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Topological Degree Methods for Partial Differential Operators in Generalized Sobolev Spaces

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abstract: The main aim of this paper is to prove, by using the topological degree methods, the existence of solutions for nonlinear elliptic equation $Au = f$ where $Au = \sum_{|\alpha|\leq m}(-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u)$ is partial differential operators of general divergence form and $f \in W^{-m,p'(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$.

Key Words: Partial differential operators, General divergence form, Topological Degree, Generalized Sobolev spaces.

Contents

1. Introduction

Topological degree theory is one of the most effective tools in solving nonlinear equations. Brouwer had published a degree theory in 1912 for continuous maps defined in finite dimensional Euclidean space [4]. Leray and Schauder developed the degree theory for compact operators in infinite dimensional Banach spaces [12]. Since then numerous generalizations and applications have been investigated in various ways of approach (see e.g. [7,15,16,19]). Browder introduced a topological degree for nonlinear operators of monotone type in reflexive Banach spaces [5,6]. The theory was constructed later by Berkovits and Mustonen by using the Leray-Schauder degree [1,2,3] which can be applied to partial differential operators of general divergence form, i.e. to operators of the form

$$Au(x) = \sum_{|\alpha|\leq m}(-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u).$$

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If the functions $A_\alpha(x,\xi)$ satisfy the polynomial growth conditions with respect to $|\xi|$ and some analytical conditions, then the differential operator will generate a mapping defined in a Sobolev space,

$$A : W_0^{m,p}(\Omega) \to W^{-m,p'}(\Omega),$$

which belongs to the class (S_+) (see Page 5, item (iv) for the definition of this class below).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an open and bounded subset with the segment property (i.e. there exist a locally finite open covering $\{O_i\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\}$ such that for $x \in \bar{\Omega} \cap O_i$, and $0 < t < 1$, $x + ty_i \in \Omega$), A be a partial differential operator of general divergence form

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u))$$

defined on a subset of $W_0^{m,p(.)}(\Omega)$ and $f \in W^{-m,p'(.)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see below for the definitions of these spaces).

The goal of the paper is to prove the existence of solutions for nonlinear elliptic equations with boundary value problems of the form

$$\begin{cases} A(u) = f & \text{in} \quad \Omega \\ D^\alpha u(x) = 0 & \text{on} \quad \partial\Omega \text{ for } |\alpha| \leq m-1. \end{cases} \tag{1.1}$$

by applying the topological degree theory.

Our paper is organized in the following way. The second section recalls some preliminary definitions and results about Generalized Lebesgue and Sobolev spaces, some classes of mappings of monotone type and defines a degree function in Sobolev spaces with variables exponents. The last section defines a new monotonicity class i.e. a class (MOD), presents some normalising maps and proves the existence of a solution for the problem (1.1) using the degree theory.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field [14].

2. Preliminary definitions and results

In the sequel, we consider a naturel number $N \geq 1$ and an open and bounded domain $\Omega \subset \mathbb{R}^N$ with segment property.

2.1. Generalized Lebesgue and Sobolev spaces

We recall in what follows some well Known properties of the generalized Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W_0^{m,p(.)}(\Omega)$ which can be found for instance in X. Fan and D. Zhao [8] or O. Kováčik and J. Rákosník [11]. In the sequel, we call exponent any measurable function: $p:\Omega\to[1,+\infty)$ and we set $p^-=\operatorname{ess}\inf_{\Omega} p$ and $p^+=\operatorname{ess}\sup_{\Omega} p$.

For every exponent $p(.)$ and for every measurable function u, we set

$$\rho_{p(.)}(u)=\int_\Omega |u(x)|^{p(x)}\,dx.$$

We define the variable exponent Lebesgue space

$$L^{p(.)}(\Omega)=\{u;\ u:\Omega\to\mathbb{R} \text{ is measurable and } \rho_{p(.)}(u)<\infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$||u||_{p(.)}=\inf\{\lambda>0/\rho_{p(.)}(\frac{u}{\lambda})\le 1\}.$$

We say that a sequence $\{u_n\}\subset L^{p(.)}(\Omega)$ converges to $u\in L^{p(.)}(\Omega)$ *in the modular sense,* denote $u_n\to u(mod)$ in $L^{p(.)}$, if there exists $\lambda>0$ such that

$$\rho_{p(.)}(\frac{u_n-u}{\lambda})\to 0,\ \text{when } n\to\infty.$$

We say that a bounded exponent $p(.)$ is log-Hölder continuous on Ω if there exists $\alpha>0$ such that

$$|p(x)-p(y)|\le\frac{\alpha}{log(e+1/|x-y|)}$$

for all $x,y\in\Omega$. Next, let m be a positive integer, we define

$$W^{m,p(.)}(\Omega)=\{u\in L^{p(.)}(\Omega):D^\alpha u\in L^{p(.)}(\Omega),|\alpha|\le m\},$$

with the norm

$$||u||_{m,p(.)}=\sum_{|\alpha|\le m}||D^\alpha u||_{p(.)}$$

and $W_0^{m,p(.)}(\Omega)$ as the closure of $\mathcal{D}(\Omega)=C_0^\infty(\Omega)$ in $W^{m,p(.)}(\Omega)$.

We say that a sequence $\{u_n\}\subset W^{m,p(.)}(\Omega)$ converges to $u\in W^{m,p(.)}(\Omega)$ *in the modular sense,* denote $u_n\to u(mod)$ in $W^{m,p(.)}$, if there exists $\lambda>0$ such that

$$\rho_{p(.)}(\frac{D^\alpha u_n-D^\alpha u}{\lambda})\to 0,\ \text{when } n\to\infty,$$

for $|\alpha|\le m$.

We define

$$W^{-m,p'(.)}(\Omega)=\{g\in\mathcal{D}'(\Omega):g=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha g_\alpha,\ \text{where } g_\alpha\in L^{p'(.)}(\Omega)\},$$

and we say that a sequence $\{u_n\}\subset W^{-m,p'(.)}(\Omega)$ converges to $u\in W^{-m,p'(.)}(\Omega)$ *in the modular sense,* denote $u_n\to u(mod)$ in $W^{-m,p'(.)}$, if u_n and u have representations

$$u_n=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha g_\alpha^{(n)},\ u=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha g_\alpha,$$

such that $g_\alpha^{(n)}, g_\alpha \in L^{p'(\cdot)}(\Omega)$ and $g_\alpha^{(n)} \to g_\alpha(mod)$ in $L^{p'(\cdot)}$ for all $|\alpha| \leq m$.

In what follows, we assume that $p(.)$ is a log-Hölder continuous exponent such that $1 < p^- \leq p(x) \leq p^+ < \infty$. Under this assumption, we have:

1. Endowed with the Luxembourg norm, $L^{p(\cdot)}(\Omega)$ is a Banach space [11, Theorem 2.5], separable, reflexive [11, Corollary 2.7], uniformly convex and
$$[L^{p(\cdot)}(\Omega)]' = L^{p'(\cdot)}(\Omega)$$
.
2. For every $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ Hölder inequality holds [11, Theorem 2.1]
$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) ||u||_{p(.)}||v||_{p'(.)}.$$
3. If p and q are variable exponents so that $q(.) \leq p(.)$ a.e. in Ω then there exists the continuous embedding $L^{p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.
4. If (u_n) and $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold true
$$||u||_{p(.)} > 1 \quad \Rightarrow \quad ||u||_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq ||u||_{p(.)}^{p^+}$$
$$||u||_{p(.)} < 1 \quad \Rightarrow \quad ||u||_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq ||u||_{p(.)}^{p^-}$$
$$\lim_{n\to\infty} u_n = 0 \text{ in } L^{p(\cdot)}(\Omega) \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(.)}(u_n) = 0$$
5. The space $(W_0^{m,p(\cdot)}(\Omega), \|\cdot\|_{m,p(.)})$ is a Banach space separable and reflexive and $[W_0^{m,p(\cdot)}(\Omega)]' = W^{-m,p'(\cdot)}(\Omega)$.
6. If $q(.)$ is an exponent with $q^+ < \infty$ then $W_0^{m,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$ (continuous embedding) if $q(.) \leq p^\star(.) = \frac{Np(.)}{N-p(.)}$. Moreover we have the compact embedding $W_0^{m,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$.
7. Norm convergence and modular convergence are equivalent.

