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Some Common Fixed Point Theorems for Four Self-Mappings Satisfying a General Contractive Condition

Manoj Kumar, Rashmi sharma, Serkan Araci

Abstract: In the paper, we derive a general case for four weakly compatible self maps satisfying a general contractive condition due to the same method introduced by Altun et al. [2]. We make use of such a study to prove common fixed point theorems for weakly compatible maps along with $E.A.$ and (CLR) properties.

Key Words: Common fixed point, Weakly compatible, $E.A.$ property, (CLR) property.

Contents

1. Introduction

The study of common fixed point of mappings satisfying contractive conditions has been a very active field of research during recent years. The most general of the common fixed point theorems pertaining to four mappings A, B, S and T of a metric space (X, d), uses either a Banach-type contractive condition [3] of the form

$$d(Ax, By) \leq km(x, y) \ (0 \leq k < 1),$$

where

$$m(x, y) = max\{d(Ax, By), d(Sx, Ax), d(Ty, By) \text{ and } \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\},$$

or a Meir - Keeler - type (ε, δ) - contractive condition [6], that is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that or a φ - contractive condition [7] of the form

$$d(Ax, By) \leq \varphi(m(x, y)),$$

involving a contractive gauge function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for each $t > 0$. Note that Banach-type contractive condition is a special case of both conditions Meir - Keeler - type (ε, δ) - contractive and φ - contractive. A φ - contractive condition does not guarantee the existence of a fixed point unless some additional condition is assumed. Moreover, a φ - contractive condition, in general, does not imply the Meir - Keeler - type (ε, δ) - contractive condition. In the paper,

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we aim to prove a common fixed point theorem for four weakly compatible self - maps satisfying a general contractive condition and also prove common fixed point theorems for weakly compatible maps along with $E.A.$ and (CLR) properties.
We are now in a position to state the following three definitions which is an important to derive our main results.

Definition 1.1. [4] Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 1.2. [1] Two self-mappings f and g of a metric space (X, d) are said to satisfy $E.A.$ property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 1.3. [8] Two self-mappings f and g of a metric space (X, d) are said to satisfy $(CLRF)$ property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some x in X.

2. Main Results

Now, we give the following theorems.
Theorem 2.1. Let A, B, S and T be self maps of a metric space (X, d) satisfying the followings:

$$SX \subseteq BX,\ TX \subseteq AX, \tag{2.1}$$

for all $x \in X$, there exists right continuous functions $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\begin{gathered}\psi(0) = 0 = \phi(0) \text{ and } \psi(s) < s \text{ for } s > 0 \text{ such that}\\ \psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \phi(m(x, y)),\end{gathered} \tag{2.2}$$

where

$$m(x, y) = \max\{d(Ax, By), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}.$$

If one of AX, BX, SX or TX is complete subspace of X, then the pair (A, S) or (B, T) have a coincidence point. Moreover, if pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point of X. from (2.1), we can construct a sequence $\{y_n\}$ in X as follows:

$$y_{(2n+1)} = Sx_{2n} = Bx_{(2n+1)}, y_{(2n+2)} = Tx_{(2n+1)} = Ax_{(2n+2)}, \tag{2.3}$$

for all $n = 0, 1, 2, \ldots$. Define $d_n = d(y_n, y_{(n+1)})$. Suppose that $d_{2n} = 0$ for some n. Then $y_{2n} = y_{(2n+1)}$, that is, $Tx_{(2n-1)} = Ax_{2n} = Sx_{2n} = Bx_{(2n+1)}$, and A and S have a coincidence point. Similarly, if $d_{(2n+1)} = 0$, then B and T have a coincidence point. Assume that $d_n \neq 0$ for each n.

