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A New Characterization of $PSL(3, q)$

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ABSTRACT: Let G be a group. In this paper, we prove that G is isomorphic to $PSL(3, q)$ if and only if $|G| = |PSL(3, q)|$ and $m(G) = m(PSL(3, q))$, where q is a prime power and $m(G)$ is the maximal order of elements in G.

Key Words: The maximal order of elements, The classification of finite simple groups.

Contents

1. Introduction

Let n be an integer. We denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the prime graph of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq (we write $p \sim q$). Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers OC_i, $i = 1, 2, \ldots, t(G)$, where $\pi(OC_i) = \pi_i$. These OC_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. Also we call $OC_2, ..., OC_{t(G)}$ the odd order components of G. Let n be a positive integer and p be a prime number. Then $|n|_p$ denotes the p-part of n.

The set of element orders of G is denoted by $\pi_e(G)$. Obviously, $\pi_e(G)$ is partially ordered by divisibility. Therefore, it is uniquely determined by $\mu(G)$, the subset of its maximal elements. We denoted by $m(G)$ the maximal order of elements in G. In [15], authors consider the characterization of simple K_3-groups and some $L_2(p)$ by using the group order and maximal element order. In [11], it is proved that $PGL(2, q)$ is characterizable by the group order and maximum element order. Also, in [10], it is shown that the simple K_4-groups of type $L_2(q)$ can be characterized by their largest element orders together with their orders. In this paper, we are going to study the characterization of projective special linear group $PSL(3, q)$ by using the group order and maximal element order. In fact, we

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prove the following theorem:

Main Theorem. Let G be a group. Then $G \cong PSL(3,q)$ if and only if $|G| = |PSL(3,q)|$ and $m(G) = m(PSL(3,q))$.

2. Preliminaries

Definition 2.1. [6]. *Let a and n be integers greater than 1. Then a Zsigmondy prime of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$. Put*

$$Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}.$$

Lemma 2.2. [4] *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) *K is nilpotent;*

(2) $|K| \equiv 1 \pmod{|H|}$.

Lemma 2.3. [4] *Let G be a 2-Frobenius group, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*
(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$.

Lemma 2.4. [13] *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*
(a) G is a Frobenius group or 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.

Lemma 2.5. [9] *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $(n+1)/2 < p_i < n$. Here*

$$s(n) = 1, \ for \ 6 \leq n \leq 13;$$

$$s(n) = 2, \ for \ 14 \leq n \leq 17;$$

$$s(n) = 3, \ for \ 18 \leq n \leq 37;$$

$$s(n) = 4, \ for \ 38 \leq n \leq 41;$$

$$s(n) = 5, \ for \ 42 \leq n \leq 47;$$

$$s(n) = 6, \ for \ n \geq 48.$$

Lemma 2.6. [7,8] *Let $S = PSL(3,q)$ and $OC_2 = \frac{q^2+q+1}{d}$, where $d = (3, q-1)$.*

(i) *Let q be an odd prime power. If $x \in \pi_1(S)$,*

$$x^\alpha \mid |S|$$

and

$$x^\alpha - 1 \equiv 0 \pmod{(OC_2)},$$

then $x^\alpha = q^3$ or $q = 5$ and $x^\alpha = 2^5$.

(ii) *Let q be an odd prime power. If $x \in \pi_1(S)$, $x^\alpha \mid |S|$, then $x^\alpha + 1 \not\equiv 0 \pmod{(OC_2)}$, for every positive integer α.*

(iii) *Let $q = 2^m$, where $m \geq 1$. If $x \in \pi_1(S)$, $x^\alpha \mid |S|$ and $x^\alpha - 1 \equiv 0 \pmod{(OC_2)}$, then $x = 2$ and $\alpha = 3m$, hence $x^\alpha = q^3$.*

(iv) *Let $q = 2^m$, where $m \geq 1$. If $3 \mid q-1$ and $\frac{q^2+q-2}{6} \mid (q-1)^2(q+1)$, then $q = 16$.*

3. Proof of the main theorem

Let G be a group such that $|G| = |PSL(3,q)|$ and $m(G) = m(PSL(3,q))$, where q is a prime power. By [14], we can see that

$$\mu(PSL(3,q)) = \begin{cases} \{q-1, \dfrac{p(q-1)}{3}, \dfrac{q^2-1}{3}, \dfrac{q^2+q+1}{3}\}, & if \text{ d=3;} \\ \{p(q-1), q^2-1, q^2+q+1\}, & if \text{ d=1,} \end{cases}$$

where $q = p^\alpha$ is odd and $d = (3, q-1)$. Also,

$$\mu(PSL(3,2^m)) = \begin{cases} \{4, 2^m-1, \dfrac{2(2^m-1)}{3}, \dfrac{2^{2m}-1}{3}, \dfrac{2^{2m}+2^m+1}{3}\}, & if \text{ d=3;} \\ \{4, 2(2^m-1), 2^{2m}-1, 2^{2m}+2^m+1\}, & if \text{ d=1,} \end{cases}$$

where $d = (3, 2^m-1)$.