Lemma 2.1. *If $\{u_n\} \subset L^{p(\cdot)}(\Omega), \{v_n\} \subset L^{p'(\cdot)}(\Omega), u_n \to u \in L^{p(\cdot)}(\Omega)$ in $L^{p(\cdot)}(\Omega)$ and $v_n \to v$ a.e. and for the weak topology $\sigma(L^{p'(\cdot)}, L^{p(\cdot)})$ with $v \in L^{p'(\cdot)}(\Omega)$, then $u_n v_n \to uv$ in $L^1(\Omega)$.*

Lemma 2.2. *If $\{u_n\} \subset L^{p(\cdot)}(\Omega), u_n \to u$ a.e. with $\in L^{p(\cdot)}(\Omega)$ and $u_n \rightharpoonup u$ in $L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, then $u_n v \to uv$ in $L^1(\Omega)$.*

Lemma 2.3. *If $\{u_n\} \subset L^{p(\cdot)}(\Omega)$ with $u_n \to u$ in $L^{p(\cdot)}(\Omega)$, then*
$$\int_\Omega |u_n(x)|^{p(x)}\, dx \to \int_\Omega |u(x)|^{p(x)}\, dx.$$

Lemma 2.4. *[9,17]*

(i) *If* $\{u_n\} \subset L^1(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^1(\Omega)$, $u_n, u \geq 0$ *a.e. and* $\int_\Omega u_n(x)\,dx \to \int_\Omega u(x)\,dx$ *, then* $u_n \to u$ *in* $L^1(\Omega)$.

(ii) *If* $\{u_n\} \subset L^1(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^1(\Omega)$, $\int_\Omega u_n(x)\,dx \to \int_\Omega u(x)\,dx$ *, and* $u_n(x) \geq -h(x)$ *a.e. for some* $h \in L^1(\Omega)$ *, then* $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 2.5. **(i)** *If* $\{u_n\} \subset L^{p(.)}(\Omega)$ *with* $u_n \to u$ *a.e. with* $u \in L^{p(.)}(\Omega)$, $u_n, u \geq 0$ *a.e. and* $|cu_n(x)|^{p(x)} \leq h(x)$ *a.e. for some* $h \in L^1(\Omega)$ *and* $c > 0$ *then* $u_n \to u$ *in* $L^{p(.)}(\Omega)$.

(ii) *If* $\{u_n\} \subset L^{p(.)}(\Omega)$ *with* $u_n \to u \in L^{p(.)}(\Omega)$*, then there exists a subsequence* $\{u_{n'}\}, c > 0$ *and* $h \in L^1(\Omega)$ *such that* $u_{n'}(x) \to u(x)$ *a.e.*
and $|cu_{n'}(x)|^{p(x)} \leq h(x)$ *a.e.*

2.2. Some classes of mappings of monotone type

Let $Y = W_0^{m,p(.)}(\Omega)$ and $Z = Y^* = W^{-m,p'(.)}(\Omega)$ and a mapping $F : D_F \subset Y \to Z$.

(i) F is bounded, denote $F \in (BD)$, if the set $F(A) \subset Z$ is bounded when $A \subset D_F$ is bounded.

(ii) F is *strongly quasibounded* , denote $F \in (QB)$, if the conditions
$\{u_n\} \subset D_F$ bounded and $limsup_{n\to\infty}\langle F(u_n), u_n - \bar{u}\rangle$ is bounded from above for some $\bar{u} \in Y_0$
imply that $\{F(u_n)\}$ is bounded in Z .

(iii) f is *continuous* , denote $F \in (CONT)$, if the conditions $\{u_n\} \subset D_F$, $u \in D_F$ and $\|u_n - u\|_Y \to 0$
imply that $\|F(u_n) - F(u)\|_Z \to 0$.

(iv) F is of the class (S_+), denote $F \in (S_+)$,if the conditions
$\{u_n\} \subset D_F$, $u_n \rightharpoonup u \in Y$ in Y and $limsup_{n\to\infty}\langle F(u_n), u_n - u\rangle \leq 0$
imply that $u \in D_F$, and $\|u_n - u\|_Y \to 0$.

(v) F is *pseudomonotone*, $F \in (PM)$, if the conditions
$\{u_n\} \subset D_F$, $u_n \rightharpoonup u$ in Y , $F(u_n) \rightharpoonup z$ in Z and
$limsup_{n\to\infty}\langle F(u_n), u_n\rangle \leq \langle z, u\rangle$
imply that $u \in D_F$, $z = F(u)$ and $\langle F(u_n), u_n\rangle \to \langle F(u), u\rangle$.

(vi) F is of the class (MOD), denote $F \in (MOD)$, if the conditions
$\{u_n\} \subset D_F$, $u_n \rightharpoonup u$ in Y , $F(u_n) \rightharpoonup z$ in Z and
$limsup_{n\to\infty}\langle F(u_n), u_n\rangle \leq \langle z, u\rangle$
imply that $u \in D_F$, $z = F(u)$ and there exists a subsequence $\{u_{n'}\}$ such that $u_{n'} \to u$ (mod) in Y and $F(u_{n'}) \to F(u)$ (mod) in Z.

Theorem 2.6. **(i)** $(S_+) \cap (CONT) \subset (MOD)$.

(ii) $(MOD) \subset (PM)$.

Proof. The same as in [17, Theorem 3.1]. □

2.3. Degree theory in Generalized Sobolev spaces

Let $Y = W_0^{m,p(.)}(\Omega)$ and $Z = Y^* = W^{-m,p'(.)}(\Omega)$. We define the class $\mathcal{F}$ of *admissible mappings* and the class $\mathcal{H}$ of *admissible homotopies* as follows:
$F : D_F \subset Y \to Z$ belongs to $\mathcal{F}$, if

(a) F is a strongly quasibounded mapping of the class (MOD).
$F : D_F \subset Y \to Z$ belongs to $\mathcal{F}^a$, if $F \in \mathcal{F}$ and the following conditions hold:

(b) if $\{u_n\} \subset D_F$ is bounded , $t_n \to 0^+$ and $\langle t_nF(u_n), u_n - \bar{u}\rangle$ is bounded from above for some $\bar{u} \in Y$, then $\{t_nF(u_n)\} \subset Z$ is bounded,

(c) if $\{u_n\} \subset D_F$, $u_n \rightharpoonup u \in Y$,$t_n \to 0^+$, $t_nF(u_n) \rightharpoonup z \in Z$ and $limsup\langle t_nF(u_n), u_n\rangle \leq \langle z, u\rangle$, then $\langle t_nF(u_n), u_n\rangle \to \langle z, u\rangle$,

(d) if $\{u_n\} \subset D_F$, $u_n \to u$ in Y, $t_n \to 0^+$, $t_nF(u_n) \rightharpoonup z$ in Z and $limsup\langle t_nF(u_n), u_n\rangle \leq \langle z, u\rangle$, then $t_nF(u_n) \to 0$ in Z.

The homotopy $H : D_H \to Z$ belongs to $\mathcal{H}$, if H is a strongly quasibounded homotopy of the class (MOD).

Lemma 2.7. *If $F, G \in \mathcal{F}^a$, then $H(t, u) = tF(u) + (1 - t)G(u)$ belongs to $\mathcal{H}$ with*

$$D_{H_t} = \begin{cases} D_F \cap D_G, & \text{if } 0 < t < 1 \\ D_G, & \text{if } t = 0 \\ D_F, & \text{if } t = 1. \end{cases}$$

Proof. The same as in [17, p.30,31] □

Theorem 2.8. *For $F \in \mathcal{F}$, $G \subset Y$ open and bounded in Y, $f \in Z$ and $f \notin F(\partial_Y G)$ there exists an integer $d(F, G, f)$ (which is the degree function) satisfying the conditions:*

1. *(Existence) if $d(F, G, f) \neq 0$, then $f \in F(G)$,*
2. *(Additivity) if $G_1, G_2 \in G$ are open and bounded, $f \notin F(\bar{G} \setminus (G_1 \cup G_2))$, $G_1 \cap G_2 = \emptyset$, then*
$$d(F, G, f) = d(F, G_1, f) + d(F, G_2, f),$$
3. *(Homotopy invariance) if $H \in \mathcal{H}$,$f \in Z$ and $f \notin H([0, 1] \times \partial_Y G)$,then*
$$d(H(t, .), G, f) = \text{constant for all } t \in [0, 1],$$
4. *(Normalization) There exists a normalising map $K \in \mathcal{F}^a$ such that if $f \in Z$,$f \notin K(\partial_Y G)$ and $f \in K(G)$, then*
$$d(K, G, f) = 1.$$

Any mapping $K \in \mathcal{F}^a$ *satisfying*

$$\langle K(u), u\rangle > 0, \text{ when } u \neq 0, \text{ and } K(0) = 0$$

can be chosen as a normalising map.