from (2.2), we have

$$\psi(d(Sx_{2n}, Tx_{(2n+1)}) \leq \psi(m(x_{2n}, x_{(2n+1)})) - \phi(m(x_{2n}, x_{2n+1})), \tag{2.4}$$

where

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(Ax_{2n}, Bx_{(2n+1)}), d(Sx_{2n}, Ax_{2n}), \\ &\quad \frac{d(Sx_{2n}, Bx_{(2n+1)}) + d(Tx_{(2n+1)}, Ax_{2n})}{2} \\ &\quad d(Tx_{(2n+1)}, Bx_{(2n+1)})\} \\ &= \max\{d_{2n}, d_{(2n+1)}\} \end{aligned} \tag{2.5}$$

Thus, from (2.4), we have

$$\psi(d(Sx_{2n}, Tx_{(2n+1)}) \leq \psi(\max\{d_{2n}, d_{2n+1}\}) - \phi(\max\{d_{2n}, d_{(2n+1)}\}). \tag{2.6}$$

Now, if $d_{(2n+1)} \geq d_{2n}$, for some n, then from (2.6), we have

$$\begin{aligned} \psi(d_{(2n+1)}) &\leq \psi(d_{(2n+1)}) - \phi(d_{(2n+1)}) \\ &< \psi(d_{(2n+1)}), \end{aligned} \tag{2.7}$$

which is a contradiction. Thus, $d_{2n} > d_{(2n+1)}$ for all n, and so, from (2.6), we have

$$\psi(d_{(2n+1)}) \leq \psi(d_{2n}) - \phi(d_{2n}), \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

Similarly,

$$\psi(d_{2n}) \leq \psi(d_{(2n-1)}) - \psi(d_{(2n-1)}),$$

$$\psi(d_{(2n-1)}) \leq \psi(d_{(2n-2)}) - \phi(d_{(2n-2)}).$$

In general, we have for all $n = 1, 2, ...,$

$$\begin{aligned} \psi(d_n) &\leq \psi(d_{(n-1)}) - \phi(d_{(n-1)}) \\ &< \psi(d_{(n-1)}). \end{aligned} \tag{2.9}$$

Hence the sequence $\{\psi(d_n)\}$ is monotonically decreasing and bounded below. Thus, there exists, $r \geq 0$, such that

$$\lim_{n\to\infty} \psi(d_n) = r. \tag{2.10}$$

From(9), we deduce that

$$0 \leq \phi(d_{(n-1)}) \leq \psi(d_{(n-1)}) - \psi(d_n).$$

Letting limit as $n \to \infty$ and using (10), we get $\lim_{n\to\infty} \phi(d_{(n-1)}) = 0$ implies that

$$\lim_{n\to\infty} \phi(d_{(n-1)}) = \lim_{n\to\infty} (d(y_{(n-1)}, y_n)) = 0, \tag{2.11}$$

or

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{(n+1)}) = 0. \tag{2.12}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Let, if possible, $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ such that for each even integer $2k$ there exists even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{(2n(k))}, y_{(2m(k))}) \geq \varepsilon. \tag{2.13}$$

For every even integer $2k$, suppose that $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (13) such that

$$d(y_{2n(k)}, y_{(2m(k)-2)}) < \varepsilon. \tag{2.14}$$

from (2.13), we have

$$\begin{aligned} \varepsilon \leq & d(y_{2n(k)}, y_{2m(k)}) \\ \leq & d(y_{2n(k)}, y_{(2m(k)-2)}) + d(y_{(2m(k)-2)}, y_{(2m(k)-1)}) + d(y_{(2m(k)-1)}, y_{2m(k)}). \end{aligned}$$

Using (12) and (14) in the above inequality, we get

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{2.15}$$

Also, by the triangular inequality,

$$|d(y_{(2n(k))}, y_{(2m(k)-1)}) + d(y_{(2n(k)} y_{(2m(k))})| \leq d_{(2m(k)-1)},$$

$$|d(y_{(2n(k)+1)}, y_{(2m(k)-1)}) + d(y_{(2n(k))}, y_{(2m(k))})| \leq d_{(2m(k)-1)} + d_{2m(k)}. \tag{2.16}$$