Since $|G| = |PSL(3,q)|$ and $m(G) = m(PSL(3,q)) = \frac{q^2+q+1}{d}$, we can conclude that $\frac{q^2+q+1}{d}$ is an odd order component of G. Also, for convenience let $r = \frac{q^2+q+1}{d}$ and $q' = p'^\alpha$, where p' is prime and α is positive integer.

Proof of the main theorem. If $G \cong PSL(3,q)$, then the conclusion is evident. Now we assume that $|G| = |PSL(3,q)|$ and $m(G) = m(PSL(3,q))$. We are going to prove the main theorem in the following steps:

Step 1. G is neither a Frobenius group nor a 2-Frobenius group.

Proof. Suppose on the contrary, G is a Frobenius group with kernel K and complement H. Since r is an odd order component of G, by Lemma 2.2, $r \in \{\pi(H), \pi(K)\}$. If $2 \mid |H|$, then $|H| = q^3(q-1)^2(q+1)$ and $|K| = r$. Now, by Lemma 2.2, $|H|$ divides $|K| - 1$, which is a contradiction. If $2 \mid |K|$, then $|K| = q^3(q-1)^2(q+1)$ and $|H| = r$. Let t be a prime dividing $|K|$, $t \nmid q$ and K_t be a Sylow t-subgroup of K. Then t divides either $(q-1)^2$ or $q+1$. Let t divides $q+1$. Since $K_t \rtimes H$

is a Frobenius group with kernel K_t and complement H, thus $|H| \mid |K_t| - 1$ and hence, $r \mid |K_t| - 1 = |q+1|_t - 1$, which is a contradiction. If t divides $(q-1)^2$, then similar to the above, $r \mid |K_t| - 1 = |(q-1)^2|_t - 1$, which is impossible.

If G is a 2-Frobenius group, Lemma 2.3 implies that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = r$ and $|G/K| \mid (|K/H| - 1)$. Now, applying the previous argument for the Frobenius group K with kernel H and complement K/H leads us to get a contradiction. □

Step 2. There exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group and r is an odd order component of K/H.
Proof. It follows immediately from Lemma 2.4 and Step 1. □

Step 3. K/H is not a sporadic simple group.
Proof. Suppose that K/H is a sporadic simple group. Thus

$$r = \frac{q^2+q+1}{d} \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71\}.$$

Let $\frac{q^2+q+1}{d} = 5$, since q is a prime power, we get a contradiction. Assume that $\frac{q^2+q+1}{d} = 7$ and $d = 1$. Thus $q = 2$, $|PSL(3,2)| = 2^3.3.7$ and $K/H \in \{M_{22}, J_1, J_2, HS\}$, so $5 \mid |K/H|$, which is a contradiction. If $d = 3$, then $q = 4$ and $|PSL(3,4)| = 2^6.3^2.5.7$. Now, if $K/H \in \{M_{22}, J_1, HS\}$, then $11 \mid |K/H|$, and for $K/H = J_2$, $2^7 \mid |K/H|$, which is a contradiction. By the same method, we can consider the other possibilities for r. □

Step 4. K/H can not be an alternating group $\mathbb{A}_m$, where $m \geq 5$.
Proof. If $K/H \cong \mathbb{A}_m$, then since $r \in \pi(K/H)$, $m \geq r$. Also, since $q \geq 2$ is a prime power, $r \geq 7$. Thus by Lemma 2.5, there exists a prime number $t \in \pi(A_m)$ such that $(r+1)/2 < t < r$ and hence, $t \mid \frac{q^3(q^3-1)(q^2-1)}{d}$. Since $t \nmid r$, $t \nmid q$ and $t \nmid q-1$, so $t \in Z_2(q)$. It follows that $t = r-2$, where $r = 7$ and $q = 4$. In this case, $|PSL(3,4)| = 2^6.3^2.5.7$. On the other hand, $m \geq 7$.

Now if $K/H \cong A_7$, then $|K| = 2^3.3^2.5.7$, $2^4.3^2.5.7$, $2^5.3^2.5.7$ or $2^6.3^2.5.7$. Let $|K| = 2^3.3^2.5.7$. Since $m(G) = 7$, we can conclude that $C_G(K) = 1$. Note that $G/C_G(K) \lesssim Aut(K)$ and $|Aut(K)| = 2^3.3^2.5.7.2$, then $|G|$ divides $2^3.3^2.5.7.2$, which is a contradiction. If $|K| = 2^4.3^2.5.7$, then $|H| = 2$. Suppose that P_7 be a Sylow 7-subgroup of G. Since P_7 acts fixed-point-freely on H, we can see that $H \rtimes P_7$ is a Frobenius group with kernel H and complement P_7. Thus, $|P_7|$ divides $|H| - 1$, namely $7 \mid 1$, which is impossible. If $|K| = 2^5.3^2.5.7$ or $2^6.3^2.5.7$, similarly we get a contradiction.