Proof. The construction of the degree function is the same as in [17] where we consider $(W_0^{m,p(.)}, W_0^{m,p(.)}, W^{-m,p'(.)}, W^{-m,p'(.)})$ as a complementary system. □

Using the conditions $(1)-(4)$ for the degree function, we can deduce, as in [17], some standard properties of the degree.

Proposition 2.9. *Let* $F, T \in \mathcal{F}^a, G \subset Y$ *open and bounded in* Y, $F/\partial_Y G = T/\partial_Y G$ *and* $f \in Z$. *If* $f \notin F(\partial_Y G)$, *then* $d(F, G, f) = d(T, G, f)$.

Proposition 2.10. *If* $F \in \mathcal{F}$ *and* $G \subset Y$ *is an open and bounded in* Y, *then* $d(F, G, .)$ *is constant on each open component in* Z *of the open set* $Z \setminus F(\partial_Y G)$.

Proposition 2.11. *Let* $F \in \mathcal{F}$, $G \subset Y$ *open and bounded in* Y *and* $u_0 \in G$. *Define a mapping* $s : Y \to Y, s(u) = u - u_0$. *If* $0 \notin F(\partial_Y G)$, *then*

$$d(F, G, 0) = d(Fos^{-1}, s(G), 0).$$

3. Differential Operators in Generalized Sobolev Spaces

3.1. Mapping of class (MOD)

Let m be a positif integer. Denote
$N_1 = \sum_{|\alpha|\leq m-1} 1$, $N_2 = \sum_{|\alpha|=m} 1$ and $N_0 = N_1 + N_2$. Let $A_\alpha(x, \xi)$ be functions which satisfy the conditions:

(A_1) $A_\alpha : \Omega \times \mathbb{R}^{N_0} \to \mathbb{R}$ is a Caratheodory function for all $|\alpha| \leq m$.

(A_2) There exist an exponent $q(.)$, $(q(x) \in (1, \infty)$ with $q << p$
(i.e. $\inf_{x\in\Omega}(p(x) - q(x)) > 0$), $a_\alpha \in L^{p'(.)}(\Omega)$ and constants $c_1, c_2 > 0$ such that:
$|A_\alpha(x, \xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2\xi_\beta|^{p(x)-1} + c_1 \sum_{|\beta|<m} |c_2\xi_\beta|^{\frac{p(x)}{q'(x)}}$
when $|\alpha| = m$,
$|A_\alpha(x, \xi)| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2\xi_\beta|^{\frac{q(x)}{p'(x)}} + c_1 \sum_{|\beta|<m} |c_2\xi_\beta|^{p(x)-1}$
when $|\alpha| < m$,
for all $\xi \in \mathbb{R}^{N_0}$ and a.e. $x \in \Omega$.

(A_3) $\sum_{|\alpha|=m}(A_\alpha(x, \eta, \rho) - A_\alpha(x, \eta, \rho')).(\rho_\alpha - \rho'_\alpha) > 0$ a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}^{N_1}$ and $\rho, \rho' \in \mathbb{R}^{N_2}, \rho \neq \rho'$.

(A_4) There exist functions $b_\alpha \in L^{p'(.)}(\Omega)$ for $|\alpha| = m$, $b \in L^1(\Omega)$ constants $d_1, d_2 > 0$ and some fixed element $\phi \in W_0^{m,p(.)}(\Omega)$ such that:

$$\sum_{|\alpha|=m} A_\alpha(x, \xi)(\xi_\alpha - D^\alpha\phi(x)) \geq d_1 \sum_{|\alpha|=m} |d_2\xi_\alpha|^{p(x)} - \sum_{|\alpha|=m} b_\alpha(x)\xi_\alpha - b(x)$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{N_0}$.

We denote in the following $\xi(u) = (D^\alpha u)_{|\alpha|\le m}$ and $\eta(u) = (D^\alpha u)_{|\alpha|<m}$. Define a mapping $A : W_0^{m,p(.)}(\Omega) \to W^{-m,p'(.)}(\Omega)$ by

$$\langle A(u), v\rangle = \sum_{|\alpha|\le m} \int_\Omega A_\alpha(x, \xi(u)) D^\alpha v(x)\, dx \text{ for all } v \in W_0^{m,p(.)}(\Omega). \tag{3.1}$$

Assume that the conditions $(A_1) - (A_4)$ hold, then the operator A defined by (3.1) is continuous and of class (S_+) (the proof is the same as in [10, Proposition 27]). By theorem 2.6, $T \in (MOD)$ and consequently A is pseudomonotone.

Lemma 3.1. *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$ is bounded, $\{t_n\} \subset [0, 1]$ and $\{\langle t_n A(u_n), u_n - \bar{u}\rangle\}$ is bounded for some $\bar{u} \in W_0^{m,p(.)}(\Omega)$, then the sequence $\{t_n A_\alpha(x, \xi(u_n))\}$ is bounded in $L^{p'(.)}(\Omega)$ for all $|\alpha| \le m$.*

Proof. For $|\alpha| < m$ we use the fact that $q << p$, which implies that for every $\epsilon > 0$ there exists a constant $k(\epsilon)$ such that $|t|^{q(x)} \le k(\epsilon)|\epsilon t|^{p(x)}$ for all $t > 0$. Therefore, by (A_2)

$$\begin{aligned} t_n|A_\alpha(x, \xi(u_n))| \quad &\le \quad a_\alpha(x) + c_1 \sum_{|\beta|=m} |k(\epsilon)|\epsilon c_2 D^\beta(u_n)|^{p(x)}|^{\frac{1}{p'(x)}} \\ &\quad + c_1 \sum_{|\beta|<m} |c_2 D^\beta(u_n)|^{p(x)-1}. \end{aligned}$$

When ϵ is sufficiently small, $\|\epsilon c_2 D^\beta(u_n)\|_{p(.)} \le 1$ uniformly for all $|\beta| \le m$,

$$\| |k(\epsilon)|\epsilon c_2 D^\beta(u_n)|^{p(x)}|^{\frac{1}{p'(x)}} \|_{p'(.)} \le 1 + k(\epsilon) \int_\Omega |\epsilon c_2 D^\beta(u_n)|^{p(x)}\, dx$$

and

$$\| |c_2 D^\beta(u_n)|^{p(x)-1} \|_{p'(.)} \le 1 + \int_\Omega |c_2 D^\beta(u_n)|^{p(x)}\, dx$$

We conclude that

$$\begin{aligned} \|t_n A_\alpha(x, \xi(u_n))\|_{p'(.)} &\le \|a_\alpha(x)\|_{p'(.)} + c_1 \sum_{|\beta|=m} (1 + k(\epsilon))\|\epsilon c_2 D^\beta(u_n)\|_{p(.)} \\ &\quad + c_1 \sum_{|\beta|<m} (1 + \int_\Omega |c_2 D^\beta(u_n)|^{p(x)}\, dx) \\ &\le cst. \end{aligned}$$

To show the same property for $|\alpha| = m$, let $w = (w_\alpha) \in (L^{p(.)}(\Omega))^{N_2}$, by condition (A_3) we have

$$t_n \sum_{|\alpha|=m} (A_\alpha(x, \xi(u_n)) - A_\alpha(x, \eta(u_n), w))(D^\alpha(u_n) - w_\alpha) \ge 0.$$

for all $x \in \Omega$ and hence

$$\int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x,\xi(u_n))(w_\alpha - D^\alpha \bar{u})\,dx \leq$$
$$\langle t_n A(u_n), u_n - \bar{u}\rangle - \int_\Omega \sum_{|\alpha|<m} t_n A_\alpha(x,\xi(u_n))(D^\alpha u_n - D^\alpha \bar{u})\,dx$$
$$+ \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x,\eta(u_n),w)(w_\alpha - D^\alpha u_n)\,dx \tag{3.2}$$

