



## Solution to Linear KdV and Nonlinear Space Fractional PDEs

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**ABSTRACT:** In this work, the author will briefly discuss applications of the Fourier and Laplace transforms in the solution of certain singular integral equations and evaluation of integrals. By combining integral transforms and operational methods we get more powerful analytical tool for solving a wide class of linear or even non-linear fractional differential or fractional partial differential equation. Numerous examples and exercises occur throughout the paper.

**Key Words:** Nonlinear differential equations, Laplace transform, Fourier transform, KdV equation, Modified Bessel's function, Kelvin's function, Caputo fractional derivative, Riemann-Liouville fractional derivative.

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### Introduction

The integral transform technique is one of the most useful tools of applied mathematics employed in many branches of science and engineering. The Fourier and Laplace transformations receive a special attention in the literature because of their importance in various applications and therefore, is considered as a standard technique in solving linear differential equations, integral equations, the solution of difference equations.

**Definition 1** The Laplace transform of a given function  $f(t)$  is defined as follows

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s). \quad (1.1)$$

If  $\mathcal{L}\{f(t)\} = F(s)$ , then  $\mathcal{L}^{-1}\{F(s)\}$  is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \quad (1.2)$$

where  $F(s)$  is analytic in the region  $\text{Re}(s) > c$ .

**Example 1** By using an appropriate integral representation for the modified Bessel's functions of the second kind of order  $\nu$ ,  $K_\nu(s)$ , show that

$$\mathcal{L}^{-1}\left\{\frac{K_\nu(s)}{s^\nu}\right\} = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu}(t^2-1)^{\nu-\frac{1}{2}}. \quad (1.3)$$

**Solution.** Upon taking the inverse Laplace transform of the given  $K_\nu(s)$ , we obtain

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \left(\frac{K_\nu(s)}{s^\nu}\right) ds, \quad (1.4)$$

at this point, using the following integral representation for  $K_\nu(s)$

$$\frac{K_\nu(s)}{s^\nu} = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} \int_0^\infty e^{-s \cosh t} \sinh^{2\nu} t dt. \quad (1.5)$$

By setting relation (1.5) in (1.4), we arrive at

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \left(\frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} \int_0^\infty e^{-s \cosh r} \sinh^{2\nu} r dr\right) ds, \quad (1.6)$$

let us change the order of integration in relation (1.6), we get

$$f(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} \int_0^\infty \sinh^{2\nu} r \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s(t-\cosh r)} ds\right) dr, \quad (1.7)$$

the value of the inner integral is  $\delta(t - \cosh r)$ , therefore

$$f(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} \int_0^\infty \delta(t - \cosh r) \sinh^{2\nu} r dr, \quad (1.8)$$

after making a change of variable  $t - \cosh r = u$ , and considerable algebra and elimination process, we obtain

$$f(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} \int_{-\infty}^{t-1} \delta(u) \frac{((t-u)^2-1)^\nu}{\sqrt{(t-u)^2-1}} du = \frac{\sqrt{\pi}}{\Gamma(\nu+0.5)2^\nu} (t^2-1)^{\nu-\frac{1}{2}}. \quad (1.9)$$

Note. Let us consider the special case  $\nu = 0$ , we get the following relations

$$\mathcal{L}^{-1}\{K_0(s)\} = (t^2-1)^{-\frac{1}{2}},$$

$$\mathcal{L}^{-1}\{K_1(s)\} = \mathcal{L}^{-1}\{-K_0'(s)\} = t(t^2-1)^{-\frac{1}{2}}.$$

The above example has been merely a new approach to a result with which we were already familiar. However, in more difficult applications the use of complex inversion formula and contour integration is often either the only or, at least, the

best way of finding an inverse Laplace transform of a given function.