Let $K/H \cong A_8$. We know that $A_8 \cong PSL(4,2)$, and by [12],

$$\mu(PSL(4,q)) = \{(q^2+1)(q+1), q^3-1, 2(q^2-1), 4(q-1)\},$$

where $q = 2^m$ and m is positive integer. Now, $m(K/H) = 15$. But $K/H \leq G$ and $m(G) = 7$, which is a impossible. If $m \geq 9$, then $3^4 \mid |K/H|$, which is a contradiction. □

Step 5. $K/H = PSL(3,q)$.
Proof. By Steps 3 and 4, and the classification theorem of finite simple groups, K/H is a simple group of Lie type such that $t(K/H) \geq 2$ and $r \in OC(K/H)$. Thus K/H is isomorphic to one of the finite simple groups:

Case 1. Let $t(K/H) = 2$. Then $OC_2(K/H) = r = \dfrac{q^2+q+1}{d}$. Thus we have:

1.1. Suppose that $K/H \cong A_s(q')$, where $(q'-1) \mid (s+1)$ and s is an odd prime, then $r = \frac{q'^s-1}{q'-1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-1) \mid q^3(q-1)^2(q+1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(q'-1)^5} \leq q'^{5s}$ and $q'^{s(s+1)-s} < q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-1) \leq q^3(q-1)^2(q+1) < r^5 \leq q'^{5s}$, which implies that $s < 5$. Hence $s = 3$, so $q'^2+q'+1 = \frac{q^2+q+1}{d}$. Since $(q'-1) \mid (s+1)$, $q' \in \{2,3,5\}$, which implies that $K/H \cong PSL(4,2)$, $K/H \cong PSL(4,3)$, $K/H \cong PSL(4,5)$. Let $K/H \cong PSL(4,3)$, then $\frac{q^2+q+1}{d} = 13$. If $d = 1$, then $q = 3$ and $|PSL(3,3)| = 2^4.3^3.13$. On the other hand, $5 \mid |K/H|$, which is a contradiction. If $d = 3$, then $q(q+1) = 38$, which is impossible. The same reasoning rules out the case when $K/H \cong PSL(4,5)$ or $K/H \cong PSL(4,2)$.
1.2. Suppose that $K/H \cong A_{s-1}(q')$, where $(s,q') \neq (3,2)$, $(3,4)$ and s is an odd prime, then $r = \frac{q'^s-1}{(s,q'-1)(q'-1)}$ and $q'^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}(q'^i-1) \mid q^3(q-1)^2(q+1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(s,q'-1)^5(q'-1)^5} \leq q'^{5s}$ and $q'^{s(s-1)-s} < q'^{\frac{s(s-1)}{2}}\prod_{i=1}^{s-1}(q'^i-1) \leq q^3(q-1)^2(q+1) < r^5$, which implies that $s(s-1)-s < 5s$. Hence $s = 3, 5$. If $s = 5$, then $\frac{q^2+q+1}{d} = \frac{q'^4+q'^3+q'^2+q'+1}{(5,q'-1)}$ and hence, $(q',q) = 1$. Also, $q'^{10}(q'-1)(q'^2-1)(q'^3-1)(q'^4-1) \mid q^3(q-1)^2(q+1)$. But $q'^{10} \nmid q^3(q-1)^2(q+1)$, which is a contradiction. If $s = 3$, then $\frac{q'^3-1}{(3,q'-1)(q'-1)} = \frac{q^2+q+1}{d}$ and hence, by Lemma 2.6(i,iii), $q' \in \{q,5\}$. Thus either $q' = q$ and $K/H \cong PSL(3,q)$ or $q' = 5$ and $K/H \cong PSL(3,5)$.
1.3. If $K/H \cong C_n(q')$, where $n = 2^u \geq 2$, then $\frac{q'^n+1}{(2,q'-1)} = \frac{q^2+q+1}{d}$. Now, if $(2,q'-1) = 1$, then $q'^n = \frac{q^2+q+1-d}{d}$ and hence, $(q',q) = 1$ and $q'^{n+1} > \frac{q^2+q+1-d}{d}$. On the other hand, $(p'^\alpha)^{n^2} = |K/H|_{p'} \leq |G|_{p'} = |\frac{q^2+q+1}{d}|_{p'}|(q-1)^2|_{p'}|q+1|_{p'} < (\frac{q^2+q+1-d}{d})^4 < (p'^\alpha)^{4(n+1)}$, thus $n \in \{2,4\}$ and hence, $r \in \{q'^2+1, q'^4+1\}$. Since $(2,q'-1) = 1$, $q' = 2^\alpha$ and $r \in \{2^{2\alpha}+1, 2^{4\alpha}+1\}$. Let $d = 1$. It follows that $q(q+1) \in \{2^{2\alpha}, 2^{4\alpha}\}$, which is a contradiction. If $d = 3$, then $\frac{q^2+q+1}{3} \in \{2^{2\alpha}+1, 2^{4\alpha}+1\}$, which implies that $\frac{(q-1)(q+2)}{3} \in \{2^{2\alpha}, 2^{4\alpha}\}$. Since $3 \mid q-1$, $3 \mid q+2$ and hence, $3 \mid 2^{2\alpha}$, which is a contradiction. If $(2,q'-1) = 2$, then similar to the above, we get a contradiction. The same reasoning completes the