The first term on the right remains bounded by the hypothesis of Lemma and the second one remains bounded by virtue of the previous argument. Moreover, by (A_2),

$$\|A_\alpha(x,\eta(u_n),w)\|_{p'(.)} \leq \|a_\alpha\|_{p'(.)} + c_1 \sum_{|\beta|=m} \||c_2 w_\beta|^{p(x)-1}\|_{p'(.)}$$
$$+ c_1 \sum_{\beta<m} \||k(\epsilon)|\epsilon c_2 D^\beta u_n|^{p(x)}|^{\frac{1}{p'(x)}}\|_{p'(.)}.$$

where

$$\||c_2 w_\beta|^{p(x)-1}\|_{p'(.)} \leq 1 + \int_\Omega |c_2 w_\beta|^{p(x)}\,dx \leq const$$

for all $|\beta| = 1$, since $w_\beta \in L^{p(.)}$. Moreover,

$$\||k(\epsilon)|\epsilon c_2 D^\beta u_n|^{p(x)}|^{\frac{1}{p'(x)}}\|_{p'(.)} \leq 1 + k(\epsilon)\int_\Omega |\epsilon c_2 D^\beta u_n|^{p(x)} \leq const,$$

when ϵ is made sufficiently small. Thus we have shown that $\{A_\alpha(x,\eta(u_n),w)\}$ is bounded in $L^{p'(.)}(\Omega)$, which implies that the third term on the right in (3.2) is also bounded. By the theorem of Banach-Steinhaus, the sequence $\{t_n A_\alpha(x,\xi(u_n))\}$ remains bounded in $L^{p'(.)}(\Omega)$ for every $|\alpha| = m$. □

Lemma 3.2. *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $u_n \to u \in W_0^{m,p(.)}$ in $W_0^{m,p(.)}$, $\{t_n\} \subset [0,1]$, $t_n \to t \in [0,1]$, $t_n A_\alpha(x,\xi(u_n)) \to t A_\alpha(x,\xi(u))(\sigma(L^{p'(.)}, L^{p(.)})$ in $L^{p'(.)}(\Omega)$ and*

$$t_n \sum_{|\alpha|\leq m} A_\alpha(x,\xi(u_n)) D^\alpha u_n \to t \sum_{|\alpha|\leq m} A_\alpha(x,\xi(u)) D^\alpha u \text{ in } L^1(\Omega),$$

then $t_n A_\alpha(x,\xi(u_n)) \to t A_\alpha(x,\xi(u))$ in $L^{p'(.)}(\Omega)$ for all $|\alpha| \leq m$.

Proof. When $|A_\alpha(x,\xi(u_n(x)))| \geq a_\alpha(x)$, we have from the condition that

$$\begin{aligned}
&|\frac{t_n|A_\alpha(x,\xi(u_n(x)))|) - t_n a_\alpha(x)}{\lambda}|^{p'(x)} \\
&\leq |\frac{t_n A_\alpha(x,\xi(u_n(x)))}{\lambda}| \times |\frac{t_n c_1}{\lambda}\sum_{|\beta|\leq m}|c_2 D^\beta u_n(x)|^{p(x)-1}|^{p'(x)-1} \\
&\leq |\frac{t_n A_\alpha(x,\xi(u_n(x)))}{\lambda}| \times |\frac{t_n c_1}{\lambda}\sum_{|\beta|\leq m}|c_2 D^\beta u_n(x)|^{p(x)}|^{\frac{1}{p(x)}} \\
&\leq \frac{c_2}{\lambda}\sum_{|\beta|\leq m} t_n|A_\alpha(x,\xi(u_n)))D^\beta u_n(x)|,
\end{aligned} \tag{3.3}$$

when $\lambda > 0$ is large enough. On the other hand , if $|A_\alpha(x,\xi(u_n(x)))| < a_\alpha(x)$, we have

$$|\frac{t_n|A_\alpha(x,\xi(u_n(x)))| - t_n a_\alpha(x)}{\lambda}|^{p'(x)} \leq |\frac{t_n a_\alpha(x)}{\lambda}|^{p'(x)}.$$

Therefore

$$\begin{aligned}
|\frac{t_n|A_\alpha(x,\xi(u_n(x)))| - t_n a_\alpha(x)}{\lambda}|^{p'(x)} \quad \leq \quad & |\frac{t_n a_\alpha(x)}{\lambda}|^{p'(x)} \\
& + \frac{c_2 t_n}{\lambda}\sum_{|\beta|\leq m}|A_\alpha(x,\xi(u_n(x)))D^\beta u_n(x)|
\end{aligned}$$

a.e. $x \in \Omega$, when $\lambda > 0$ is large enough. If $|\beta| \leq m-1$, then, by the compact imbedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, $D^\beta u_n \to D^\beta u$ in $L^{p(.)}$.By Lemma 2.1, $t_n A_\alpha(x,\xi(u_n))D^\beta u_n \to t A_\alpha(x,\xi(u))D^\beta u$ in $L^1(\Omega)$, when $|\beta| \leq m-1$.
Let $|\alpha|, |\beta| = m$ be arbitrary. Denote $\rho' = (0,0,...,0,\frac{D^\beta u_n(x)}{\lambda},0,...,0)$, where $\frac{D^\beta u_n(x)}{\lambda}$ is the α^{the} coordinate of the vector ρ'. By condition (A_3),

$$t_n\sum_{|\gamma|=m}(A_\gamma(x,\eta(u_n),\rho(u_n)) - A_\gamma(x,\eta(u_n),\pm\rho'))(D^\gamma(u_n) \mp \rho'_\gamma) \geq 0,$$

and hence

$$\begin{aligned}
& t_n\sum_{|\gamma|=m}(A_\gamma(x,\xi(u_n))D^\gamma u_n) \pm \frac{t_n}{\lambda}A_\alpha(x,\eta(u_n),\pm\rho')D^\beta u_n \geq \\
& t_n\sum_{|\gamma|=m}(A_\gamma(x,\eta(u_n),\pm\rho')D^\gamma(u_n) \mp \frac{t_n}{\lambda}A_\alpha(x,\xi(u_n))D^\beta u_n.
\end{aligned} \tag{3.4}$$

Let $|\gamma_1| = |\gamma_2| = m$ be arbitrary. Then

$$|A_{\gamma_1}(x,\eta(u_n),\pm\rho')D^{\gamma_2}u_n| \leq [a_{\gamma_1}(x)+c_1|\frac{c_2}{\lambda}D^\beta u_n|^{p(x)-1}+c_1\sum_{|\delta|\leq m-1}v_n^\delta(x)]|D^{\gamma_2}u_n|, \tag{3.5}$$

where $v_n^\delta(x) = |c_2 D^\delta u_n(x)|^{\frac{p(x)}{q'(x)}}$. Using Lemma 2.5, we obtain $D^\delta u_{n'} \to D^\delta u$ a.e. and $|c_2 D^\delta u_{n'}|^{p(x)} \le h$ for all $|\delta| \le m-1$ a.e. for some $h \in L^1(\Omega)$ and for some subsequence $\{u_{n'}\}$. We have v_n^δ and $v^\delta = |c_2 D^\delta u(x)|^{\frac{p(x)}{q'(x)}}$ belong to the space $L^{q'(\cdot)}(\Omega)$, $v_{n'}^\delta \to v^\delta$ a.e. and $|v_{n'}^\delta|^{q'(x)} \le |c_2 D^\delta u_{n'}|^{p(x)} \le h$ a.e.
By using the Lemma 2.5 we can see that $v_{n'}^\delta \to v^\delta$ in $L^{q'(\cdot)}$. Hence $v_{n'}^\delta \to v^\delta$ in $L^{p'(\cdot)}$. By contradiction argument, $v_n^\delta \to v^\delta$ in $L^{p'(\cdot)}$ and $D^{\gamma_2} u_n \rightharpoonup D^{\gamma_2} u$ in $L^{p(\cdot)}$. Consequently, $a_{\gamma_1} D^{\gamma_2} u_n \to a_\alpha D^{\gamma_2} u$ in L^1 by Lemma 2.2 and $v_n^\delta D^{\gamma_2} u_n \to v^\delta D^{\gamma_2} u$ in L^1 by Lemma 2.1. Moreover,

$$\begin{aligned}
|\frac{c_2}{\lambda} D^\beta u_n|^{p(x)-1} |D^{\gamma_2} u_n| &\le |\frac{c_2}{\lambda} D^\beta u_n|^{p(x)-1} |D^\beta u_n| + |\frac{c_2}{\lambda} D^{\gamma_2} u_n|^{p(x)-1} |D^{\gamma_2} u_n| \\
&\le \frac{2\lambda}{c_2} |\frac{c_2}{\lambda} D^\beta u_n|^{p(x)} + \frac{2\lambda}{c_2} |\frac{c_2}{\lambda} D^{\gamma_2} u_n|^{p(x)} \\
&\le h_\beta + h_\gamma \in L^1(\Omega),
\end{aligned}$$

when λ is large enough, because $D^\beta u_n \to D^\beta u$ and $D^{\gamma_2} u_n \to D^{\gamma_2} u$ in $L^{p(\cdot)}$. Hence, by (3.5),

