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Mapping Properties of Certain Linear Operator Associated with Hypergeometric Functions

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abstract: The main object of the present paper is to find some sufficient conditions in terms of hypergeometric inequalities so that the linear operator denoted by $H^{a,b,c}_{\mu,\delta}$ maps a certain subclass of close-to-convex function $\mathcal{R}^{\tau}(A,B)$ into subclasses of k-uniformly starlike and k-uniformly convex functions $k-\mathcal{ST}(\beta)$ and $k-\mathcal{UCV}(\beta)$ respectively. Further, we consider an integral operator and discuss its properties. Our results generalize some relevant results.

Key Words: Analytic function, Starlike function, Convex function, k-uniformly convex function, k-uniformly starlike function, Integral operator.

Contents

1. Introduction and Definition

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and having normalized power series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \le \alpha < 1)$, denoted by $\mathcal{S}^*(\alpha)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}), \tag{1.2}$$

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and convex of order α $(0 \leq \alpha < 1)$, denoted by $\mathcal{CV}(\alpha)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{1.3}$$

It is an established fact that $f \in \mathcal{CV}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.
Note that for $\alpha = 0$, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{CV}(0) = \mathcal{CV}$, the well-known standard class of starlike and convex functions respectively (see [19]).

Furthermore, a function $f(z)$ of the form (1.1) is said to be k-uniformly convex in $\mathbb{U}$, denoted by $k-\mathcal{UCV}$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (0 \leq k < \infty; z \in \mathbb{U}). \tag{1.4}$$

The class $k-\mathcal{ST}$, consisting of k-uniformly starlike functions is defined via $k-\mathcal{UCV}$ (see [11]) by usual Alexander's relation i.e.,

$$f \in k-\mathcal{ST} \Longleftrightarrow g \in k-\mathcal{UCV}, \text{ where } g(z) = \int_0^z \frac{f(t)}{t}dt.$$

The class $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ were introduced by Kanas and Wisniowska where its geometric definition and connection with conic domain were considered (see [10,11]).
In particular, if $k = 0$ and $k = 1$, we get

$$0-\mathcal{UCV} \equiv \mathcal{CV},\ 0-\mathcal{ST} \equiv \mathcal{S}^*,\ 1-\mathcal{UCV} \equiv \mathcal{UCV} \text{ and } 1-\mathcal{ST} \equiv \mathcal{SP}$$

where $\mathcal{CV}$, $\mathcal{S}^*$, $\mathcal{UCV}$ and $\mathcal{SP}$ are respectively the familiar classes of univalent convex functions, univalent starlike functions (see [6]), uniformly convex functions ([8]) (also, see [14,20]) and parabolic starlike functions [20]. Recently many researchers have generalized the classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$. Bharti et al. [2] (also, see [21]) introduced the classes $k-\mathcal{UCV}(\beta)$ and $k-\mathcal{ST}(\beta)$ to be the classes of functions $f \in \mathcal{A}$ satisfying the condition:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \beta \tag{1.5}$$

and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \tag{1.6}$$

respectively for some $k \geq 0$ and β $(0 \leq \beta < 1)$. Note that $f(z) \in k-\mathcal{UCV}(\beta) \Longleftrightarrow zf'(z) \in k-\mathcal{ST}(\beta)$. Clearly, $1-\mathcal{UCV}(0) = \mathcal{UCV},\ 1-\mathcal{ST}(0) = \mathcal{SP},\ 0-\mathcal{UCV}(0) = \mathcal{CV},\ 0-\mathcal{ST}(0) = \mathcal{S}^*$. It has been shown that (see [2])

$$k-\mathcal{ST}(\beta) = \mathcal{S}^*\left(\frac{k+\beta}{1+k}\right),\ k-\mathcal{UCV}(\beta) = \mathcal{CV}\left(\frac{k+\beta}{1+\beta}\right).$$

For two analytic functions f and g in $\mathbb{U}$, we say that the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [15]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In 1995, Dixit and Pal (see [5]) has introduced the class $\mathcal{R}^\tau(A,B)$ as follows:
A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$ if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B(f'(z)-1)}\right| < 1 \quad (z \in \mathbb{U}; \tau \in \mathbb{C}\setminus\{0\}, -1 \le B < A \le 1). \tag{1.7}$$