proof in the case when either $K/H \cong B_n(q')$ or $K/H \cong {}^2D_n(q')$, where $n = 2^u \geq 4$.

1.4. If $K/H \cong B_s(3)$, where s is an odd prime, then $\frac{3^s-1}{2} = \frac{q^2+q+1}{d}$. So $3^s = \frac{2}{d}(q^2+q+\frac{d+2}{2})$ and hence, $(3,q)=1$ and $3^{s+1} > q^2+q+\frac{d+2}{2}$. Since $3^{s^2} = |K/H|_3 \leq |G|_3 = |\frac{q^2+q+1}{d}|_3|(q-1)^2|_3|q+1|_3 < (q^2+q+\frac{d+2}{2})^4 < 3^{4(s+1)}$, thus $s^2 < 4(s+1)$ and hence, $s=3$, which implies that $\frac{3^3-1}{2} = \frac{q^2+q+1}{d}$, which has already been considered. By the same method, we can prove that K/H cannot be a simple group $C_s(3)$.

1.5. If $K/H \cong C_s(2)$, where s is an odd prime, then $2^s - 1 = \frac{q^2+q+1}{d}$ and hence, $2^s = \frac{q^2+q+1+d}{d}$. Now, if $q \in \{2,4\}$, then $\frac{q^2+q+1}{d} = 7$ and hence, $s=3$. In these cases, $|PSL(3,2)| = 2^3.3.7$ and $|PSL(3,4)| = 2^6.3^2.5.7$. On the other hand, $2^9 \mid |C_s(2)|$, which is a contradiction. If $q \notin \{2,4\}$, then $(2,q)=1$ and $2^{s+1} > \frac{q^2+q+1+d}{d}$. We know that $2^{s^2} = |K/H|_2 \leq |G|_2 = |(q-1)^2|_2|q+1|_2 < (\frac{q^2+q+1+d}{d})^2 < 2^{2(s+1)}$ and hence, $s^2 < 2(s+1)$, which implies that $s<3$, which is impossible.

1.6. If $K/H \cong D_s(q')$, where $s \geq 5$ is prime and $q' = 2,3,5$, then $\frac{q'^s-1}{q'-1} = r$. Thus $q'^{s(s-1)}\prod_{i=1}^{s-1}(q'^{2i}-1) \mid q^3(q-1)^2(q+1)$. On the other hand, $r^5 = \frac{(q'^s-1)^5}{(q'-1)^5} \leq q'^{5s}$ and $q'^{s(s-1)}.q'^{\frac{s(s-1)}{2}} < q'^{s(s-1)}\prod_{i=1}^{s-1}(q'^{2i}-1) \leq q^3(q-1)^2(q+1) < r^5$, which implies that $q'^{s(s-1)+\frac{s(s-1)}{2}} < q'^{5s}$ and hence, $s<5$, which is a contradiction.

1.7. If $K/H \cong D_{s+1}(q')$, where s is an odd prime and $q' = 2,3$, then $\frac{q'^s-1}{(2,q'-1)} = r$. Thus $\frac{1}{(2,q'-1)}q'^{s(s+1)}(q'^s+1)(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^{2i}-1) \mid q^3(q-1)^2(q+1)$. Also, $r^5 = \frac{(q'^s-1)^5}{(2,q'-1)^5} \leq q'^{5s}$ and $q'^{s(s+1)+\frac{s(s+1)}{2}} < \frac{1}{(2,q'-1)}q'^{s(s+1)}(q'^s+1)(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^{2i}-1) \leq q^3(q-1)^2(q+1) < r^5$, which implies that $q'^{\frac{3s(s+1)}{2}} < q'^{5s}$ and hence, $s<3$, which is a contradiction.

1.8. If $K/H \cong E_6(q')$, then $r = \frac{q'^6+q'^3+1}{(3,q'-1)}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^3-1)(q'^2-1) \mid q^3(q-1)^2(q+1)$. On the other hand, $r^5 = \frac{(q'^6+q'^3+1)^5}{(3,q'-1)^5} \leq (q'^9-1)^5 \leq q'^{45}$ and $q'^{36}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^5-1)(q'^3-1)(q'^2-1) \leq q^3(q-1)^2(q+1) < r^5 < q'^{45}$, which is impossible. If $K/H \cong {}^2E_6(q')$, where $q' > 2$, then similar to the above, we get a contradiction.