$$|A_{\gamma_1}(x, \eta(u_n), \pm\rho') D^{\gamma_2} u_n| \le h \in L^1$$

and we obtain from (3.4)

$$t_n |A_\alpha(x, \xi(u_n)) D^\beta u_n| \le h_1 \in L^1.$$

Consequently, we can find that

$$t_n |A_\alpha(x, \xi(u_n))| - t_n a_\alpha(x) \to t |A_\alpha(x, \xi(u))| - t a_\alpha(x) \;\text{ in } L^{p'(\cdot)}$$

by using lemma 2.5 and (3.3). Which implies that

$$t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u)) \text{ in } L^{p'(\cdot)}(\Omega)$$

for every $|\alpha| \le m$. □

Lemma 3.3. *Assume that the conditions $(A_1) - (A_3)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(\cdot)}(\Omega)$, $D^\alpha u_n \to D^\alpha u \in L^{p(\cdot)}$ a.e. and $D^\alpha u_n \rightharpoonup D^\alpha u$ in $L^{p(\cdot)}(\Omega)$ for all $|\alpha| \le m$, $\{t_n\} \subset [0,1]$, $t_n \to t \in [0,1]$,*

$$t_n A_\alpha(x, \xi(u_n)) \to t A_\alpha(x, \xi(u)) (\sigma(L^{p'(\cdot)}, L^{p(\cdot)})$$

in $L^{p'(\cdot)}(\Omega)$ for all $|\alpha| \le m$ and $\langle t_n A(u_n), u_n \rangle \to \langle t A(u), u \rangle$, then

$$t_n \sum_{|\alpha| \le m} A_\alpha(x, \xi(u_n)) D^\alpha u_n \to t \sum_{|\alpha| \le m} A_\alpha(x, \xi(u)) D^\alpha u$$

in $L^1(\Omega)$.

Proof. Denote $f_n(x) = t_n \sum_{|\alpha|=m} A_\alpha(x, \xi(u_n(x))D^\alpha u_n(x)$ and $f(x) = t \sum_{|\alpha|=m} A_\alpha(x, \xi(u(x))D^\alpha u(x)$.
It is enough to prove that $f_n \to f$ in $L^1(\Omega)$ for a subsequence. By lemma 2.1, $a_\alpha D^\alpha u_n \to a_\alpha D^\alpha u$ in $L^1(\Omega)$, when $|\alpha| = m$.Consequently, for every $|\alpha| = m$, there exists $h_\alpha \in L^1(\Omega)$ such that $|a_\alpha(x)D^\alpha u_n(x)| \leq h_\alpha(x)$ a.e. for a subsequence. Denote $v_n^\beta(x) = |c_2 D^\beta u_n(x)|^{\frac{p(x)}{q'(x)}}$ and $v^\beta(x) = |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$, when $|\beta| < m$. On account of compactness of the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, we have $D^\beta u_n \to D^\beta u$ in $L^{p(.)}(\Omega)$, when $|\beta| \leq m-1$, which implies by lemma 2.3 and 2.5 that

$$\||v_n^\beta|^{q'(x)}\|_1 \to \||v^\beta|^{q'(x)}\|_1$$

and

$$|v_{n'}^\beta(x)|^{q'(x)} \to |v^\beta(x)|^{q'(x)} \text{ a.e.}$$

for some subsequence. By lemma 2.4, $|v_{n'}^\beta|^{q'(x)} \to |v^\beta|^{q'(x)}$ in $L^1(\Omega)$, and so for avery $|\beta| < m$ there must exist $h_\beta \in L^1$ such that

$$|v_n^\beta(x)|^{q'(x)} \leq h_\beta(x) \text{ a.e.}$$

Therefore $|v_n^\beta(x)| \leq |h_\beta(x)|^{\frac{1}{q'(x)}} \in L^{q'(.)}(\Omega) \subset L^1(\Omega)$. Condition (A_3) implies that

$$t_n \sum_{|\alpha|=m} (A_\alpha(x, \eta(u_n), \rho(u_n)) - A_\alpha(x, \eta(u_n), \bar{0}))D^\alpha u_n \geq 0.$$

Consequently,

$$\begin{aligned} f_n(x) &\geq \sum_{|\alpha|=m} t_n A_\alpha(x, \eta(u_n), \bar{0}))D^\alpha u_n(x) \\ &\geq -t_n \sum_{|\alpha|=m} (|a_\alpha(x)D^\alpha u_n(x)| + c_1 \sum_{|\beta|<m} |c_2 D^\beta u_n(x)|^{\frac{p(x)}{q'(x)}}) \\ &\geq -\sum_{|\alpha|=m} (h_\alpha(x) + c_1 \sum_{|\beta|<m} |h_\beta(x)|^{\frac{1}{q'(x)}}) = -h(x) \in L^1(\Omega). \end{aligned}$$

Since $D^\alpha u_n \to D^\alpha u$ in $L^{p(.)}$ for $|\alpha| < m$, we know from Lemma 2.1 that

$$t_n \sum_{|\alpha|<m} A_\alpha(x, \xi(u_n))D^\alpha u_n \to t \sum_{|\alpha|<m} A_\alpha(x, \xi(u))D^\alpha u$$

in $L^1(\Omega)$. Moreover, the assumption $\langle t_n A(u_n), u_n\rangle \to \langle tA(u), u\rangle$ implies that

$$\int_\Omega f_n(x)\, dx \to \int_\Omega f(x)\, dx.$$

By lemma 2.4, $f_n \to f$ in $L^1(\Omega)$, which completes the proof. □

Lemma 3.4. *Assume that the conditions* $(A_1)-(A_3)$ *hold. If the sequence* $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $u_n \rightharpoonup u$ *in* $W_0^{m,p(.)}(\Omega)$, $\{t_n\} \subset [0,1]$, $t_n \to t$, $t_nA(u_n) \to z \in W^{-m,p'(.)}(\Omega)(\sigma(W^{-m,p'(.)}, W_0^{m,p(x)})$ *in* $W^{-m,p'(.)}(\Omega)$ *and* $limsup\langle t_nA(u_n), u_n\rangle \le \langle z, u\rangle$, *then*

$$\langle t_nA(u_n), u_n\rangle \to \langle z, u\rangle.$$

Proof. By lemma 3.1, the sequence $\{t_nA_\alpha(x, \xi(u_n)\}$ is bounded in $L^{p'(.)}(\Omega)$ for all $|\alpha| \le m$. We can thus assume that

$$t_nA_\alpha(x, \xi(u_n)) \to h_\alpha \in L^{p'(.)}(\Omega)(\sigma(L^{p'(.)}, L^{p(.)})) \text{ in } L^{p'(.)}(\Omega)$$

for $|\alpha| \le m$. It is clear that

$$\begin{aligned} \langle z, w\rangle &= \lim\langle t_nA(u_n), w\rangle \\ &= \lim \sum_{|\alpha|\le m} \int_\Omega t_nA_\alpha(x, \xi(u_n))D^\alpha w(x)\, dx \\ &= \sum_{|\alpha|\le m} \int_\Omega h_\alpha(x)D^\alpha w(x)\, dx \end{aligned}$$

for all $w \in W_0^{m,p(.)}(\Omega)$. By the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, we have

$$D^\alpha u_n \to D^\alpha u \text{ in } L^{p(.)} \text{ for } |\alpha| < m.$$

Hence

$$\sum_{|\alpha|<m} \int_\Omega t_nA_\alpha(x, \xi(u_n))D^\alpha u_n \to \sum_{|\alpha|<m} \int_\Omega h_\alpha(x)D^\alpha u.$$

Moreover, by the assumption,

$$\limsup \sum_{|\alpha|\le m} \int_\Omega t_nA_\alpha(x, \xi(u_n))D^\alpha u_n \le \sum_{|\alpha|\le m} \int_\Omega h_\alpha D^\alpha u.$$

Therefore

$$\limsup \sum_{|\alpha|=m} \int_\Omega t_nA_\alpha(x, \xi(u_n))D^\alpha u_n \le \sum_{|\alpha|=m} \int_\Omega h_\alpha D^\alpha u,$$

So that it is enough to prove that

$$\liminf \sum_{|\alpha|=m} \int_\Omega t_nA_\alpha(x, \xi(u_n))D^\alpha u_n \ge \sum_{|\alpha|=m} \int_\Omega h_\alpha D^\alpha u.$$