By giving particular values of A, B and τ, the class $\mathcal{R}^\tau(A,B)$ includes several interesting subclasses of $\mathcal{S}$ studied by different researchers (see [3,17,18]).
The generalized hypergeometric function ${}_pF_q$ $(p,\ q \in \mathbb{N}_0 := \{0,1,2,\cdots\})$ with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1,2,3,\cdots p)$ and q denominator parameters $\beta_k \in \mathbb{C}\setminus \mathbb{Z}_0^-$ $(\mathbb{Z}_0^- := \{0,-1,-2\cdots\}, k = 1,2,\cdots q)$ is defined by (see, for example, [24]), p. 19):

$$\begin{aligned}{}_pF_q(z) &= {}_pF_q(\alpha_1,\alpha_2,\cdots\alpha_p;\beta_1,\beta_2,\cdots\beta_q;z)\\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \quad (z \in \mathbb{U}),\end{aligned} \tag{1.8}$$

where $(\lambda)_n$ is the Pochhammer symbol (or shifted factorial) defined in terms of the gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0, \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\cdots\}, \lambda \in \mathbb{C}). \end{cases}$$

Note that ${}_pF_q(z)$ is an entire function if $p < q+1$. However, if $p = q+1$, then ${}_pF_q(z)$ is analytic in $\mathbb{U}$. Also, if $p = q+1$ and $\Re(\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j) > 0$, then ${}_pF_q(z)$ converges on $\partial\mathbb{U}$. In particular, the function

$$\begin{aligned}{}_2F_1(a,b;c;z) &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n \quad (z \in \mathbb{U})\\ &= 1 + \frac{ab}{1.c}z + \frac{a(a+1)b(b+1))}{1.2c(c+1)}z^2 + \cdots\end{aligned} \tag{1.9}$$

is the familiar Gaussian hypergeometric function. The hypergeometric function ${}_2F_1(a,b;c;z)$ has been extensively studied by various authors and play an important role in the Geometric Function Theory. It is useful in unifying various functions by givien appropriate values to the parameters a, b and c. Further, the series (1.9) may be regarded as a generalization of the elementary geometric series. It reduces to the geometric series in two cases. When $a = c$ and $b = 1$ and when $b = c$ and $a = 1$. It is worthy to mention here that the function ${}_2F_1(a, b; c; z)$ is symmetric in a and b and the series (1.9) terminates if at least one of the numerator parameters a and b is zero or negative integer. For recent expository work on hypergeometric function see [4,7,12,22]. It is well-known that ${}_2F_1(a, b; c; z)$ is the solution of the second order homogeneous differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0 \tag{1.10}$$

Note that the behavior of the hypergeometric function ${}_2F_1(a, b; c; z)$ near $z = 1$ is classified into three cases according as $\Re(c-a-b)$ is positive, zero or negative. By Gauss summation formula we get

$${}_2F_1(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{1.11}$$

provided $\Re(c-a-b) > 0$. The normalized hypergeometric function $z{}_2F_1(a, b; c; z)$ has a series expansion of the form

$$z{}_2F_1(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n. \tag{1.12}$$

Using normalized hypergeometric function $z{}_2F_1(a, b; c; z)$ consider the function (see [25],with p=1)

$$\begin{aligned} \mathcal{J}_{\mu,\delta}(a, b; c; z) =& (1-\mu+\delta)[z\ {}_2F_1(a, b; c; z)] + (\mu-\delta)z[z\ {}_2F_1(a, b; c; z)]' \\ &+ \mu\delta z^2[z\ {}_2F_1(a, b; c; z)]'' \end{aligned} \tag{1.13}$$

with $\mu, \delta \geq 0$, $\mu \geq \delta$, $z \in \mathbb{U}$. For a function $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.14}$$

We consider the linear operator $H_{\mu,\delta}^{a,b,c} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by mean of Hadamard product as

$$H_{\mu,\delta}^{a,b,c}(f)(z) = \mathcal{J}_{\mu,\delta}(a, b; c; z) * f(z). \tag{1.15}$$

Thus, for a function $f \in \mathcal{A}$ of the form (1.1), we have

$$(H_{\mu,\delta}^{a,b,c}(f))(z) = z + \sum_{n=2}^{\infty} [1+(n-1)(\mu-\delta+n\mu\delta)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n \quad (z \in \mathbb{U}). \tag{1.16}$$