1.9. If $K/H \cong G_2(q')$, where $2 < q' \equiv \epsilon \pmod 3$ and $\epsilon = \pm 1$, then $q'^2 - \epsilon q' + 1 = \frac{q^2+q+1}{d}$. We know that $|K/H| \mid |G|$. Since $|K/H| = q'^6(q'^2-1)(q'^6-1)$ and $q'^2 - \epsilon q' + 1 = r$, it follows that $|K/H| > |G|$, which is a contradiction. By the same method, we can prove that K/H cannot be a simple group $G_2(q')$, where $q' \equiv 0 \pmod 3$.

1.10. If $K/H \cong {}^2A_s(q')$, where $(s,q') \neq (3,3),\ (5,2),\ (q'+1) \mid (s+1)$ and s is an odd prime, then $r=\frac{q'^s+1}{q'+1}$ and $q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-(-1)^i)\mid q^3(q-1)^2(q+1)$. Also, $r^5=\frac{(q'^s+1)^5}{(q'+1)^5}<q'^{5s}$ and $q'^{\frac{s(s+1)}{2}}.q'^{\frac{s(s-1)}{2}+s}<q'^{\frac{s(s+1)}{2}}(q'^{s+1}-1)\prod_{i=1}^{s-1}(q'^i-(-1)^i)\leq q^3(q-1)^2(q+1)<r^5$, which implies that $s+1<5$ and hence, $s=3$. In this state, $r=q'^2-q'+1$ and $q'^6(q'^4-1)(q'^2-1)(q'+1)\mid q^3(q-1)^2(q+1)$. But $q'^6(q'^4-1)(q'^2-1)(q'+1)>q^3(q-1)^2(q+1)$, which is a contradiction. By the same method, we can prove that K/H cannot be a simple group ${}^2A_{s-1}(q')$.

1.11. If $K/H\cong{}^2A_3(2)$, then $\frac{q^2+q+1}{d}=5$, which is impossible.

1.12. If $K/H\cong{}^2D_n(2)$, where $n=2^m+1\geq 5$, then $2^{n-1}+1=\dfrac{q^2+q+1}{d}$. Thus $2^{n-1}=\frac{q^2+q+1-d}{d}$ and hence, $(2,q)=1$. We know that $2^{n(n-1)}=|K/H|_2\leq|G|_2=|(q-1)^2|_2|q+1|_2<(\frac{q^2+q+1-d}{d})^3<2^{3n}$, so $n-1<3$, which is impossible.

1.13. If $K/H\cong{}^2D_s(3)$, where $5<s\neq 2^m+1$ and s is an odd prime, then $\frac{3^s+1}{4}=r$ and $3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1)\mid q^3(q-1)^2(q+1)$. Also, $r^5=\frac{(3^s+1)^5}{1024}\leq 3^{5s}$ and $3^{s(s-1)}<3^{s(s-1)}\prod_{i=1}^{s-1}(3^{2i}-1)\leq q^3(q-1)^2(q+1)<r^5$, which implies that $s(s-1)<5s$ and hence, $s-1<5$, which is a contradiction.

1.14. If $K/H\cong{}^2D_n(3)$, where $9\leq n=2^m+1$ and n is not prime, then $\frac{3^{n-1}+1}{2}=\frac{q^2+q+1}{d}$. Thus $(3,q)=1$ and $3^{n-1}=\frac{2}{d}(q^2+q+\frac{2-d}{2})$ and hence, $3^n>q^2+q+\frac{2-d}{2}$. Since $3^{n(n-1)}=|K/H|_3\leq|G|_3=|\frac{q^2+q+1}{d}|_3|(q-1)^2|_3|q+1|_3<(q^2+q+\frac{2-d}{2})^4<3^{4n}$, we obtain $n-1<4$, which is impossible.

1.15. If $K/H\cong{}^3D_4(q')$, then $r=q'^4-q'^2+1$ and $q'^{12}(q'^4+q'^2+1)(q'^6-1)(q'^2-1)\mid q^3(q-1)^2(q+1)$. Also, $r^5=(q'^4-q'^2+1)^5<(q'^4)^5=q'^{20}$ and $q'^{12}(q'^4+q'^2+1)(q'^6-1)(q'^2-1)\leq q^3(q-1)^2(q+1)<r^5<q'^{20}$, which is a contradiction.

1.16. If $K/H\cong{}^2F_4(2)'$, then $|K/H|=2^{11}\cdot 3^3\cdot 5^2\cdot 13$. Thus $\frac{q^2+q+1}{d}=13$. If $d=1$, then $q=3$ and $|PSL(3,3)|=2^4.3^3.13$. On the other hand, $5\mid|K/H|$, which is a contradiction. If $d=3$, then $q(q+1)=38$, which is impossible.