Denote

$$\Omega_k = \{x \in \Omega / |D^\alpha u(x)| \le k \text{ for all } |\alpha| = m\}$$

and

$$E_k(x) = \begin{cases} 1 & , \text{when } x \in \Omega_k \\ 0 & , \text{otherwise.} \end{cases}$$

By condition (A_3) we have

$$\int_\Omega \sum_{|\alpha|=m} t_n[A_\alpha(x, \eta(u_n), E_k(x)\rho(u)) - A_\alpha(x, \xi(u_n))].[E_k(x)D^\alpha u - D^\alpha u_n] \geq 0.$$

Consequently,

$$\begin{aligned} \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \xi(u_n))D^\alpha u_n \quad \geq \quad & -\int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \eta(u_n), E_k(x)\rho(u))E_k(x)D^\alpha u \\ & + \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \xi(u_n))E_k(x)D^\alpha u \\ & + \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x, \eta(u_n), E_k(x)\rho(u))D^\alpha u_n. \quad (3.6) \end{aligned}$$

By compact embedding, $D^\beta u_{n'} \to D^\beta u$ a.e. and $|c_2 D^\beta u_{n'}(x)|^{p(x)} \leq h(x)$ a.e for some $h \in L^1(\Omega)$ and for $|\beta| \leq m-1$ for some subsequence. Consequently,

$$(|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}})^{q'(x)} \leq h(x)\text{a.e.}$$

By lemma 2.5,

$$|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}} \to |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$$

in $L^{q'(.)}(\Omega)$. Since $q << p$, we have

$$|c_2 D^\beta u_{n'}(x)|^{\frac{p(x)}{q'(x)}} \to |c_2 D^\beta u(x)|^{\frac{p(x)}{q'(x)}}$$

in $L^{p'(.)}(\Omega)$. By (A_2),we obtain

$$|A_\alpha(x, \eta(u_n), E_k(x)\eta(u))| \leq a_\alpha(x) + c_1 \sum_{|\beta|=m} |c_2 k|^{p(x)-1} + c_1 \sum_{|\beta|\leq m-1} |c_2 D^\beta u_{n'}|^{\frac{p(x)}{q'(x)}}),$$

for $|\alpha| = m$. Now the right-hand side converges in $L^{p'(.)}(\Omega)$ and the left-hand side converges a.e.,so that it is easy to deduce that the left-hand side also converges in $L^{p'(.)}(\Omega)$. The first term on the right in (3.6) therefore tend towards

$$-\int_\Omega \sum_{|\alpha|=m} t A_\alpha(x, \eta(u), E_k(x)\eta(u)E_k(x)D^\alpha u,$$

and the third term on the right in (3.6) will tend towards

$$\int_\Omega \sum_{|\alpha|=m} t A_\alpha(x, \eta(u), E_k(x)\rho(u))D^\alpha u,$$

when n approaches infinity. Consequently,

$$\begin{aligned}\liminf \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x,\xi(u_n))D^\alpha u_n &\geq \liminf \int_\Omega \sum_{|\alpha|=m} t_n A_\alpha(x,\xi(u_n))E_k(x)D^\alpha u \\ &\quad + \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|=m} tA_\alpha(x,\eta(u),\bar{0})D^\alpha u \\ &= \sum_{|\alpha|=m} \int_{\Omega_k} h_\alpha D^\alpha u \\ &\quad + \int_{\Omega\setminus\Omega_k} \sum_{|\alpha|=m} tA_\alpha(x,\eta(u),\bar{0})D^\alpha u,\end{aligned}$$

as $E_k(x)D^\alpha u \in L^{p(.)}(\Omega)$. Letting $k \to \infty$ we prove the lemma, since $h_\alpha D^\alpha u \in L^1(\Omega)$ and $tA_\alpha(x,\eta(u),\bar{0})D^\alpha u \in L^1(\Omega)$ for $|\alpha| = m$. □

In the sequel we shall use the following well-known fact: if $u_n \rightharpoonup v$ in $L^{p(.)}(\Omega)$ and $u_n \to w$ a.e. in Ω, then $v = w$ a.e. in Ω.

Lemma 3.5. *Assume that the conditions $(A_1)-(A_4)$ hold. If the sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$, $u_n \rightharpoonup u \in W_0^{m,p(.)}(\Omega)$ in $W_0^{m,p(.)}(\Omega)$, $A(u_n) \to z \in W^{-m,p'(.)}(\Omega)(\sigma(W^{-m,p'(.)}, W_0^{m,p(.)})$ in $W^{-m,p'(.)}(\Omega)$ and $limsup\langle A(u_n), u_n\rangle \leq \langle z, u\rangle$, then $u_{n'} \to u$ in $W_0^{m,p(.)}(\Omega)$ for some subsequence.*

Proof. We deduce as in [13] that $D^\alpha u_n(x) \to D^\alpha u(x)$ a.e. for $|\alpha| \leq m$ and for some subsequence. According to lemma 3.1 we may assume that

$$A_\alpha(x,\xi(u_n)) \to w_\alpha(x)(\sigma(L^{p'(.)}, L^{p(.)})) \text{ in } L^{p'(.)}$$

for every $|\alpha| \leq m$. Since

$$A_\alpha(x,\xi(u_n)) \to A_\alpha(x,\xi(u)) \text{ a.e.,}$$

we know that $w_\alpha(x) = A_\alpha(x,\xi(u))$ a.e. A is pseudomonotone.
Hence

$$z = A(u) \text{ and } \langle A(u_n), u_n\rangle \to \langle A(u), u\rangle.$$

By lemma 3.3,

$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi(u_n))D^\alpha u_n \to \sum_{|\alpha|\leq m} A_\alpha(x,\xi(u))D^\alpha u$$

in $L^1(\Omega)$, and by condition (A_4),

$$\begin{aligned}d_1 \sum_{|\alpha|=m} |d_2 D^\alpha u_n(x)|^{p(x)} &\leq \sum_{|\alpha|=m} A_\alpha(x,\xi(u_n(x))D^\alpha u_n(x) \\ &\quad - \sum_{|\alpha|=m} A_\alpha(x,\xi(u_n(x))D^\alpha \varphi(x) \\ &\quad + \sum_{|\alpha|=m} b_\alpha(x)D^\alpha u_n(x) + b(x).\end{aligned}$$

The right hand side converges in $L^1(\Omega)$ in accordance with Lemma 2.1. Lemma 2.5 implies $D^\alpha u_n \to D^\alpha u$ in $L^{p(.)}(\Omega)$ for all $|\alpha| = m$. By compact embedding, $D^\alpha u_n \to D^\alpha u$ in $L^{p(.)}$, when $|\alpha| < m$, which completes the proof. □

Theorem 3.6. *If the conditions $(A_1) - (A_4)$ hold, then the mapping A defined by (3.1) belong to the class $\mathcal{F}^a$.*

Proof. Strong quasiboundedness and condition b) of class $\mathcal{F}^a$ follow immediately from Lemma 3.1.
Lemma 3.4 implies condition c), and condition d) follows from lemmas 3.4, 3.3 and 3.2. Suppose that
$\{u_n\} \subset W_0^{m,p(.)}(\Omega), u_n \rightharpoonup u$ in $W_0^{m,p(.)}(\Omega), A(u_n) \to z \in Z(\sigma(W^{-m,p'(.)}, W_0^{m,p(.)}))$ in $W^{-m,p'(.)}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n\rangle \leq \langle z, u\rangle$.
By pseudomonotonicity, $z = A(u)$ and $\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle$. If (A_4) holds, then, by Lemma 3.5, $u_n \to u$ in $W_0^{m,p(.)}(\Omega)$ for some subsequence. Choosing $t_n = 1$ in Lemma 3.3 and Lemma 3.2 we may deduce that $A(u_n) \to A(u)$ (mod) in $W^{-m,p'(.)}(\Omega)$. Hence $A \in (MOD)$. □

3.2. Normalising maps

Let $Y = W_0^{m,p(.)}(\Omega)$, $Z = W^{-m,p'(.)}(\Omega)$. We start with an abstract existence theorem.