Using (12), we get

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{(2m(k)-1)}) = \lim_{k\to\infty} d(y_{(2n(k)+1)}, y_{(2m(k)-1)}) = \varepsilon. \tag{2.17}$$

from (2.2), we have

$$\begin{aligned} \psi(d(Sx_{2n(k)}, Tx_{(2m(k)-1)}) \quad \leq \quad & \psi(m(x_{(2n(k))}, x_{(2m(k)-1)}) \\ & -\phi(m(x_{(2n(k)}, x_{(2m(k)-1)}), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} m(x_{2n(k)}, x_{(2m(k)-1)}) = & \max\{d(Ax_{2n(k)}, Bx_{(2m(k)-1)}), d(Sx_{2n(k)}, Ax_{2n(k)}), \\ & \frac{(d(Sx_{2n(k)}, Bx_{(2m(k)-1)}) + d(Tx_{2n(k)}, Ax_{(2m(k)-1)})}{2}, \\ & d(Tx_{(2m(k)-1)}, Bx_{(2m(k)-1)})\} \\ = & \max\{d(y_{2n(k)}, y_{(2m(k)-1)}), d(y_{2n(k)}, y_{(2n(k)+1)}), \\ & \frac{(d(y_{(2n(k)+1)}, y_{(2m(k)-1)}) + d(y_{2n(k)}, y_{(2m(k)-1)})}{2}, \\ & d(y_{(2m(k)-1)}, y_{2m(k)})\}. \end{aligned}$$

Letting limit as $k \to \infty$ and using (17), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction, since $\varepsilon > 0$. Thus, $\{y_{2n}\}$ is a Cauchy sequence and so $\{y_n\}$ is a Cauchy sequence. Now, suppose that $A(X)$ is complete. Note that $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(X)$, say u, that is, $\lim_{n\to\infty} y_{2n} = u$. Let $v \in A^{(-1)}u$. Then $Av = u$. Now, we shall prove that $Sv = u$. Let, if possible, $Sv \neq u$, that is, $d(Sv, u) = p > 0$.

Putting $x = v$ and $y = x_{(2n-1)}$ in (1.2), we have

$$\psi(d(Sv, Tx_{(2n-1)}) \leq \psi(m(v, x_{(2n-1)}) - \phi(m(v, x_{(2n-1)}).$$

Letting limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} \psi(d(Sv, Tx_{(2n-1)}) \quad &\leq \quad \lim_{n\to\infty} \psi(m(v, x_{(2n-1)}) \\ &\quad - \lim_{n\to\infty} \phi(m(v, x_{(2n-1)}), \end{aligned} \tag{2.19}$$

where,

$$\begin{aligned}\lim_{n\to\infty} m(v, x_{(2n-1)}) = &\lim_{n\to\infty} [\max\{d(u, y_{(2n-1)}), d(Sv, u), d(y_{2n}, y_{(2n-1)}), \\ &\frac{(d(Sv, y_{(2n-1)}) + d(y_{2n}, u))}{2}\}] \\ = &\max\{d(u, u), d(Sv, u), d(u, u), \frac{1}{2}(d(Sv, u) + d(u, u))\} \\ = &d(Sv, u) = p.\end{aligned}$$

Thus, from (2.19), we have

$$\psi(d(Sv, u) \leq \psi(p) - \phi(p),$$

that is

$$\psi(p) \leq \psi(p) - \phi(p),$$

which is a contradiction, since $p > 0$. Thus, $Sv = u = Av$. Hence u is the coincidence point of the pair (A, S). Since $SX \subseteq BX$, $Sv = u$, implies that, $u \in BX$. Let $w \in B^{(-1)}u$. Then $Bw = u$. By using the same arguments as above, one can easily verify that, $Tw = u = Bw$, that is, u is the coincidence point of the pair (B, T). The same result holds, if we assume that BX is complete instead of AX. Now, if TX is complete, then by (1), $u \in TX \subseteq AX$. Similarly, if SX is complete, then $u \in SX \in BX$. Now, since the pairs (A, S) and (B, T) are weakly compatible, so

$$u = Sv = Av = Tw = Bw,$$

then

$$\begin{aligned}Au = ASv = SAv = Su, \\ Bu = BTw = TBw = Tu.\end{aligned} \tag{2.20}$$