Case 2. Let $t(K/H)=3$. Then $r\in\{OC_2(K/H),\ OC_3(K/H)\}$:

2.1. If $K/H\cong A_1(q')$, where $4\mid q'$, then the odd order components of K/H are $q'+1$ and $q'-1$. If $q'+1=r$, then $q'=r-1=\frac{q^2+q+1}{d}-1$ and hence, either $q'=q(q+1)$ or $q'=\frac{(q-1)(q+2)}{3}$, which are impossible. If $q'-1=r$, then by Lemma 2.6(i,iii), $q'=q^3$. Since $q'(q'-1)(q'+1)=|K/H|\mid|G|=r.q^3(q-1)^2(q+1)$, we can conclude that $(q^2-q+1)\mid(q^2-2q+1)$, which is a contradiction.

2.2. If $K/H\cong A_1(q')$, where $4\mid q'-1$, then $q'=r$ or $\frac{q'+1}{2}=r$. Now, we consider the following cases:

(i) Let $q=2^m$ and $q'=r$. If $d=3$, then $\frac{q'+1}{2}=\frac{q^2+q+4}{6}$, thus $2\mid\frac{q'+1}{2}$. Since $\frac{q'+1}{2}$ is an odd order component of K/H, we get a contradiction. If $d=1$, then since

$|K/H| \mid |G|$, we can conclude that $\frac{q^2+q+2}{2} \mid q^2(q-1)^2$, which implies that $q = 2$ and hence, $q' = 7$. But $4 \mid q'-1$, which is a contradiction. If $\frac{q'+1}{2} = r$ and $d = 3$, then $|A_1(q')| = (\frac{q^2+q+1}{3})(\frac{2q^2+2q-1}{3})(\frac{2q^2+2q-4}{3})$. Since $|K/H| \mid |G|$, $\frac{2q^2+2q-1}{3} \mid (q-1)^2$. Also, $(q-1)^2 = q^2-2q+1 = \frac{2q^2+2q-1}{3} + \frac{q^2-8q+4}{3}$, thus $\frac{2q^2+2q-1}{3} \mid \frac{q^2-8q+4}{3}$. But $\frac{2q^2+2q-1}{3} > \frac{q^2-8q+4}{3}$, which is a contradiction. If $d = 1$, then since $|K/H| \mid |G|$, we can conclude that $2(2q^2+2q+1) \mid q^2(q-1)^2$, which is a contradiction.
(ii) Let q be an odd prime power, $q' = r$ and $d = 1$, then $q' = q^2+q+1$ and $|K/H| = (q^2+q+1)(q^2+q)(\frac{q^2+q+2}{2})$. Since $|K/H| \mid |G|$, we can conclude that $\frac{q^2+q+2}{2} \mid q^2(q-1)^2$, which implies that $q = 5$ and hence, $q' = 31$. But $4 \mid q'-1$, which is a contradiction. If $d = 3$, then $q' = \frac{q^2+q+1}{3}$ and since $|K/H| \mid |G|$, $\frac{q^2+q-2}{6} \mid q^3(q-1)^2(q+1)$. On the other hand, $(q+2, q-1) = 3$ and $\frac{q^2+q-2}{6} = \frac{(q-1)(q+2)}{6}$, which implies that $\frac{(q+2)}{6} \mid q-1$, which is a contradiction. If $\frac{q'+1}{2} = r$, then by 2.6(ii), we get a contradiction. The same reasoning rules out the case when $K/H \cong A_1(q')$, where $4 \mid q'+1$.
2.3. If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2t+1} > 3$, then $q' - \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$ or $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$. Let $(3, q) = 1$. If $q' - \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$, then $q' > \frac{q^2+q+1}{d}$. Also, $(3^{2t+1})^3 = |K/H|_3 \le |G|_3 < (\frac{q^2+q+1}{d})^2 < (3^{2t+1})^3$, which is a contradiction. Let $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{d}$, and $d = 1$, thus $3^{t+1}(3^t+1) = q(q+1)$. Now, since $(3, q) = 1$, $3 \nmid q$ and hence, $|q+1|_3 = 3^{t+1}$. Thus $|G|_3 = |\frac{q^2+q+1}{d}|_3(|q+1|_3) < 3^{3t+3}$. On the other hand, $3^{3(2t+1)} = |K/H|_3 \le |G|_3 < 3^{3t+3}$, which is a contradiction. If $d = 3$, then $q' + \sqrt{3q'} + 1 = \frac{q^2+q+1}{3}$ and hence, $3^{t+2}(3^t+1) = (q-1)(q+2)$. Thus either $3^{t+1} \mid (q-1)$ and $(q+2) \mid 3(3^t+1)$ or $3^{t+1} \mid (q+2)$ and $(q-1) \mid 3(3^t+1)$. This forces $q-1 = 3^{t+1}$ and $q+2 = 3(3^t+1)$. This guarantees that $|G|_3 < 3^{3t+3}$. Also, $3^{3(2t+1)} = |K/H|_3 \le |G|_3 < 3^{3t+3}$, which is a contradiction. Assume that $(3, q) \ne 1$. So $d = 1$ and $q' \pm \sqrt{3q'} + 1 = q^2+q+1$, this forces $q = 3^{t+1}$ and $q+1 = 3^t \pm 1$, which is a contradiction.
2.4. If $K/H \cong {}^2D_s(3)$, where $s = 2^t + 1 \ge 5$, then $\frac{3^s+1}{4} = \frac{q^2+q+1}{d}$ or $\frac{3^{s-1}+1}{2} = \frac{q^2+q+1}{d}$. If $\frac{3^s+1}{4} = r$, then $3^{s(s-1)}(3^{s-1}-1)(3^{s-1}+1)\prod_{i=1}^{s-2}(3^{2i}-1) \mid q^3(q-1)^2(q+1)$. On the other hand, $r^5 = \frac{(3^s+1)^5}{1024} \le 3^{5s}$ and $3^{2s(s-1)-s} < 3^{s(s-1)}(3^{s-1}-1)(3^{s-1}+1)\prod_{i=1}^{s-2}(3^{2i}-1) \le q^3(q-1)^2(q+1) < r^5$, which implies that $2s(s-1) < 6s$ and hence, $s < 4$, which is a contradiction. If $\frac{3^{s-1}+1}{2} = \frac{q^2+q+1}{d}$, then similar to the above, we get a contradiction.
2.5. If $K/H \cong {}^2D_{s+1}(2)$, where $s = 2^n - 1$ and $n \ge 2$, then $2^s + 1 = \dfrac{q^2+q+1}{d}$ or $2^{s+1} + 1 = \dfrac{q^2+q+1}{d}$. If $2^s + 1 = r$, then $2^s = q(q+1)$ or $2^s = \dfrac{(q-1)(q+2)}{3}$, which is impossible. The same reasoning rules out the case when $2^{s+1} + 1 = r$.