Theorem 3.7. *Let $G \subset Y$ be open and bounded in Y, $0 \in G$, $f \in Z$ and $F \in \mathcal{F}^a$. Suppose that there exists a normalising map $K \in \mathcal{F}^a$ such that $K(0) = 0$ and $\langle K(u), u\rangle \geq b > 0$ for all $u \in \partial_Y G$. Choose a constant $a \geq 0$ such that*

$$a \leq \inf_{u\in\partial_Y G} \frac{\langle K(u), u\rangle}{\|K(u)\|}.$$

If $\langle F(u) - f, u\rangle + \|F(u) - f\|a > 0$ for all $u \in \partial_Y G$, then $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. Since $\langle K(u), u\rangle \geq b > 0$ for all $u \in \partial_Y G$, it is clear that $0 \notin \overline{K(\partial_Y G)}$. Define by

$$H(t, u) = tF(u) + (1 - t)K(u).$$

a homotopy which belong to the class $\mathcal{H}$ by Lemma 2.7. We show that $H(t, u) \neq 0$ for all $t \in [0, 1]$,$u \in \partial_Y G$. If $0 \in H([0, 1] \times \partial_Y G)$, then

$$tF(u) + (1 - t)K(u) = 0$$

for some $u \in \partial_Y G$ and $t \in [0, 1]$. It is clear that $t \neq 0$. Thus

$$t||F(u)|| = (1 - t)||K(u)||,$$

implying

$$1 - \frac{1}{t} = -\frac{||F(u)||}{||K(u)||}.$$

On the other hand,

$$t\langle F(u), u\rangle + (1-t)\langle K(u), u\rangle = 0,$$

and therefore

$$\langle F(u), u\rangle = (1-\frac{1}{t})\langle K(u), u\rangle = -||F(u)||\frac{\langle K(u), u\rangle}{||K(u)||} \leq -||F(u)||a,$$

which is a contradiction. Hence $H(t,u) \neq 0$ for all $t \in [0,1]$ and $u \in \partial_Y G$. By homotopy invariance,

$$d(F, G, 0) = f(K, G, 0).$$

Since $0 \in K(G)$, we have $d(K, G, 0) = 1$. □

Corollary 3.8. *Let $G \subset Y$ be open and bounded in Y, $\bar{u} \in G$, $f \in Z$ and $F \in \mathcal{F}^a$. Suppose that there exists a normalising map $K \in \mathcal{F}^a$ such that $\tilde{K}(u) = K(u+\bar{u}) - K(\bar{u})$ is also a normalising map in the class $\mathcal{F}^a$ and*

$$\langle K(u) - K(\bar{u}), u - \bar{u}\rangle \geq b > 0 \text{ for all } u \in \partial_Y G.$$

Choose a constant $a \geq 0$ such that

$$a \leq \inf_{u \in \partial_Y G} \frac{\langle K(u) - K(\bar{u}), u - \bar{u}\rangle}{\|K(u) - K(\bar{u})\|}.$$

If $\langle F(u) - f, u - \bar{u}\rangle + \|F(u) - f\|a > 0$ for all $u \in \partial_Y G$, then $d(F, G, f) = 1$.

Proof. We may assume that $f = 0$. It follows from the above assumptions that the degree $d(F, G, 0)$ is defined. Define $s(u) = u - \bar{u}$. Then, by property 2.11,

$$d(F, G, 0) = d(Fos^{-1}, s(G), 0).$$

Let $u \in \partial_Y s(G)$ be arbitrary. Denote $u = u' - \bar{u}$, where $u' \in \partial_Y G$.
Now $\langle \tilde{K}(u), u\rangle = \langle K(u') - K(\bar{u}), u' - \bar{u}\rangle \geq b > 0$ and

$$\frac{\langle \tilde{K}(u), u\rangle}{||\tilde{K}(u)||} = \frac{\langle K(u') - K(\bar{u}), u - \bar{u}\rangle}{\|K(u') - K(\bar{u})\|} \geq a \geq 0.$$

Moreover,

$$\langle Fos^{-1}(u), u\rangle + ||Fos^{-1}(u)||a = \langle F(u'), u' - \bar{u}\rangle + ||F(u')||a.$$

Therefore the assumptions of Theorem 3.7 hold. Hence

$$d(Fos^{-1}(u), s(G), 0) = 1.$$

□

Next we shall present two mappings of class $\mathcal{F}^a$, which can be used as normalising maps.

Theorem 3.9. *The mapping* $K : W_0^{m,p(.)}(\Omega) \to W^{-m,p'(.)}(\Omega)$ *defined by*

$$\langle Ku, v\rangle = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)-1} sgn D^\alpha u(x) D^\alpha v(x)\, dx \text{ for all } v \in W_0^{m,p(.)}(\Omega),$$

belongs to the class $\mathcal{F}^a$, $\langle K(u), u\rangle > 0$ *for* $u \in W_0^{m,p(.)}(\Omega)$, $u \neq 0$ *and* $K(0) = 0$.

Proof. Denote $A_\alpha(x, \eta, \rho) = |\rho_\alpha|^{p(x)-1} sgn\rho_\alpha$ when $|\alpha| = m$ and $A_\alpha(x, \eta, \rho) = 0$ when $|\alpha| \leq m - 1$. The mappings A_α are clearly continuous with respect to η and ρ. Hence condition (A_1) is satisfied. It is obvious that condition (A_2) holds. Since the function $|\rho_\alpha|^{p(x)-1} sgn\rho_\alpha$ is strictly increasing, we obtain condition (A_3). Condition (A_4) is reduced to

$$\sum_{|\alpha|=m} A_\alpha(x, \xi).\xi_\alpha \geq \sum_{|\alpha|=m} |\xi_\alpha|^{p(x)},$$

when we choose $\varphi \equiv 0$,$b_\alpha = b = 0$ and $d_1 = d_2 = 1$. Moreover,

$$\begin{aligned} \langle K(u), u\rangle &= \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)-1} sgn D^\alpha u(x) D^\alpha u(x)\, dx \\ &= \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^{p(x)}\, dx \geq 0, \end{aligned}$$

and the equality holds if and only if $D^\alpha u(x) = 0$ a.e. for every $|\alpha| = m$, which implies that $u \equiv 0$. □

Theorem 3.10. *The mapping, so-called* $p(.)$*-Laplacian,*

$$K : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega)$$

defined by

$$K(u) = -\Delta_{p(.)}(u) = -div(|\nabla u(x)|^{p(x)-2} \nabla u(x)).$$

i.e.

$$\langle Ku, v\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x).\nabla v(x)\, dx \text{ for all } v \in W_0^{1,p(.)}(\Omega),$$

belongs to the class $\mathcal{F}^a$, $\langle K(u), u\rangle > 0$ *for* $u \in W_0^{1,p(.)}(\Omega)$, $u \neq 0$ *and* $K(0) = 0$.

Proof. Denote $A_i(x, \eta, \rho) = |\rho|^{p(x)-2}\rho_i, i = 1, 2, ..., N$, when $\rho \neq \bar{0}, A_i(x, \eta, \bar{0}) = 0$. Since $|A_i(x, \eta, \rho)| \leq |\rho|^{p(x)-1}$, the function A_i is continuous with respect to η and ρ. Hence condition (A_1) is satisfied. Condition (A_2) is easily seen to hold. Let $f(t) = |t|^{p(x)-1}/t$, when $t \neq 0$, $f(0) = 0$. The function $f(t)t$ is strictly increasing in $[0, \infty)$, which gives

$$(f(|\rho|)|\rho| - f(|\rho'|)|\rho'|)(|\rho| - |\rho'|) > 0,$$

when $\rho, \rho' \in \mathbb{R}^{N_2}, |\rho| \neq |\rho'|$. Hence

$$\sum_{i=1}^{N}[f(|\rho|){\rho_i}^2 + f(|\rho'|){\rho_i'}^2] - [f(|\rho|) + f(|\rho'|)]|\rho||\rho'| > 0.$$

By the Cauchy-Schwarz inequality,$|\rho||\rho'| \geq \sum_{i=1}^{N} \rho_i \rho_i'$, implying

$$\sum_{i=1}^{N}[f(|\rho|){\rho_i}^2 + f(|\rho'|){\rho_i'}^2] - \sum_{i=1}^{N}[f(|\rho|) + f(|\rho'|)]\rho_i \rho_i' > 0.$$

Consequently,

$$\sum_{i=1}^{N}[f(|\rho|)\rho_i - f(|\rho'|)\rho_i'][\rho_i - \rho_i'] > 0,$$

when $|\rho| \neq |\rho'|$. If $|\rho| = |\rho'|$, we have equality in the Cauchy-Schwarz inequality only if $\rho = \rho'$, and hence strict inequality follows. Therefore condition (A_3) holds. Condition (A_4) follows as in Theorem 3.9. Moreover,

$$\langle K(u), u\rangle = \int_\Omega |\nabla u(x)|^{p(x)}\, dx \geq 0,$$

and equality holds only if $\nabla u(x) = 0$ a.e., which implies that $u = 0$ a.e. □