**Definition 2** The error function is defined by the following integral

$$Erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp -\xi^2 d\xi.$$

The error function is encountered in probability theory, the theory of errors, the theory of heat conduction and various branches of applied mathematics and mathematical physics. In some applications, it is useful to introduce the complementary error function

$$Erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty \exp -\xi^2 d\xi.$$

Using elementary properties of integrals, it follows that

$$Erf(t) = 1 - Erfc(t)$$

**Corollary 1** Let  $\mathcal{L}(\phi(t)) = \Phi(s)$  then the following identity holds true.

$$\mathcal{L}\left(\int_0^\infty Erfc\left(\frac{\xi}{2\sqrt{t}}\right)\phi(\xi)d\xi\right) = \frac{1}{s}\Phi(\sqrt{s}). \tag{1.10}$$

**Proof.** See [6].

**Definition 3** The Fourier transform of function  $f(t)$  is defined as follows

$$\mathcal{F}\{f(t)\} = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} e^{i\omega t} f(t)dt := F(\omega). \tag{1.11}$$

If  $\mathcal{F}\{f(t)\} = F(\omega)$ , then  $\mathcal{F}^{-1}\{F(\omega)\}$  is given by

$$\mathcal{F}^{-1}\{F(\omega)\} = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} e^{-it\omega} F(\omega)d\omega = f(t). \tag{1.12}$$

**Definition 4** If the function  $\phi(x)$  belongs to  $C[a, b]$  and  $a < x < b$ , then the left Riemann-Liouville fractional integral of order  $\alpha > 0$  is defined as [8]

$$I_a^{RL,\alpha}\{\phi(x)\} = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\phi(\xi)}{(x-\xi)^{1-\alpha}} d\xi. \tag{1.13}$$

The left Riemann-Liouville fractional derivative of order  $\alpha > 0$  is defined as follows [4,8]

$$D_a^{RL,\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{\phi(\xi)}{(x-\xi)^\alpha} d\xi. \tag{1.14}$$

It follows that  $D_a^{RL,\alpha}\phi(x)$  exists for all  $\phi(x)$  belongs to  $C[a, b]$  and  $a < x < b$ .

Note: It is well-known that, the R - L operators is that they satisfy semi group properties of fractional integrals. The special case of fractional derivative when  $\alpha = 0.5$  is called semi - derivative.

**Definition 5** The left Caputo fractional derivative of order  $\alpha$  ( $0 < \alpha < 1$ ) of  $\phi(x)$  is as follows[3,8]

$$D_a^{C,\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{1}{(x-\xi)^\alpha} \phi'(\xi) d\xi. \quad (1.15)$$

**Corollary 2** Let  $\mathcal{F}\{f(t)\} = F(\omega)$  then the following identities hold true.

1.  $\mathcal{F}(D_{-\infty,t}^{C,\alpha}\phi(x)) = (-i\omega)^\alpha F(\omega),$
2.  $\mathcal{F}(D_{-\infty,t}^{RL,\alpha}\phi(x)) = (-i\omega)^\alpha F(\omega).$

**Proof.** See [ 5, 10].

**Lemma 1** Let  $\mathcal{F}\{f(t)\} = F(\omega)$  then the following identities hold true.

1.  $\mathcal{F}^{-1}(\sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}) = \mathcal{U}(1 - |t|),$
2.  $\mathcal{F}^{-1}(\sqrt{\frac{2}{\pi}} \frac{i}{\omega}) = \text{sgn}(t),$
3.  $\mathcal{F}^{-1}(-\sqrt{\frac{2}{\pi}} \frac{1}{\omega^2}) = t \text{sgn}(t),$
4.  $\mathcal{F}^{-1}(-\sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\alpha)}{|\omega|^{1-\alpha}} \sin(\frac{\pi\alpha}{2})) = |t|^{-\alpha}.$

Note.  $\mathcal{U}(t)$  and  $\text{sgn}(t)$  stand for the Heaviside unit step function and signum function respectively.

**Proof.** See [9].

**Example 2** Let us solve the following fractional Fredholm singular integral equation of convolution type. The Fourier transform provides a useful technique for the solution of such fractional singular integro- differential equations.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} D^\alpha \phi(\xi) D^\beta \phi(t - \xi) d\xi = \text{sgn}(t), \alpha, \beta > 0, \alpha + \beta = 1. \quad (1.16)$$