Now, we claim that $Tu = u$. Let, if possible, $Tu \neq u$.
from (2.2), we have

$$\begin{aligned}\psi(d(u,Tu) =&\psi(d(Sv,Tu)\\ \leq&\psi(m(v,u)) - \phi(m(v,u)),\end{aligned}$$

where

$$\begin{aligned}m(v,u) =&\max\{d(Av,Bu), d(Sv,Av), d(Tu,Bu), \frac{1}{2}(d(Sv,Bu) + d(Tu,Av))\}\\ =&\max\{d(u,Tu), d(u,u), 0, \frac{1}{2}(d(u,Tu) + d(Tu,u))\}\\ =&d(u,Tu).\end{aligned}$$

Thus, we have

$$\begin{aligned}\psi(d(u,Tu) \leq&\psi(d(u,Tu)) - \phi(d(u,Tu))\\ <&\psi(d(u,Tu)),\end{aligned}$$

which is a contradiction. So, $Tu = u$. Similarly, $Su = u$. Thus, we get $Au = Su = Bu = Tu = u$. Hence u is the common fixed point of A, B, S and T. For the uniqueness, let z be another common fixed point of A, B, S and T.

Now, we claim that $u = z$. Let, if possible, $u \neq z$.
from (2.2), we have

$$\begin{aligned}\psi(d(u,z) =&\psi(d(Su,Tz)\\ \leq&\psi(m(u,z)) - \phi(m(u,z))\\ =&\psi(d(u,z)) - \phi(d(u,z)),\end{aligned}$$

since

$$\begin{aligned}m(u,z) =&d(u,z)\\ <&\psi(d(u,z)),\end{aligned}$$

a contradiction. Thus, $u = z$, and the uniqueness follows.

Theorem 2.2. Let A, B, S and T be self mappings of a metric space (X,d) satisfying (1), (2) and the followings:

$$\text{pairs } (A,S) \text{ and } (B,T) \text{ are weakly compatible,} \tag{2.21}$$

$$\text{pair } (A,S) \text{ or } (B,T) \text{ satisfy the E.A. property.} \tag{2.22}$$

If any one of AX, BX, SX and TX is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof: Suppose that (A,S) satisfies the $E.A.$ property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some z in

X. Since $SX \subseteq BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = z$. We shall show that $\lim_{n\to\infty} Ty_n = z$. Let, if possible, $\lim_{n\to\infty} Ty_n = t = z$.

from (2.2), we have

$$\psi(d(Sx_n, Ty_n) \leq \psi(m(x_n, y_n)) - \phi(m(x_n, y_n)).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d(Sx_n, Ty_n) \leq \lim_{n\to\infty} \psi(m(x_n, y_n)) - \lim_{n\to\infty} \phi(m(x_n, y_n)), \tag{2.23}$$

where,

$$\begin{aligned}\lim_{n\to\infty} m(x_n, y_n) = &\lim_{n\to\infty}[\max\{d(Ax_n, By_n), d(Sx_n, Ax_n), d(Ty_n, By_n),\\ &\frac{1}{2}(d(Sx_n, By_n) + d(Ty_n, Ax_n))\}]\\ =&\max\{d(z,z), d(z,z), d(t,z), \frac{1}{2}(d(z,z) + d(t,z))\}\\ =&d(t,z).\end{aligned}$$