The previous theorems imply the existence of a normalising map in Sobolev space with variables exponents which satisfy the conditions of the previous section. We shall equip the space $Y = W_0^{m,p(.)}(\Omega)$ with the norm

$$||u||_Y = \sqrt{\sum_{|\alpha|=m} ||D^\alpha u||^2_{p(.)}}.$$

Let $G \subset W_0^{m,p(.)}(\Omega)$ be open and bounded in $W_0^{m,p(.)}(\Omega)$ and $0 \in G$. Since the set $\partial_Y G$ is closed, we have

$$\inf_{u \in \partial_Y G} ||u||_Y > 0.$$

We may therefore choose a constant $c \in \mathbb{R}$ such that

$$0 < c < \frac{1}{\sqrt{N_2}} \inf_{u \in \partial_Y G} ||u||_Y.$$

Moreover, we have

$$\begin{aligned} c &< \frac{1}{\sqrt{N_2}}\sqrt{\sum_{|\alpha|=m} ||D^\alpha u||^2_{p(.)}} - \epsilon \\ &\leq \frac{1}{\sqrt{N_2}}\sqrt{N_2 \max_{|\alpha|=m}\{||D^\alpha u||^2_{p(.)}\}} - \epsilon \\ &= \max_{|\alpha|=m}\{||D^\alpha u||_{p(.)}\} - \epsilon \end{aligned}$$

for some $\epsilon > 0$ and for all $u \in \partial_Y G$. Define a mapping $K \in \mathcal{F}^a$,

$$Ku(x) = \sum_{|\alpha=m} (-1)^{|\alpha|} D^\alpha (|\frac{D^\alpha u(x)}{c}|^{p(x)-1} sgn D^\alpha u(x)).$$

Let $u \in \partial_Y G$ and $||D^{\alpha_0} u||_{p(.)} = \max_{|\alpha|=m} \|D^\alpha u\|_{p(.)}$. Now

$$\begin{aligned}\langle K(u), u\rangle &= c\sum_{|\alpha|=m}\int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)-1}|\frac{D^\alpha u(x)}{c}|\,dx\\ &= c\sum_{|\alpha|=m}\int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)}\,dx\\ &\geq c\int_\Omega |\frac{D^{\alpha_0} u(x)}{||D^{\alpha_0}u||_{p(.)}-\epsilon}|^{p(x)}\,dx\\ &\geq c.\end{aligned}$$

Moreover,

$$\begin{aligned}\|K(u)\|_Z = \sup_{\|v\|_Y\leq 1}\langle K(u), v\rangle &= \sup_{\|v\|_Y\leq 1}\sum_{|\alpha|=m}\int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)-1} sgn D^\alpha u(x) D^\alpha v(x)\,dx\\ &\leq \sup_{\|v\|_Y\leq 1} 2\sum_{|\alpha|=m}\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)}\|D^\alpha v\|_{p(.)}\\ &\leq 2\sum_{|\alpha|=m}\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)}\end{aligned}$$

Consequently,

$$\frac{\langle K(u), u\rangle}{\|K(u)\|_Z} \geq \frac{c\sum_{|\alpha|=m}\int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)}\,dx}{2\sum_{|\alpha|=m}\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)}}.$$

If $\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)} \leq 1$ for all $|\alpha| = m$, then

$$\frac{\langle K(u), u\rangle}{\|K(u)\|_Z} \geq \frac{c}{2N_2}\int_\Omega |\frac{D^{\alpha_0} u(x)}{c}|^{p(x)} \geq \frac{c}{2N_2}.$$

If

$$\max_{|\alpha|=m}\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)} = \||\frac{D^{\alpha_1} u(x)}{c}|^{p(x)-1}\|_{p'(.)} > 1$$

for some $|\alpha_1| = m$, then we obtain

$$\int_\Omega |\frac{D^{\alpha_1} u(x)}{c}|^{p(x)}\,dx = \int_\Omega ||\frac{D^{\alpha_1} u(x)}{c}|^{p(x)-1}|^{p'(x)}\,dx \geq \||\frac{D^{\alpha_1} u(x)}{c}|^{p(x)-1}\|_{p'(.)}.$$

Hence

$$\frac{\langle K(u), u\rangle}{\|K(u)\|_Z} \geq \frac{c\sum_{|\alpha|=m}\int_\Omega |\frac{D^\alpha u(x)}{c}|^{p(x)}\,dx}{2\sum_{|\alpha|=m}\||\frac{D^\alpha u(x)}{c}|^{p(x)-1}\|_{p'(.)}} \geq \frac{c\int_\Omega |\frac{D^{\alpha_1} u(x)}{c}|^{p(x)}\,dx}{2N_2\||\frac{D^{\alpha_1} u(x)}{c}|^{p(x)-1}\|_{p'(.)}} \geq \frac{c}{2N_2}.$$

In other words, we have constructed a normalising map K for which the conditions of this section hold. For example, if $G = B_R(0)$, we may choose $c = \frac{R}{2\sqrt{N_2}}$, $b = \frac{R}{2\sqrt{N_2}}, a = \frac{R}{4N_2\sqrt{N_2}}$ and

$$Ku(x) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (|\frac{D^\alpha u(x)}{c}|^{p(x)-1} sgn D^\alpha u(x))$$

in Theorem 3.7. While, if $G = B_R(\bar{u})$ for some $\bar{u} \in W_0^{m,p(.)}$, we may choose a, b and c as above and

$$\begin{aligned} Ku(x) &= \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (|\frac{D^\alpha u(x) - D^\alpha \bar{u}}{c}|^{p(x)-1} sgn D^\alpha u(x))) \\ &+ \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (|\frac{D^\alpha \bar{u}(x)}{c}|^{p(x)-1} sgn D^\alpha \bar{u}(x)) \end{aligned}$$

in Corollary 3.8.

3.3. Existence results

Theorem 3.11. *Assume that the conditions $(A_1)-(A_4)$ hold. Define the mapping A as in (3.1). Let $f \in W^{-m,p'(.)}(\Omega)$. If*

$$\liminf_{\|u\|_{m,p(.)}\to\infty} \langle A(u) - f, u - \bar{u}\rangle \geq 0 \text{ for some } \bar{u} \in W_0^{m,p(.)}(\Omega), \tag{3.7}$$

then the problem (1.1) is almost solvable, i.e., $f \in \overline{A(W_0^{m,p(.)}(\Omega))}$. If

$$\liminf_{\|u\|_{m,p(.)}\to\infty} \langle A(u) - f, u - \bar{u}\rangle > 0 \text{ for some } \bar{u} \in W_0^{m,p(.)}(\Omega), \tag{3.8}$$

then the problem (1.1) is solvable, i.e., $f \in A(W_0^{m,p(.)}(\Omega))$.

Proof. Suppose first that (3.7) holds. By theorem 3.6, $A \in \mathcal{F}^a$. If

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R} (\langle A(u) - f, u - \bar{u}\rangle + \|A(u) - f\| \frac{R}{4N_2\sqrt{N_2}}) \leq 0 \text{ for all } R > 0,$$

then, by (3.7), there exists a sequence $\{u_n\} \subset W_0^{m,p(.)}(\Omega)$ such that $\|u_n\|_{m,p(.)} \to \infty$ and $\|A(u_n) - f\| \to 0$. Hence $f \in \overline{A(W_0^{m,p(.)}(\Omega))}$.Suppose that

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R} (\langle A(u) - f, u - \bar{u}\rangle + \|A(u) - f\| \frac{R}{4N_2\sqrt{N_2}}) > 0 \text{ for some } R > 0.$$

As indicated in the previous section, there exists a normalising map K satisfying the assumptions of Corollary 3.8 with $b = \frac{R}{2\sqrt{N_2}}$ and $a = \frac{R}{4N_2\sqrt{N_2}}$. Denote $B_R(\bar{u}) = \{u \in W_0^{m,p(.)}(\Omega) / \|u - \bar{u}\|_Y \leq R\}$. By Corollary 3.8, $d(A, B_R(\bar{u}), f) = 1$. By property (1) of the degree function, $f \in A(B_R(\bar{u})) \subset \overline{A(W_0^{m,p(.)}(\Omega))}$. If (3.8) holds, then we clearly have

$$\inf_{\|u-\bar{u}\|_{m,p(.)}=R}(\langle A(u)-f, u-\bar{u}\rangle + \|A(u)-f\|\frac{R}{4N_2\sqrt{N_2}}) > 0 \text{ for some } R>0,$$

and proceeding as above we obtain $f \in A(W_0^{m,p(.)}(\Omega))$. □

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