**Solution.** Upon taking the Fourier transform of the given integral equation, yields

$$(-i\omega)^\alpha \Phi(\omega) (-i\omega)^\beta \Phi(\omega) = -i\omega \Phi^2(\omega) = \sqrt{\frac{2}{\pi}} \frac{i}{\omega}, \quad (1.17)$$

solving the above equation, leads to

$$\Phi(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega}, \quad (1.18)$$

taking the inverse Fourier transform and using part four of Lemma 1, on the right hand side, after simplifying we get the following result

$$\phi(t) = \sqrt[6]{\frac{2}{\pi}} \sqrt{\frac{1}{t}}. \quad (1.19)$$

**Theorem 1** Let  $\mathcal{F}(\phi(x)) = \Phi(\omega)$  then the following identity holds true.

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{2}{\pi}} \frac{\mu}{\omega} \sin\left(\frac{\omega b}{a}\right) e^{-\frac{i\omega\lambda}{a}} \Phi\left(\frac{\mu\omega}{a}\right). \quad (1.20)$$

**Proof.**

By definition of the Fourier transforms we have

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega x} \left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) dx, \quad (1.21)$$

changing the order of integration leads to

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \phi\left(\frac{\xi+\lambda}{\mu}\right) \left(\int_{x=\frac{\xi-b}{a}}^{x=\frac{\xi+b}{a}} e^{i\omega x} dx\right) d\xi, \quad (1.22)$$

after calculation of the inner integral we arrive at

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \phi\left(\frac{\xi+\lambda}{\mu}\right) \frac{1}{i\omega} \left(e^{i\omega\left(\frac{\xi+b}{a}\right)} - e^{i\omega\left(\frac{\xi-b}{a}\right)}\right) d\xi, \quad (1.23)$$

after simplifying we get the following

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{\omega b}{a}\right)}{\omega} \int_{-\infty}^{+\infty} e^{-\frac{i\omega\xi}{a}} \phi\left(\frac{\xi+\lambda}{\mu}\right) d\xi, \quad (1.24)$$

Let us introduce a change of variable  $\frac{\xi+\lambda}{\mu} = z$ , after performing easy calculation, we obtain

$$\mathcal{F}\left(\int_{\xi=ax+b}^{\xi=ax-b} \phi\left(\frac{\xi+\lambda}{\mu}\right)d\xi\right) = \sqrt{\frac{2}{\pi}} \frac{\mu \sin\left(\frac{\omega b}{a}\right)}{\omega} e^{-\frac{i\omega\lambda}{a}} \Phi\left(\frac{\mu\omega}{a}\right), \quad (1.25)$$

**Lemma 2** Let  $L\{f(t)\} = F(s)$  then the following identities hold true.

1.  $e^{-a\sqrt{s}} = \frac{k}{(2\sqrt{\pi})} \int_0^\infty e^{-s\xi - \frac{a^2}{4\xi}} \xi^{-\frac{3}{2}} d\xi$ ,
2.  $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi} \int_0^\infty f(u) \int_0^\infty e^{-tr - ur^\alpha \cos\alpha\pi} \sin(ur^\alpha \sin\alpha\pi) dr du$ ,
3.  $\mathcal{L}^{-1}(F(\sqrt{s})) = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty u e^{-\frac{u^2}{4t}} f(u) du$ ,

$$4. \mathcal{L}^{-1}\left(\frac{1}{s(\sqrt{s+a^2})}\right) = \frac{1}{a} \text{Erf}(a\sqrt{t}).$$

**Proof.** See [1, 2, 6].

The Laplace transform is useful tool in applied mathematics, for instance for solving singular integral equations, partial differential equations, and in automatic control, where it defines a transfer function.

**Example 3** Let us solve the following singular integral equation. The Laplace transform provides a useful technique for the solution of such singular integral equations.

$$\int_0^{+\infty} \text{Erfc}\left(\frac{\xi}{2\sqrt{t}}\right)\phi(\xi)d\xi = \frac{1}{a} \text{Erf}(a\sqrt{t}). \quad (1.26)$$

**Solution.** Upon taking the Laplace transform of the given integral equation, yields

$$\frac{1}{s}\Phi(\sqrt{s}) = \frac{1}{s\sqrt{s+a^2}}, \quad (1.27)$$

solving the above equation, leads to

$$\Phi(\sqrt{s}) = \frac{1}{\sqrt{s+a^2}}, \quad (1.28)$$

or

$$\Phi(s) = \frac{1}{\sqrt{s^2+a^2}}, \quad (1.29)$$

so that upon taking the inverse Laplace transform, we arrive at the solution

$$\phi(t) = \mathcal{L}^{-1}\frac{1}{\sqrt{s^2+a^2}} = J_0(at). \quad (1.30)$$

Lastly, the substitution of the obtained solution into the integral equation (1.26) yields the following integral identity

$$\int_0^{+\infty} \text{Erfc}\left(\frac{\xi}{2\sqrt{t}}\right)J_0(a\xi)d\xi = \frac{1}{a} \text{Erf}(a\sqrt{t}). \quad (1.31)$$