Thus, from (2.23), we get

$$\begin{aligned}\psi(d(z,t) \leq &\psi(d(z,t)) - \phi(d(z,t))\\ <&\psi(d(z,t)),\end{aligned}$$

which is a contradiction. Therefore, $t = z$, that is, $\lim_{n\to\infty} Ty_n = z$. Suppose that BX is a complete subspace of X. Then $z = Bu$ for some u in X. Subsequently, we have

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z = Bu.$$

Now, we shall show that $Tu = Bu$. Let, if possible, $Tu \neq Bu$.

from (2.2), we have

$$\psi(d(Sx_n, Tu) \leq \psi(m(x_n, u)) - \phi(m(x_n, u)).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d(Sx_n, Tu) \leq \lim_{n\to\infty} \psi(m(x_n, u)) - \lim_{n\to\infty} \phi(m(x_n, u)), \tag{2.24}$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x_n, u) = &\lim_{n\to\infty}[\max\{d(Ax_n, Bu), d(Sx_n, Ax_n), d(Tu, Bu),\\ &\frac{1}{2}(d(Sx_n, Bu) + d(Tu, Ax_n))\}]\\ =&\max\{d(z,z), d(z,z), d(Tu,z), \frac{1}{2}(d(z,z) + d(Tu,z))\}\\ =&d(Tu,z).\end{aligned}$$

Thus, from (2.24), we have

$$\begin{aligned}\psi(d(z,Tu) \leq& \psi(d(z,Tu)) - \phi(d(z,Tu))\\ <& \psi(d(z,Tu)),\end{aligned}$$

which is a contradiction. Therefore, $Tu = z = Bu$. Since B and T are weakly compatible, therefore, $BTu = TBu$, implies that, $TTu = TBu = BTu = BBu$. Since $TX \subseteq AX$, there exists $v \in X$, such that, $Tu = Av$.
Now, we claim that $Av = Sv$. Let, if possible, $Av \neq Sv$.
from (2.2), we have

$$\psi(d(Sv,Tu) \leq \psi(m(v,u)) - \phi(m(v,u)), \tag{2.25}$$

where

$$\begin{aligned}m(v,u) =& \max\{d(Av,Bu), d(Sv,Av), d(Tu,Bu), \frac{1}{2}(d(Sv,Bu) + d(Tu,Av))\}\\ =& d(Sv,Av) = d(Sv,Tu).\end{aligned}$$

Thus, from (2.25), we have

$$\begin{aligned}\psi(d(Sv,Tu) \leq& \psi(d(Sv,Tu)) - \phi(d(Sv,Tu)\\ <& \psi(d(Sv,Tu)),\end{aligned}$$

which is a contradiction. Therefore, $Sv = Tu = Av$. Thus, we have, $Tu = Bu = Sv = Av$. The weak compatibility of A and S implies that $ASv = SAv = SSv = AAv$. Now, we claim that Tu is the common fixed point of A, B, S and T. Suppose that, $TTu \neq Tu$.
from (2.2), we have

$$\begin{aligned}\psi(d(Tu,TTu) &= \psi(d(Sv,TTu)\\ &\leq \psi(m(v,Tu)) - \psi(m(v,Tu)),\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}m(v,Tu) =& \max\{d(Av,BTu), d(Sv,Av), d(BTu,TTu),\\ & \frac{1}{2}(d(Sv,BTu) + d(TTu,Av))\}\\ =& \max\{d(Tu,TTu), 0, 0, d(Tu,TTu)\}\\ =& d(Tu,TTu).\end{aligned}$$

Thus, from (2.26), we have

$$\begin{aligned}\psi(d(Tu,TTu) \leq& \psi(d(Tu,TTu)) - \phi(d(Tu,TTu))\\ <& \psi(d(Tu,TTu)),\end{aligned}$$

which is a contradiction. Therefore, $Tu = TTu = BTu$. Hence Tu is the common fixed point of B and T. Similarly, we prove that Sv is the common fixed point of

A and S. Since $Tu = Sv$, Tu is the common fixed point of A, B, S and T. The proof is similar when AX is assumed to be a complete subspace of X. The cases in which or SX is a complete subspace of X are similar to the cases in which AX or BX, respectively is complete subspace of X, since $TX \subseteq AX$ and $SX \subseteq BX$.