2.6. If $K/H \cong F_4(q')$, where q' is even, then $q'^4 + 1 = \frac{q^2+q+1}{d}$ or $q'^4 - q'^2 + 1 = \frac{q^2+q+1}{d}$. If $q'^4 + 1 = r$, then $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 - q'^2 + 1) \mid q^3(q-1)^2(q+1)$. Also, $r^5 = (q'^4 + 1)^5 < (q'^5)^5 = q'^{25}$ and $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 - q'^2 + 1) \leq q^3(q-1)^2(q+1) < r^5 < q'^{25}$, which is a contradiction. If $q'^4 - q'^2 + 1 = \frac{q^2+q+1}{d}$, then similar to the above, we get a contradiction. By the same method, we can prove that K/H cannot be a simple group $F_4(q')$, where q' is odd.
2.7. If $K/H \cong E_7(2)$, then $r \in \{73, 127\}$. Therefore, either $r = 73$ and $q = 8$ or $r = 127$ and $q = 19$. So either $|PSL(3, 8)| = 2^9.3^2.7^2.73$ or $|PSL(3, 19)| = 2^4 \cdot 3^4 \cdot 5 \cdot 19^3 \cdot 127$. On the other hand, $13 \mid |E_7(2)|$, which is a contradiction.
2.8. If $K/H \cong E_7(3)$, then $r \in \{757, 1093\}$. Let $\frac{q^2+q+1}{d} = 757$ and $d = 1$. Thus $q(q+1) = 756$ and hence, $q = 27$. We know that $|PSL(3, 27)| = 2^4.3^9.7.13^2.757$. On the other hand, $5 \mid |E_7(3)|$, which is a contradiction. If $d = 3$, then $q(q+1) = 2270$, which is impossible. If $\frac{q^2+q+1}{d} = 1093$, then $q(q + 1) \in \{1092, 3278\}$, which is impossible.
2.9. If $K/H \cong A_2(2)$, then $\frac{q^2+q+1}{d} \in \{3, 7\}$. Since q is a prime power, $\frac{q^2+q+1}{d} = 7$, which implies that $K/H \cong PSL(3, 2)$.
2.10. If $K/H \cong A_2(4)$, then $\frac{q^2+q+1}{d} \in \{5, 7, 9\}$. Since q is a prime power, $\frac{q^2+q+1}{d} = 7$, which implies that $K/H \cong PSL(3, 4)$.
2.11. If $K/H \cong {}^2A_5(2)$, then $\frac{q^2+q+1}{d} \in \{5, 7, 11\}$. Since q is a prime power, $\frac{q^2+q+1}{d} = 7$. In this state, $|PSL(3, 2)| = 2^3.3.7$, but $5 \mid |K/H|$, which is a contradiction.
2.12. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2t+1} \geq 2$, then $r = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$. In both cases, we can see at once that $|K/H| > |G|$, which is a contradiction.
Case 3. Let $t(K/H) \in \{4, 5\}$. Then