**Lemma 3** The following exponential identities hold true.

1.  $\exp(\pm\lambda\frac{d}{dt})\Phi(t) = \Phi(t\pm\lambda),$
2.  $\exp(\pm\lambda t\frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda}),$
3.  $\exp(\lambda q(t)\frac{d}{dt})\Phi(t) = \Phi(Q(F(t) + \lambda)),$
4.  $\exp(\lambda(t^2 - a^2)\frac{d}{dt})\Phi(t) = \Phi\left(a\left(\frac{(t+a)+(t-a)e^{2a\lambda}}{(t+a)-(t-a)e^{2a\lambda}}\right)\right),$
5.  $\exp\left(\frac{k^2}{2t}\frac{d}{dt}\right)\Phi(t) = \Phi(\sqrt{t^2 + k^2}).$

where  $F(t)$  is the primitive function of  $(q(t))^{-1}$  and  $Q(t)$  is the inverse function of  $F(t)$ .

**Proof.** See [7, 8].

**Corollary 3** Let us consider the following PDE with non - constant coefficients

$$u_t + (x^2 - a^2)u_x = bvt^{\nu-1}u, \quad u(x, 0) = \phi(x). \quad \nu > 0.$$

In view of the Lemma 3, the above boundary value problem has the following formal solution

$$u(x, t) = e^{bt^\nu} \phi\left(a \left( \frac{(x+a) + (x-a)e^{-2at}}{(x+a) - (x-a)e^{-2at}} \right)\right).$$

**Lemma 4** The following exponential identity holds true.

$$\exp\left(\frac{1}{3}\left(\frac{k^2}{2t} \frac{d}{dt}\right)^3\right)\phi(t) = \int_{-\infty}^{+\infty} Ai(\xi)\phi(\sqrt{t^2 + k^2\xi})d\xi, \quad (1.32)$$

**Proof.** It is well known that

$$\mathcal{F}\{Ai(t)\} = \frac{1}{\sqrt{2\pi}} \exp\left(i\frac{(\omega)^3}{3}\right), \quad (1.33)$$

in other words, we have the following relation

$$\mathcal{F}\{Ai(t)\} = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} e^{i\omega t} Ai(t)dt := \frac{1}{\sqrt{2\pi}} \exp\left(i\frac{(\omega)^3}{3}\right). \quad (1.34)$$

Let us introduce a change of parameter as follows

$$\left(\sqrt[3]{3\lambda}\right)\beta = iw, \quad (1.35)$$

after substitution of (1.35) in (1.34) and simplifying, we arrive at

$$\mathcal{F}\{Ai(\xi)\} = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} e^{\sqrt[3]{3\lambda}\beta\xi} Ai(\xi)d\xi := \frac{1}{\sqrt{2\pi}} \exp(\lambda\beta^3), \quad (1.36)$$

if we set  $\beta = \frac{k^2}{2t} \frac{d}{dt}$ , and  $\lambda = \frac{1}{3}$  in relation (1.36), we get the following operational identity,

$$\left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} Ai(\xi)d\xi e^{\xi \frac{k^2}{2t} \frac{d}{dt}} = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{3}\left(\frac{k^2}{2t} \frac{d}{dt}\right)^3\right). \quad (1.37)$$

In view of first part of the Lemma 1, we obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{3}\left(\frac{k^2}{2t} \frac{d}{dt}\right)^3\right)\phi(t) &= \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} Ai(\xi)d\xi e^{\xi \frac{k^2}{2t} \frac{d}{dt}} \phi(t) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{+\infty} Ai(\xi)\phi(\sqrt{t^2 + k^2\xi})d\xi, \end{aligned} \quad (1.38)$$

after simplifying the above relation, we get

$$\exp\left(\frac{1}{3}\left(\frac{k^2}{2t} \frac{d}{dt}\right)^3\right)\phi(t) = \int_{-\infty}^{+\infty} Ai(\xi)\phi(\sqrt{t^2 + k^2\xi})d\xi. \quad (1.39)$$

Note: In the above identity,  $Ai(\cdot)$  stands for the Airy function [11].