Taking $\delta = 0$ in (1.16) we have $H^{a,,b,c}_{\mu,0}(f) \equiv L_\mu(f)$ considered by Kim and Shon (see [13]) while taking $\mu = \delta = 0$ we get $H^{a,b,c}_{0,0}(f)(z) = (\mathcal{I}^{a,b}_c f)(z)$ where $\mathcal{I}^{a,b}_c$ is known as Hohlov operator (see [9]).
Motivated by Sharma et al. [23] (also, see [1,16]), in this paper sufficient conditions in term of hypergeometric inequalities are found so that the linear operator defined by (1.16) maps a certain subclass of close-to-convex function $\mathcal{R}^\tau(A,B)$ into subclasses of k-uniformly starlike and k-uniformly convex functions $k-\mathcal{ST}(\beta)$ and $k-\mathcal{UCV}(\beta)$ respectively. Further, we consider an integral operator and discuss its properties.

2. Preliminaries Lemmas

To investigate our main results, we need each of the following lemmas:

Lemma 2.1. *(see [2,21]) Let $f \in \mathcal{A}$ be of the form (1.1). If*

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\beta)]|a_n| \le 1-\beta, \tag{2.1}$$

then $f \in k-\mathcal{ST}(\beta)$.

Lemma 2.2. *(see [2,21]) A function f of the form (1.1) is in $k-\mathcal{UCV}(\beta)$ if it satisfies the inequality*

$$\sum_{n=2}^{\infty}n[n(1+k)-(k+\beta)]|a_n| \le 1-\beta. \tag{2.2}$$

Lemma 2.3. *(see [5]) Let the function f, given by (1.1) be a member of $\mathcal{R}^\tau(A,B)$. Then*

$$|a_n| \le (A-B)\frac{|\tau|}{n} \quad (n \in \mathbb{N}\setminus\{1\}). \tag{2.3}$$

The estimate in (2.3) is sharp.

3. Main Results

Unless otherwise mentioned, we assume throughout the sequel that
$-1 \le B < A \le 1,\ \tau \in \mathbb{C}\setminus\{0\},\ k \ge 0,\ \mu,\ \delta \ge 0$ and $\mu \ge \delta$.

Theorem 3.1. *Let $a,\ b,\ c \in \mathbb{R}$ be such that $a,b > 1$ and $c > a+b+2$. If the hypergeometric inequality*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[1+k-(\mu-\delta)(k+\beta)+\{(1+k)(\mu-\delta)-(k+\beta)\mu\delta+2(1+k)\mu\delta\}\\ &\frac{ab}{c-a-b-1}+\frac{(\mu-\delta-1)(k+\beta)(c-a-b)}{(a-1)(b-1)}+\frac{(1+k)\mu\delta(a)_2(b)_2}{(c-a-b-2)_2}\Big]\\ &\le(1-\beta)+\left[\frac{(k+\beta)(\mu-\delta-1)(c-1)}{(a-1)(b-1)}\right]+\frac{1-\beta}{(A-B)|\tau|}\end{aligned} \tag{3.1}$$

is satisfied, then $H^{a,b,c}_{\mu,\delta}$ maps the class $\mathcal{R}^\tau(A,B)$ into $k-\mathcal{ST}(\beta)$.

Proof: Let the function f given by (1.1) be a member of $\mathcal{R}^{\tau}(A,B)$. By (1.16) we have

$$H_{\mu,\delta}^{a,b,c}(f)(z)=z+\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n \quad (z\in\mathbb{U}).$$

In view of Lemma 2.1, it is sufficient to show that

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\beta)][1+(n-1)(\mu-\delta+n\mu\delta)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 1-\beta.$$

Since $f\in\mathcal{R}^{\tau}(A,B)$ by virtue of Lemma 2.3, it is again sufficient to show that

$$\begin{aligned} S_1 &= \sum_{n=2}^{\infty}\left[\frac{n(1+k)-(k+\beta)}{n}\right][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &\leq \frac{1-\beta}{(A-B)|\tau|}. \end{aligned} \tag{3.2}$$

