function**). *A positive function κ defined on $\mathbb{R}^n$ is called a temperate weight function if there exists positive constants C and N such that*

$$\kappa(\xi + \eta) \le (1 + C|\xi|)^N \kappa(\eta) \ ; \ \xi, \eta \in \mathbb{R}^n.$$

The set of all such functions κ is denoted by $\mathscr{K}$. Certain properties of temperate weight functions can be found in [2].

Definition 4.2. *The generalised Sobolev space $B_\kappa^p([0,1]^n)$ is a space of all infinitely differentiable complex valued function f on $[0,1]^n$ such that*

$$\left\| \prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right\|_{B_\kappa^p([0,1]^n)}^p$$

$$= \int_{[0,1]^n} |\kappa(\xi)|^p \left| \mathscr{F} \left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right) (\xi) \right|^p d\xi < \infty,$$

where $1 \leq p < \infty$ *and* $\alpha \in \mathbb{N}_0^n$, *and* κ *is the temperate weight function.*

Definition 4.3. *The generalised Sobolev space* $V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)$ *is a space of all infinitely differentiable complex valued function* $\phi(a,b)$ *on* $\mathbb{R}_+^n \times [0,1]^n$ *such that*

$$\left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \phi(a,b) \right\|_{V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)}^p$$
$$= \int_{\mathbb{R}_+^n} \left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \phi(a,b) \right\|_{B_\kappa^p([0,1]^n)}^p \frac{da}{a_1 a_2 a_3 \cdots a_n} < \infty.$$

Theorem 4.4. *Let* ψ *be a wavelet such that*

$$C_\psi^p = \int_{\mathbb{R}_+^n} \frac{\left|\hat{\psi}(a \cdot \xi)\right|^p}{a_1 a_2 \cdots a_n} da < \infty, \quad \forall \ \xi \in \mathbb{R}^n, .$$

Then the wavelet transform $W_{\tilde{\psi}} : B_\kappa^p([0,1]^n) \to V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)$ *is one-one onto continuous linear map.*

Proof. Let $\left(W_{\tilde{\psi}} f\right) \in V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)$. Then by using the techniques of [6], we have

$$\left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \left(W_{\tilde{\psi}} f\right)(a,b) \right\|_{V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)}^p$$
$$= \left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \left(W_{\psi} f\right)(a,b) \right\|_{V_\kappa^p(\mathbb{R}_+^n \times [0,1]^n)}^p$$
$$= \int_{\mathbb{R}_+^n} \left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \left(W_{\psi} f\right)(a,b) \right\|_{B_\kappa^p([0,1]^n)}^p \frac{da}{a_1 a_2 \cdots a_n}$$
$$= \int_{\mathbb{R}_+^n} \left\| W_\psi \left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right)(a,b) \right\|_{B_\kappa^p([0,1]^n)}^p \frac{da}{a_1 a_2 \cdots a_n}$$

Now by using equation (3.5), we have

$$\begin{aligned}
&\left\| \prod_{j=1}^{n} \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right)^{\alpha_j} \left(W_{\tilde{\psi}} f \right)(a,b) \right\|^p_{V^p_\kappa(\mathbb{R}^n_+ \times [0,1]^n)} \\
&= \int_{\mathbb{R}^n_+} \left(\int_{[0,1]^n} |\kappa(\xi)|^p \left| \mathscr{F}\left(W_\psi \left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right)(a,b) \right)(\xi) \right|^p d\xi \right) \frac{da}{a_1 a_2 \cdots a_n} \\
&= \int_{\mathbb{R}^n_+} \left(\int_{[0,1]^n} |\kappa(\xi)|^p \left| \hat{\psi}(a\xi) \right|^p \left| \mathscr{F}\left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right)(\xi) \right|^p d\xi \right) \frac{da}{a_1 a_2 \cdots a_n} \\
&= \int_{[0,1]^n} |\kappa(\xi)|^p \left| \mathscr{F}\left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right)(\xi) \right|^p \left(\int_{\mathbb{R}^n_+} \left| \hat{\psi}(a \cdot \xi) \right|^p \frac{da}{a_1 a_2 \cdots a_n} \right) d\xi \\
&= C^p_\psi \int_{[0,1]^n} |\kappa(\xi)|^p \left| \mathscr{F}\left(\prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right)(\xi) \right|^p d\xi \\
&= C^p_\psi \left\| \prod_{j=1}^{n} \left(t_j \frac{\partial}{\partial t_j} \right)^{\alpha_j} f(t) \right\|^p_{B^p_\kappa([0,1]^n)}.
\end{aligned}$$

This completes the proof of theorem. □

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6. Disclosure statement

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