**Example 4** By using an appropriate integral representation for the Bessel's functions of the first kind of order zero,  $J_0(s)$ , show that

$$\mathcal{L}^{-1}\left\{\frac{J_0\left(\frac{1}{s}\right)}{s}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} ber(2\sqrt{t \sin \phi}) d\phi. \quad (1.40)$$

Note: In the above relation  $ber(t) = \Re(J_0(i\sqrt{t}))$ , stands for Kelvin's function of order zero.

**Solution.** Let us consider the following well - known integral identity for  $J_0(s)$

$$J_0(s) = \frac{2}{\pi} \int_0^1 \frac{\cos s\xi}{\sqrt{1-\xi^2}} d\xi.$$

In view of the above identity and upon taking the inverse Laplace transform of the given  $\frac{1}{s}J_0\left(\frac{1}{s}\right)$ , we obtain

$$\mathcal{L}^{-1}\left\{\frac{J_0\left(\frac{1}{s}\right)}{s}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \mathcal{L}^{-1}\left(\frac{1}{s} \cos\left(\frac{\xi}{s}\right)\right) \frac{d\xi}{\sqrt{1-\xi^2}}. \quad (1.41)$$

but, the value of the inverse Laplace transform under the integral sign is as below

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cos\left(\frac{\xi}{s}\right)\right) = ber(2\sqrt{t\xi}).$$

therefore, we we get the following

$$\mathcal{L}^{-1}\left\{\frac{J_0\left(\frac{1}{s}\right)}{s}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{ber(2\sqrt{t\xi})}{\sqrt{1-\xi^2}} d\xi. \quad (1.42)$$

At this stage, let us introduce a change of variable  $\xi = \sin \phi$ , after simplifying, we get

$$\mathcal{L}^{-1}\left\{\frac{J_0\left(\frac{1}{s}\right)}{s}\right\} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} ber(2\sqrt{t \sin \phi}) d\phi. \quad (1.43)$$

### 1. Linearized KdV with Variable Coefficients

The KdV equations are attracting many researchers, and a great deal of research work has been done in some of these equations. Linearized KdV with variable coefficients often provide more powerful and realistic model than their constant coefficient counterparts when the non-homogeneities of media are considered. In this section, we will implement the exponential operator method to construct an exact solution for a variety of the linearized KdV equation with non - constant coefficients.

**Problem 1** Let us solve the following linearized KdV with non-constant coefficients

$$\frac{1}{t^2} \frac{\partial u(x, t)}{\partial t} + \frac{1}{8x^3} \frac{\partial^3 u}{\partial x^3} = -3\lambda u(x, t), \quad (1.1)$$



subject to the boundary conditions and the initial condition

$$u(x, 0) = \phi(x), -\infty < x < \infty, t > 0.$$

**Solution:** At this point, in order to solve the above linearized KdV, we may rewrite the equation in the following exponential operator form

$$\frac{1}{t^2} \frac{\partial u(x, t)}{\partial t} = -\frac{1}{8x^3} \frac{\partial^3 u}{\partial x^3} - 3\lambda u(x, t), \tag{1.2}$$

In order to obtain a solution for equation (1) first by solving the first order PDE with respect to t, we arrive at

$$\frac{du}{u} = \left(-3\lambda - \frac{1}{8x^3} \frac{\partial^3}{\partial x^3}\right) t^2 dt, \tag{1.3}$$

solving the first order PDE with respect to t, and applying the initial condition, we get the following

$$u(x, t) = \exp\left(-\lambda t^3 - \frac{t^3}{24x^3} \frac{\partial^3}{\partial x^3}\right) \phi(x), \tag{1.4}$$

using (1.34), we obtain the solution as follows

$$\begin{aligned} u(x, t) &= \exp(-\lambda t^3) \int_{-\infty}^{+\infty} Ai(\xi) d\xi \left(e^{-\frac{t}{2x} \frac{\partial}{\partial x}}\right) \phi(x) \\ &= \exp(-\lambda t^3) \int_{-\infty}^{+\infty} Ai(\xi) \phi(\sqrt{x^2 - t\xi}) d\xi. \end{aligned} \tag{1.5}$$

It is easy to verify that  $u(x, 0) = \phi(x)$ .