Now

$$\begin{aligned} S_1 &= \sum_{n=2}^{\infty}\left[\frac{n(1+k)-(k+\beta)}{n}\right][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \sum_{n=2}^{\infty}\Bigg[(1+k)+(n-1)(1+k)(\mu-\delta)+n(n-1)(1+k)\mu\delta-\frac{k+\beta}{n} \\ &\quad -\frac{(n-1)(k+\beta)(\mu-\delta)}{n}-(n-1)(k+\beta)\mu\delta\Bigg]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \sum_{n=1}^{\infty}\Bigg[(1+k)+n(1+k)(\mu-\delta)+(n^2+n)(1+k)\mu\delta-\frac{k+\beta}{n+1} \\ &\quad -\frac{n(k+\beta)(\mu-\delta)}{n+1}-n(k+\beta)\mu\delta\Bigg]\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= (1+k)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]+(1+k)(\mu-\delta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} \\ &\quad +(1+k)\mu\delta\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-2}}+2(1+k)\mu\delta\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} \\ &\quad -(k+\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}-(k+\beta)(\mu-\delta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &\quad +(k+\beta)(\mu-\delta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n(n+1)}-(k+\beta)\mu\delta\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} \\ &= (1+k)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_1(1)_n}-(1+k)+(1+k)(\mu-\delta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} \end{aligned}$$

$$+ (1+k)\mu\delta \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + 2(1+k)\mu\delta \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}$$
$$- (k+\beta) \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} - (k+\beta)(\mu-\delta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$+ (k+\beta)(\mu-\delta) \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} - (k+\beta)\mu\delta \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}. \tag{3.3}$$

Repeated applications of the relation

$$(d)_m = d(d+1)_{m-1} \quad (d \in \mathbb{C},\ m \in \mathbb{N})$$

in (3.3) give

$$S_1 = (1+k) \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_1(1)_n} - (1+k) + (1+k)(\mu-\delta)\frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$
$$+ (1+k)\mu\delta\frac{(a)_2(b)_2}{(c)_2} \sum_{n=0}^{\infty} \frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + 2(1+k)\mu\delta\frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$
$$- (k+\beta)\frac{(c-1)}{(a-1)(b-1)} + \left\{ \sum_{n=0}^{\infty} \frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n} - 1 - \frac{(a-1)(b-1)}{c-1} \right\}$$
$$- (k+\beta)(\mu-\delta) \left\{ \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} - 1 \right\}$$
$$+ \frac{(k+\beta)(\mu-\delta)(c-1)}{(a-1)(b-1)} \left\{ \sum_{n=0}^{\infty} \frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n} - 1 - \frac{(a-1)(b-1)}{c-1} \right\}$$
$$- (k+\beta)\mu\delta\frac{ab}{c}$$
$$\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} = [(1+k) - (k+\beta)(\mu-\delta)]\,{}_2F_1(a,b;c;1)$$
$$+ [(1+k)(\mu-\delta) - (k+\beta)\mu\delta + 2(1+k)\mu\delta]\frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;1)$$
$$+ (1+k)\mu\delta\frac{(a)_2(b)_2}{(c)_2}\,{}_2F_1(a+2,b+2;c+2;1)$$
$$+ (\mu-\delta-1)\frac{(k+\beta)(c-1)}{(a-1)(b-1)}\,{}_2F_1(a-1,b-1;c-1;1)$$
$$+ \left[\frac{(k+\beta)(c-1)}{(a-1)(b-1)} - \frac{(k+\beta)(\mu-\delta)(c-1)}{(a-1)(b-1)} - (1-\beta) \right]. \tag{3.4}$$

Since the condition $c > a+b+2$ holds we use the Gauss summation formula (1.11) in (3.4) and get

$$S_1 = [(1+k) - (k+\beta)(\mu-\delta)]\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$
$$+ [(1+k(\mu-\delta) - (k+\beta)\mu\delta + 2(1+k)\mu\delta]\frac{ab}{c}$$
$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1+k)\mu\delta\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}$$
$$+ (\mu-\delta-1)\frac{(k+\beta)(c-1)}{(a-1)(b-1)}\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}$$
$$+ \left[\beta - 1 - \frac{(k+\beta)(\mu-\delta)(c-1)}{(a-1)(b-1)} + \frac{(k+\beta)(c-1)}{(a-1)(b-1)}\right]$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[(1+k) - (\mu-\delta)(k+\beta) + \{(1+k)(\mu-\delta)$$
$$- (k+\beta)\mu\delta + 2(1+k)\mu\delta\}\frac{ab}{c-a-b-1} + \frac{(\mu-\delta-1)(k+\beta)(c-a-b)}{(a-1)(b-1)}$$
$$+ (1+k)\mu\delta\frac{(a)_2(b)_2}{(c-a-b-2)_2}\Big] - \left[\frac{(k+\beta)(\mu-\delta-1)(c-1)}{(a-1)(b-1)} + (1-\beta)\right].$$

Thus, in view of (3.2) if the hypergeometric inequality (3.1) is satisfied, the

$$H^{a,,b,c}_{\mu,\delta}(f) \in k - \mathcal{ST}(\beta)$$

as asserted. This ends the proof of Theorem 3.1. □

Putting $\delta = 0$ in Theorem 3.1, we get the following result due to Sharma et al. (see [23] Theorem 1).