Now, we shall prove that the common fixed point is unique. If possible, let p and q be two common fixed points of A, B, S and T, such that, $p \neq q$.

from (2.2), we have

$$\begin{aligned}\psi(d(p,q) &= \psi(d(Sp,Tq)\\ &\leq \psi(m(p,q)) - \phi(m(p,q)),\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}m(p,q) =& \max\{d(Ap,Bq), d(Sp,Aq), d(Bq,Tq), \frac{1}{2}(d(Sp,Bq) + d(Tq,Ap))\}\\ =& \max\{d(p,q), 0, 0, d(p,q)\}\\ =& d(p,q).\end{aligned}$$

Thus, from (2.27), we have

$$\begin{aligned}\psi(d(p,q) \leq& \psi(d(p,q)) - \phi(d(p,q))\\ <& \psi(d(p,q)),\end{aligned}$$

which is a contradiction. Therefore, $p = q$, and the uniqueness follows.

Theorem 2.3. Let A, B, S and T be self maps of a metric space (X,d) satisfying (2), (21) and the following:

$$\begin{aligned}&SX \subseteq BX \text{ and the pair } (A,S) \text{ satisfies } (CLR_A) \text{ property or}\\ &TX \subseteq AX \text{ and the pair } (B,T) \text{ satisfies } (CLR_B) \text{ property.}\end{aligned} \tag{2.28}$$

Then A, B, S and T have a unique common fixed point.

Proof: Without loss of generality, assume that $SX \subseteq BX$ and the pair (A,S) satisfies (CLR_A) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Ax$, for some x in X. Since $SX \subseteq BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = Ax$. We shall show that $\lim_{n\to\infty} Ty_n = Ax$. Let, if possible, $\lim_{n\to\to} Ty_n = z \neq Ax$.

from (2.2), we have

$$\psi(d(Sx_n, Ty_n) \leq \psi(m(x_n,y_n)) - \phi(m(x_n,y_n)).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d(Sx_n, Ty_n) \leq \lim_{n\to\infty} \psi(m(x_n,y_n)) - \lim_{n\to\infty} \phi(m(x_n,y_n)), \tag{2.29}$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x_n, y_n) = &\lim_{n\to\infty}[\max\{d(Ax_n, By_n), d(Sx_n, Ax_n), d(Ty_n, By_n),\\ &\frac{1}{2}(d(Sx_n, By_n) + d(Ty_n, Ax_n))\}]\\ =&\max\{d(Ax, Ax), d(Ax, Ax), d(z, Ax), \frac{1}{2}(d(z, z) + d(z, Ax))\}\\ =&d(z, Ax).\end{aligned}$$

Thus, from (2.29), we get

$$\begin{aligned}\psi(d(Ax, z) \leq &\psi(d(Ax, z)) - \phi(d(Ax, z))\\ <&\psi(d(Ax, z)),\end{aligned}$$

which is a contradiction. Therefore, $Ax = z$, that is, $lim_{n\to\infty}Tyn = Ax$. Subsequently, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Ax = z.$$

Now, we shall show that $Sx = z$. Let, if possible, $Sx \neq z$. from (2.2), we have

$$\psi(d(Sx, Ty_n) \neq \psi(m(x, y_n)) - \phi(m(x, y_n)).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d(Sx, Ty_n) \leq \lim_{n\to\infty} \psi(m(x, y_n)) - \lim_{n\to\infty} \phi(m(x, y_n)), \tag{2.30}$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x, y_n) = &\lim_{n\to\infty}[\max\{d(Ax, By_n), d(Sx, Ax), d(Ty_n, By_n),\\ &\frac{1}{2}(d(Sx, By_n) + d(Ty_n, Ax))\}]\\ =&\max\{d(z, z), d(Sx, z), d(z, z), \frac{1}{2}(d(Sx, z) + d(z, z))\}\\ =&d(Sx, z).\end{aligned}$$