## 2. Nonlinear Space Fractionl Partial Differential Equations.

In this section, the author implemented the exponential operational method for solving a nonlinear space fractional partial differential equations with non-constant coefficients.

**Problem 2** Let us solve the following space-fractional PDE with non-constant coefficients, where fractional derivative is in the Riemann-Liouville sense

$$\frac{t^{-(\nu-1)}}{\nu} \frac{\partial u(x, t)}{\partial t} = \left(\eta - \sqrt{\lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}}\right) u(x, t), \tag{2.1}$$

subject to the boundary conditions and the initial condition

$$u(x, 0) = \phi(x), -\infty < x < \infty, t > 0.$$

**Solution:** At this point, in order to solve the above linear space fractional PDE, we may rewrite the equation in the following exponential operator form

$$\frac{\partial u(x, t)}{\partial t} = \nu t^{\nu-1} \left(\eta - \sqrt{\lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}}\right) u(x, t), \tag{2.2}$$

In order to obtain a solution for equation (1) first by solving the first order PDE with respect to  $t$ , and applying the initial condition, we get the following

$$u(x, t) = \exp(t^\nu \eta - t^\nu \sqrt{\lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}}) \phi(x), \quad (2.3)$$

the above equation may be rewritten as follows

$$u(x, t) = \exp \eta t^\nu \exp(-t^\nu \sqrt{\lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}}) \phi(x), \quad (2.4)$$

in order to find the result of the action of exponential operator, we may use part one of Lemma 1.2 by choosing  $a = t^\nu$  and  $s = \lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}$ , to obtain

$$u(x, t) = \frac{t^\nu \exp(\eta t^\nu)}{(2\sqrt{\pi})} \int_0^\infty e^{-\xi \lambda - \frac{t^{2\nu}}{4\xi}} \xi^{-\frac{3}{2}} d\xi [e^{-\frac{\xi k^2}{2x} \frac{\partial}{\partial x}} \phi(x)], \quad (2.5)$$

In view of part five of the Lemma 3, we get the following formal solution to non-linear fractional partial differential equation as below

$$u(x, t) = \frac{t^\nu \exp(\eta t^\nu)}{(2\sqrt{\pi})} \int_0^\infty e^{-\xi \lambda - \frac{t^{2\nu}}{4\xi}} \xi^{-\frac{3}{2}} \phi(\sqrt{x^2 + k^2 \xi}) d\xi. \quad (2.6)$$

**Example 5** Let us consider the following special case

$$\frac{t^{-(\nu-1)}}{\nu} \frac{\partial u(x, t)}{\partial t} = (\eta - \sqrt{\lambda - \frac{k^2}{2x} \frac{\partial}{\partial x}}) u(x, t), \quad (2.7)$$

subject to the initial condition

$$u(x, 0) = \phi(x) = \exp(-x^2), \quad -\infty < x < \infty, t > 0.$$

**Solution** In the solution of problem 2, if we put  $\phi(x) = \exp(-x^2)$ , after simplifying we obtain

$$u(x, t) = \frac{t^\nu \exp(\eta t^\nu - x^2)}{(2\sqrt{\pi})} \int_0^\infty e^{-(\lambda+k^2)\xi - \frac{t^{2\nu}}{4\xi}} \xi^{-\frac{3}{2}} d\xi. \quad (2.8)$$

Let us recall the integral representation for the modified Bessel's function of the second kind of order  $\mu$  as below

$$\frac{2\alpha}{\beta} K_\mu(2\sqrt{\alpha\beta}) = \int_0^\infty e^{-\alpha\xi - \frac{\beta}{\xi}} \frac{d\xi}{\xi^{\mu+1}}. \quad (2.9)$$

Finally, in terms of the modified Bessel's functions of the second kind we obtain the formal solution as follows

$$u(x, t) = \frac{\exp(\eta t^\nu - x^2)}{\sqrt{\pi}} \frac{(k^2 + \lambda)}{t^\nu} K_{\frac{1}{2}}(t^\nu \sqrt{k^2 + \lambda}) \quad (2.10)$$

### 3. Conclusions

The article is intended for scientists and researchers of different disciplines of engineering and science dealing with the solutions of fractional integro - differential and fractional PDEs. The results reveal that the transforms method is very convenient and effective.

### Aknowledgments

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