Corollary 3.2. *Let $a,\ b,\ c \in \mathbb{R}$ be such that $a, b > 1$ and $c > a + b + 1$. If the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{\mu(1+k)ab}{(c-a-b-1)} + \{1+k-\mu(k+\beta)\} + \frac{(k+\beta)(\mu-1)(c-a-b)}{(a-1)(b-1)}\right]$$
$$\leq \frac{1-\beta}{(A-B)|\tau|} + \frac{(k+\beta)(\mu-1)(c-1)}{(a-1)(b-1)} + (1-\beta),$$

is satisfied, then L_μ maps the class $\mathcal{R}^\tau(A,B)$ into $k - \mathcal{ST}(\beta)$.

Further, by taking $\mu = 0$ in Corollary 3.2, we get the following result:

Corollary 3.3. *Let $a, b, c \in \mathbb{R}$ be such that $a, b > 1$ and $c > a + b$. If the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+k-\frac{(k+\beta)(c-a-b)}{(a-1)(b-1)}\right] \leq \frac{1-\beta}{(A-B)|\tau|} - \frac{(k+\beta)(c-1)}{(a-1)(b-1)}$$
$$* + (1-\beta),$$

is satisfied, then $\mathcal{I}^{a,b}_c$ maps the class $\mathcal{R}^\tau(A,B)$ into $k - \mathcal{ST}(\beta)$.

Remark 3.4. *Letting $\beta = 0$ and $\tau = (1-\alpha)\cos\lambda e^{-i\lambda}$ $(0 \le \alpha < 1, |\lambda| < \frac{\pi}{2})$ in Corollary 3.3 we get the result of Aouf et al. ([1], Theorem 2.9).*

Letting $k = \beta = 0$ in Theorem 3.1 we have the following result in form of a corollary:

Corollary 3.5. *Let $a, b, c \in \mathbb{R}$ be such that $a, b > 0$ and $c > a + b + 2$. If the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + (\mu - \delta + 2\mu\delta)\frac{ab}{c-a-b-1} + \frac{\mu\delta(a)_2(b)_2}{(c-a-b-2)_2}\right] \le 1 + \frac{1}{(A-B)|\tau|},$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $\mathcal{R}^\tau(A,B)$ into the class $\mathcal{S}^*$.

Remark 3.6. *Letting $\delta = 0$ in Corollary 3.5 we get the result of Sharma et al. ([23],Corollary 1).*

Putting $k = 1$ and $\beta = 0$ in Theorem 3.1, we get the following result:

Corollary 3.7. *Let $a, b, c \in \mathbb{R}$ be such that $a, b > 1$ and $c > a + b + 2$. If the hypergeometric inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}&\left[2 - \mu + \delta + (2\mu - 2\delta + 3\mu\delta)\frac{ab}{c-a-b-1}\right.\\ &\left. + \frac{(\mu-\delta-1)(c-a-b)}{(a-1)(b-1)} + 2\mu\delta\frac{(a)_2(b)_2}{(c-a-b-2)_2}\right]\\ &\le 1 + (\mu-\delta-1)\frac{(c-1)}{(a-1)(b-1)} + \frac{1}{(A-B)|\tau|},\end{aligned}$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $\mathcal{R}^\tau(A,B)$ into $\mathcal{SP}$.