Thus, from (2.30), we get

$$\begin{aligned}\psi(d(Sx, z) \leq &\psi(d(Sx, z)) - \phi(d(Sx, z))\\ <&\psi(d(Sx, z)),\end{aligned}$$

which is a contradiction. Therefore, $Sx = z = Ax$. Since, the pair (A, S) is weakly compatible, it follows that $Az = Sz$. Also, since $SX \subseteq BX$, there exists some y in X such that $Sx = By$, that is, $By = z$. Now, we show that $Ty = z$. Let, if possible, $Ty \neq z$.

from (2.2), we have

$$\psi(d(Sx_n, Ty) \leq \psi(m(x_n, y)) - \phi(m(x_n, y)).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d(Sx_n, Ty) \leq \lim_{n\to\infty} \psi(m(x_n, y)) - \lim_{n\to\infty} \phi(m(x_n, y)), \tag{2.31}$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x_n, y) = &\lim_{n\to\infty}[\max\{d(Ax_n, By), d(Sx_n, Ax_n), d(Ty, By),\\ &\frac{1}{2}(d(Sx_n, By) + d(Ty, Ax_n))\}]\\ =&\max\{d(z,z), d(z,z), d(z,Ty), \frac{1}{2}(d(z,z) + d(Ty,z))\}\\ =&d(z,Ty).\end{aligned}$$

Thus, from (2.31), we get

$$\begin{aligned}\psi(d(z, Ty) \leq &\psi(d(z,Ty)) - \phi(d(z,Ty))\\ < &\psi(d(z,Ty)),\end{aligned}$$

which is a contradicition. Thus, $z = Ty = By$. Since the pair (B, T) is weakly compatible, it follows that $Tz = Bz$. Now, we claim that $Sz = Tz$. Let, if possible, $Sz \neq Tz$.

from (2.2), we have

$$\psi(d(Sz, Tz) \leq \psi(m(z,z)) - \phi(m(z,z)), \tag{2.32}$$

where

$$\begin{aligned}m(z,z) = &\max\{d(Az, Bz), d(Sz, Az), d(Bz, Tz), \frac{1}{2}(d(Sz, Bz) + d(Tz, Az)\\ =&d(Sz, Tz).\end{aligned}$$

Thus, from (2.32), we have

$$\begin{aligned}\psi(d(Sz, Tz) \leq &\psi(d(Sz,Tz)) - \phi(d(Sz,Tz))\\ < &\psi(d(Sz,Tz)),\end{aligned}$$

which is a contradiction. Therefore, $Sz = Tz$, that is, $Az = Sz = Tz = Bz$. Now, we shall show that $z = Tz$. Let, if possible, $z \neq Tz$.

from (2.2), we have

$$\psi(d(Sx, Tz) \leq \psi(m(x,z)) - \phi(m(x,z)), \tag{2.33}$$

where

$$\begin{aligned} m(x,z) =& \max\{d(Ax,Bz), d(Sx,Ax), d(Bz,Tz), \frac{1}{2}(d(Sx,Bz)+d(Tz,Ax))\} \\ =& d(Sx,Tz) = d(z,Tz). \end{aligned}$$

Thus, from (2.33), we have

$$\begin{aligned} \psi(d(z,Tz) \leq& \psi(d(z,Tz)) - \phi(d(z,Tz)) \\ <& \psi(d(z,Tz)), \end{aligned}$$

which is a contradicition. Therefore, $z = Tz = Bz = Az = Sz$. Hence z is the common fixed point of A, B, S and T. Now, we shall prove that the common fixed point is unique. Let u be another common fixed point of A, B, S and T. Let, if possible, $z \neq u$.

from (2.2), we have

$$\begin{aligned} \psi(d(u,z) =& \psi(d(Su,Tz) \\ \leq& \psi(m(u,z)) - \phi(m(u,z)) \\ =& \psi(d(u,z)) - \phi(d(u,z)), \text{ since } m(u,z) = d(u,z) \\ <& \psi(d(u,z)), \end{aligned}$$

which is a contradiction. Thus, $u = z$, and hence the uniqueness follows.