$$r \in \{OC_2(K/H),\ OC_3(K/H),\ OC_4(K/H),\ OC_5(K/H)\},$$

as follows:
3.1. If $K/H \cong {}^2B_2(q')$, where $q' = 2^{2t+1}$ and $t \geq 1$, then $r \in \{q' - 1, q' \pm \sqrt{2q'} + 1\}$. Let $q' - 1 = r$ and $d = 1$. Thus $2(2^{2t} - 1) = q(q + 1)$. Now, if $|q|_2 = 2$, then $q + 1 = 3$. It follows that $t = 1$, $q' = 8$ and $|K/H| = 2^6.5.7.13$. On the other hand, $|PSL(3, 2)| = 2^3.3.7$. But $5 \mid |K/H|$, which is a contradiction. Thus $|q + 1|_2 = 2$ and since $|q - 1|_2 \leq 2^t$, $|G|_2 \leq 2^{2t+1}$. Moreover, $2^{2(2t+1)} = |K/H|_2 \leq |G|_2 \leq 2^{2t+1}$, which is a contradiction.

If $q' - 1 = r$ and $d = 3$, then we can see that $2^2(3.2^{2t-1} - 1) = q(q + 1)$. Now, if $|q|_2 = 2^2$, then $q + 1 = 5$. It follows that $t = 1$, $q' = 8$ and $|K/H| = 2^6.5.7.13$. On the other hand, $|PSL(3, 4)| = 2^6.3^2.5.7$. But $13 \mid |K/H|$, which is a contradiction. Thus $|q + 1|_2 = 2^2$ and hence, $|q - 1|_2 = 2$. Moreover, $2^{2(2t+1)} = |K/H|_2 \leq |G|_2 = |(q - 1)^2|_2|q + 1|_2 \leq 2^{2t+2}$, which is a contradiction.
Assume that $q' + \sqrt{2q'} + 1 = r$ and $d = 3$. Thus $\frac{q^2+q-2}{3} = 2^{t+1}(2^t + 1)$ and hence, $(q - 1)(q + 2) = 3.2^{t+1}(2^t + 1)$. Since $3 \mid q - 1$, $q - 1 = 3k$ for some positive integer k. Thus $3k(k+1) = 2^{t+1}(2^t+1)$ and hence, $k(k+1) = 2^{t+1}(\frac{2^t+1}{3})$. Now, if $2^{t+1} \mid k$, then $k + 1 < \frac{2^t+1}{3}$ and if $2^{t+1} \mid k+1$, then $k \leq \frac{2^t+1}{3}$, which are impossible. If $d = 1$,

then $q^2+q+1 = q'+\sqrt{2q'}+1$ and $q(q+1) = 2^{t+1}(2^t+1)$, which is impossible. The same reasoning rules out the case when $q'-\sqrt{2q'}+1 = r$.

3.2. If $K/H \cong A_2(4)$ then $\frac{q^2+q+1}{d} \in \{5,7,9\}$. Since q is a prime power, $\frac{q^2+q+1}{d} = 7$, which implies that $K/H \cong PSL(3,4)$.

3.3. If $K/H \cong {}^2E_6(2)$, then $\frac{q^2+q+1}{d} \in \{13,17,19\}$. Let $\frac{q^2+q+1}{d} = 19$ and $d=1$, thus $q(q+1) = 18$, which is a contradiction. If $d=3$, then $q(q+1) = 56$, which implies that $q=7$. Thus $|PSL(3,7)| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$. On the other hand $13 \mid |{}^2E_6(2)|$, which is a contradiction. For $r \in \{13,17\}$, similar to the above we get a contradiction.

3.4. If $K/H \cong E_8(q')$, then $r \in \{q'^8-q'^7+q'^5-q'^4+q'^3-q'+1, q'^8+q'^7-q'^5-q'^4-q'^3+q'+1, q'^8-q'^6+q'^4-q'^2+1, q'^8-q'^4+1\}$. If $q'^8-q'^7+q'^5-q'^4+q'^3-q'+1 = r$, then $r < q'^9$. On the other hand, $r^5 < q'^{45}$ and $|G| < r^5$. Since $q'^{120} \mid |K/H|$ and $|K/H| \mid |G|$, we get a contradiction. For other cases, similarly we get a contradiction.

Step 6. $G \cong M$.

Proof: By the above steps, we have $K/H \cong PSL(3,q)$, i.e., $|K/H| = |PSL(3,q)| = |G|$, thus by Step 2, $H=1$ and $K=G$. Therefore $G \cong PSL(3,q)$. □

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