Remark 3.8. *Letting $\delta = 0$ in Corollary 3.7 we get the result due to Sharma et al. ([23], Corollary 2, page 330).*

Theorem 3.9. *Let $a, b, c \in \mathbb{R}$ be such that $a, b > 0$ and $c > a + b + 3$. If the hypergeometric inequality*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[\{(1+k)(1+2\mu-2\delta+4\mu\delta) - (\mu-\delta)(k+\beta)\\ &- 2\mu\delta(k+\beta)\}\frac{ab}{c-a-b-1}\\ &+ (1-\beta) + \{(1+k)(\mu-\delta+5\mu\delta) - \mu\delta(k+\beta)\}\frac{(a)_2(b)_2}{(c-a-b-2)_2}\\ &+ (1+k)\mu\delta\frac{(a)_3(b)_3}{(c-a-b-3)_3}\Bigg]\\ &\le \frac{1-\beta}{(A-B)|\tau|} + (1-\beta),\end{aligned} \tag{3.5}$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $\mathcal{R}^\tau(A,B)$ into $k-\mathcal{UCV}(\beta)$.

Proof: Let the function f given by (1.1) be a member of $\mathcal{R}^{\tau}(A,B)$. By virtue of Lemma 2.2 and the coefficient inequality (2.3) it is sufficient to show that

$$(A-B)|\tau|S_2 \leq 1-\beta, \tag{3.6}$$

where

$$\begin{aligned}
S_2 &= \sum_{n=2}^{\infty}[n(1+k)-(k+\beta)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= (1+k)\sum_{n=1}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n} - (k+\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&+ (1+k)(\mu-\delta)\sum_{n=1}^{\infty}n(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n} + (1+k)\mu\delta\sum_{n=1}^{\infty}n(n+1)^2\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&- (\mu-\delta)(k+\beta)\sum_{n=1}^{\infty}n\frac{(a)n(b)_n}{(c)_n(1)_n} - \mu\delta(k+\beta) \\
&\sum_{n=1}^{\infty}n(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n} = (1+k)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+!}(1)_n} + (1+k) \\
&\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right) - (k+\beta)\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right) \\
&+ (1+k)(\mu-\delta)\sum_{n=1}^{\infty}(n-1+2)\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} \\
&+ (1+k)\mu\delta\sum_{n=1}^{\infty}\{(n-1)(n-2)+5(n-1)+4\}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} \\
&- (\mu-\delta)(k+\beta)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} \\
&- \mu\delta(k+\beta)\left(\sum_{n=1}^{\infty}(n-1+2)\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}}\right) \\
&= [1+k+2(1+k)(\mu-\delta)+4(1+k)\mu\delta-(\mu-\delta)(k+\beta) \\
&- 2\mu\delta(k+\beta)]\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} \\
&+ [(1+k)(\mu-\delta)+5(1+k)\mu\delta-\mu\delta(k+\beta)]\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} \\
&+ (1+k)\mu\delta\sum_{n=0}^{\infty}\frac{(a)_{n+3}(b)_{n+3}}{(c)_{n+3}(1)_n} - (1-\beta) + (1-\beta)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.
\end{aligned} \tag{3.7}$$

Repeated applications of the relation

$$(d)_m = d(d+1)_{m-1} \quad (m \in \mathbb{N}\setminus\{1\})$$

in (3.7) give

$$\begin{aligned}S_2 =&[(1+k)(1+2(\mu-\delta)+4\mu\delta)-(\mu-\delta)(k+\beta)\\&-2\mu\delta(k+\beta)]\frac{ab}{c}{}_2F_1(a+1,b+1;c+1;1)\\&+(1+k)[\mu-\delta+5\mu\delta-\mu\delta(k+\beta)]\frac{(a)_2(b)_2}{(c)_2}{}_2F_1(a+2,b+2;c+2;1)\\&+(1+k)\mu\delta\frac{(a)_3(b)_3}{(c)_3}{}_2F_1(a+3,b+3;c+3;1)\\&-(1-\beta)+(1-\beta){}_2F_1(a,b;c;1).\end{aligned}$$

Since the condition $c > a+b+3$ holds we use the Gauss summation formula (1.11) and after simplification we get

$$\begin{aligned}S_2 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[&\{(1+k)(1+2(\mu-\delta)+4\mu\delta)-(\mu-\delta)(k+\beta)-2\mu\delta(k+\beta)\}\\&\frac{ab}{c-a-b-1}+(1-\beta)+\{(1+k)(\mu-\delta+5\mu\delta-\mu\delta(k+\beta)\}\frac{(a)_2(b)_2}{(c-a-b-2)_2}\\&+\frac{(1+k)\mu\delta(a)_3(b)_3}{(c-a-b-3)_3}\Big]-(1-\beta).\end{aligned}$$

Thus, in view of (3.6), if the hypergeometric inequality (3.5) is satisfied, then $H_{\mu,\delta}^{a,b,,c}(f) \in k-\mathcal{UCV}(\beta)$ as asserted.The proof of Theorem 3.9 is complete. □

Putting $\delta = 0$ in Theorem 3.9 we get the following result due to Sharma et al. ([23], Theorem 2).