Example 2.4. Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$. Let the self maps A, B, S and T be defined by

$$Sx = \frac{x}{8}, Bx = \frac{x}{4}, Tx = \frac{x}{2}, Ax = x.$$

Clearly,

$$\begin{aligned} SX = [0, \frac{1}{8}] \subseteq [0, \frac{1}{4}] = BX, \\ TX = [0, \frac{1}{2}] \subseteq [0,1] = AX. \end{aligned}$$

Also AX is complete subspace of X and pairs (A,S), (B,T) are weakly compatible. Now,

$$\begin{aligned} d(Sx,Ty) =& |\frac{x}{8} - \frac{y}{2}| = \frac{x}{8}|x-4y|. \\ d(Ax,By) =& |x - \frac{y}{4}| = \frac{1}{4}|4x-y|. \\ d(Sx,Ax) =& |\frac{x}{8} - x| = \frac{7}{8}x. \\ d(By,Ty) =& |\frac{y}{4} - \frac{y}{2}| = \frac{y}{4}. \\ \frac{(d(Sx,By)+d(Ty,Ax))}{2} =& \frac{1}{2}[|\frac{x}{8}n - \frac{y}{4}| + |\frac{y}{2} - x|] \\ =& \frac{1}{16}[|x-2y| + 4|y-2x|]. \end{aligned}$$

Let $\psi(t) = \frac{t}{3}$ and $\phi(t) = \frac{t}{6}$. Thus, we have

$$\psi(d(Sx, Ty)) = \frac{1}{24}|x - 4y|.$$

$$\begin{aligned} m(x,y) =& \max\{d(Ax, By), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\} \\ =& d(Sx, Ax). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(d(Sx, Ax)) =& \frac{1}{3}(\frac{7}{8}x) = \frac{7}{24}x. \\ \phi(d(Sx, Ax)) =& \frac{1}{6}(\frac{7}{8}x) = \frac{7}{48}x. \end{aligned}$$

Thus, we have

$$\psi(m(x,y)) - \phi(m(x,y)) = \frac{7}{24}x - \frac{7}{48}x = \frac{7}{48}x.$$

Therefore,

$$\psi(d(Sx, Ty) \leq \psi(m(x,y)) - \phi(m(x,y)).$$

Hence condition (2) is satisfied. If, we consider the sequence $\{x_n\} = \{\frac{1}{n}\}$, then

$$\begin{aligned} \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0. \\ \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{\frac{n}{8}} = \lim_{n\to\infty} \frac{1}{8n} = 0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0, \text{ where } 0 \in X.$$

So the pair (A, S) satisfies the $E.A.$ property. Also,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ax_n = 0 = A(0).$$

So the pair (A, S) satisfies the (CLR_A) property. Hence all the conditions of above Theorems are satisfied. Here 0 is the unique common fixed point of A, S, B and T.

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Manoj Kumar (Corresponding Author),
Department of Mathematics,
Lovely Professional University,
Phagwara, Punjab,
India.
E-mail address: manojantil18@gmail.com, manoj.19564@lpu.co.in

and

Rashmi Sharma,
Department of Mathematics,
Lovely Professional University, Phagwara, Punjab,
India.
E-mail address: rashmisharma.lpu@gmail.com

and

Serkan Araci,
Faculty of Economics,
Administrative and Social Sciences,Department of Economics,
Hasan Kalyoncu University, TR-27410 Gaziantep,
Turkey.
E-mail address: mtsrkn@hotmail.com