Corollary 3.10. *Let $a, b, c \in \mathbb{R}$ be such that $a, b > 0$ and $c > a+b+2$. If $f \in \mathcal{R}^\tau(A, B)$ and the hypergeometric inequality*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\mu\frac{(1+k)(a)_2(b)_2}{(c-a-b-2)_2}+(1+k+2\mu+k\mu-\mu\beta)\frac{ab}{c-a-b-1}\right.\\&\left.+(1-\beta)\right]\\&\le\frac{1-\beta}{(A-B)|\tau|}+(1-\beta)\end{aligned}$$

is satisfied, the $L_\mu(f) \in k-\mathcal{UCV}(\beta)$.

Taking $\mu = 0$ in Corollary 3.10 we have the following result:

Corollary 3.11. *Let $a, b, c \in \mathbb{R}$ be such that $c > a+b+1$. If $f \in \mathcal{R}^\tau(A, B)$ and the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1+k)\frac{ab}{c-a-b-1}+(1-\beta)\right]\le\frac{1-\beta}{(A-B)|\tau|}+(1-\beta),$$

is satisfied, then $\mathcal{I}_c^{a,b}(f) \in k-\mathcal{UCV}(\beta)$.

Letting $k=\beta=0$ in Corollary 3.10 we have the following result due to Sharma et al. ([23], Corollary 3).

Corollary 3.12. *Let $a,b,c\in\mathbb{R}$ be such that $a,b>0$ and $c>a+b+2$. If $f\in\mathcal{R}^{\tau}(A,B)$ and the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1+2\mu)\frac{ab}{c-a-b-1}+\mu\frac{(a)_2(b)_2}{(c-a-b-2)_2}+1\right]\leq\frac{1}{(A-B)|\tau|}+1,$$

is satisfied, the $L_{\mu}(f)\in\mathcal{CV}$.

Taking $k=1$ and $\beta=0$ in Corollary 3.10, we improve the result of Sharma et al. ([23], Corollary 4).

Corollary 3.13. *Let $a,b,c\in\mathbb{R}$ be such that $a,b>0$ and $c>a+b+2$. If $f\in\mathcal{R}^{\tau}(A,B)$ and the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(3\mu+2)\frac{ab}{c-a-b-1}+2\mu\frac{(a)_2(b)_2}{(c-a-b-2)_2}+1\right]\leq\frac{1}{(A-B)|\tau|}+1,$$

is satisfied, then $L_{\mu}(f)\in\mathcal{UCV}$.

4. Integral operator

In this section, we define a particular integral operator $M_{\mu,\delta}(a,b,c;z)$ acting on $\mathcal{J}_{\mu,\delta}(a,b,c;z)$ as

$$M_{\mu,\delta}(a,b,c;z)=\int_0^z\frac{\mathcal{J}_{\mu,\delta}(a,b,c;t)}{t}dt, \tag{4.1}$$

and investigate its geometric properties.

Theorem 4.1. *Let $a,b,c\in\mathbb{R}$ be such that $a,b>1$ and $c>a+b+2$. If $f\in\mathcal{R}^{\tau}(A,B)$ and the hypergeometric inequality given by (3.1) is satisfied, then $M_{\mu,\delta}(a,b,c;z)*f(z)\in k-\mathcal{UCV}(\beta)$.*

Proof: Let the function f given by (1.1) be a member of $\mathcal{R}^{\tau}(A,B)$. By (4.1) we have,

$$M_{\mu,\delta}(a,b,c;z)*f(z)=z+\sum_{n=2}^{\infty}\left[\frac{1+(n-1)(\mu-\delta+n\mu\delta)}{n}\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n, \tag{4.2}$$

with $z\in\mathbb{U}$). In view of Lemma 2.2, it is sufficient to show that

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\beta)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}|a_n|\leq 1-\beta$$

Making use of Lemma 2.3, it is sufficient to show that

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-(k+\beta)]}{n}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq\frac{1-\beta}{(A-B)|\tau|}.$$

The rest part of the proof of Theorem 4.1 is similar to that of Theorem 3.1. We choose to omit the detail. This ends the proof of Theorem 4.1. □

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